The centenary of Mark Kac (1914–1984)

Rafael D. Benguria
Instituto de Física
Pontificia Universidad Católica de Chile
Casilla 306, Santiago 22, Chile, rbenguri@puc.cl

After completing my Ph.D. in Physics at Princeton University, I obtained a post-doctoral position at Rockefeller University. In the early sixties, Detlev Bronk, who was president (1953–1968) of the then *Rockefeller Institute for Medical Research* hired George Uhlenbeck, Mark Kac and Theodore Berlin (1961), among others, to establish a Mathematical Physics group, and Abraham Pais (1962) to lead a group in High Energy Physics. In fact Detlev Bronk successfully made the transition from a research institute to the Rockefeller University (1965). Kenneth Case joined the Mathematical Physics group at Rockefeller in 1969, and so did James Glimm in 1974. I stayed in Kenneth Case's Lab at Rockefeller University from 1979 to 1981. There I had the chance to work with Ken Case and Mark Kac [2], to meet many visitors in Mathematics and Physics and to enjoy the friendly atmosphere of the 14th floor of the Tower Building where the labs of Ken Case and Eddie Cohen were housed. This year marks the hundredth anniversary of the birth of Mark Kac (1914-1984), who was a prominent figure in mathematics and physics of the twentieth century, and I think it is appropriate to remember his life and work in the Bulletin of the IAMP.

Mark Kac was born three weeks after the beginning of the First World War (August 16) in Krzemieniec, in Central Europe. Krzemieniec, at the foot of Mountain Bona, is a city that has belonged to different countries in recent history. Even during the early years of Mark Kac it was part of the Austro–Hungarian Empire, then part of the Russian Empire, later a Polish Territory and finally part of the Ukraine, where its is known as Kremenets). At the time of Mark Kac's birth, Krzemieniec was part of Vohlynia, and a typical cultural city of Central Europe. The romantic Polish poet Juliusz Słowacki was born there in the early XIXth century, and a contemporary of Mark Kac, the violinist Isaac Stern was born in Krzemieniec in 1920. Although during the two World Wars suffered enormously (specially during the Holocaust in the Second World War), Krzemieniec enjoyed a quiet and stimulating atmosphere in the interbellum. In 1922, the polish leader Jósef Piłsudski, reopened the Lyceum of Krzemieniec, who had been founded in the XIXth century under the supervision of Vilnius University. The Lyceum was closed by the Soviet occupation army in September 1939, at the beginning of the Second World War. A measure of the Lyceum reputation is the fact that it was soon known as the "Athens of Volhynia". Mark Kac entered the Lyceum in 1925. In the summer of 1930 Kac had his first experience with research. Acquainted with the Cardano solution of the cubic equation, he wanted to find an alternative way of finding that solution. He exploited the invariance of the equation under bilinear transformations. Doing so, he found a two parameter flow of cubic equations and determined the parameters of the flow that yielded a trivial cubic, to finally determine the solution of the original equation. One can certainly repeat that procedure for the quartic as well, and use the two parameters to reduce the quartic into a quadratic in x^2. This

Mark Kac, circa 1980

was the content of Mark Kac's first paper, which appeared in the "Młody Mathematyk" (i.e., *The Young Mathematician*). In 1931, after completing High School at the Lyceum, Kac moved to Lvov (at that time a city of the region of Galicia, then part of Poland, today known as Lviv, a city in western Ukraine) to attend the Jan Kazimierz University, which was the name of the University of Lvov in the period 1919–1939 (today is known as the Ivan Franko University of Lviv) in honour of its founder, the King John Casimir (1661). Between the wars, Lvov (with its two universities, namely the Jan Kazimierz and the Technical University) was the site of the famous "Lvov School of Mathematics", which played a major role in the development of Functional Analysis. Stephan Banach at the Technical University, and Hugo Steinhaus at the Jan Kazimierz university were the leading figures of this School, which also included Stanisław Mazur, Juliusz Schauder, Stan Ulam, and many others. The mathematical atmosphere in Lvov at that time is recollected by Ulam in the Introduction of [42]. In Ulam's words, *"the mathematical life was very intense in Lwów. Some of us met practically every day, ... to discuss problems of common interest, communicating to each other the latest work and results. Apart from the more official meetings of the local sections of the Mathematical Society, there were frequent informal discussions mostly held in one of the coffee houses located near the University building –one of them a coffee housed named "Roma", and the other "The Scottish Coffee House".* There are many recollections (see, e.g.,

[33]) about “The Scottish Coffee House” (“Kawiarnia Szkocka” in Polish, located in 27 Taras Shevchenko Prospekt, Lvov). According to Ulam [42], Stephan Banach in 1935 suggested keeping track of the problems occupying the group of mathematicians that were gathering there. A version of this book in English is available online in [42] where Ulam edited, in 193 entries, the problems discussed by the group between 1935 and May 1941, the year Steinhaus had to leave Lvov. Every entry has the name of the proposer of the problem. Among the proposers one finds names like Banach, Steinhaus, Ulam, Mazur, Kac, Marcinkiewicz, Kaczmarz, Sobolev, Ljusternik, von Neumann, Eilenberg, Zygmund, Auerbach, Sierpinski and several others. Several proposers offered special “prizes” for the solutions. Among the prizes offered for the solutions one finds: “five small beers”, “a bottle of wine” and even “a fondue in Geneva”. While still a graduate student, Mark Kac participated in this group, and in the “Scottish Book of Problems” he has four entries, which shed some light about his mathematical concerns at that time. For example in the problem 126, he asks: If $\int_0^1 f(x)\,dx = 0$ and $\int_0^1 f^2(x)\,dx = \infty$, prove that

$$\lim_{n\to\infty} \left(\int_0^1 \exp(i\frac{f(x)}{\sqrt{n}})\,dx \right)^n = 0.$$

Kac asserts that when $\int_0^1 f^2(x)\,dx = A$ (finite), the resulting limit is known and in fact equal to $1/\sqrt{e}$. This problem 126 was soon solved by A. Khinchin, who published his result in Studia Mathematica. In the entry 161 (dated June 10, 1937), there is the following Theorem of Mark Kac: Let r_n be a sequence of of integers such that

$$\lim_{n\to\infty} (r_n - \sum_{k=1}^{n-1} r_k) = \infty.$$

Prove that,

$$\lim_{n\to\infty} \left[\mathbb{E}_{0\le x\le 1} \left(a < \frac{\sin(2\pi r_1 x) + \cdots + \sin(2\pi r_n x)}{\sqrt{n}} < b \right) \right] = \frac{1}{\sqrt{\pi}} \int_a^b \exp(-y^2)\,dy.$$

One can put, e.g., $r_n = 2^{n^2}$. And he asks wether the same result hold if $r_n = 2^n$ (where one can see that the condition imposed on the sequence r_n does not hold. As described in the biography [34], *in short sinusoids of independent frequency behave as if they were statistically independent though strictly speaking they are not.* See also [23, 24]. Mark Kac got his Ph.D. in 1937 [22] under the supervision of Hugo Steinhaus. He acquired from his advisor the interest and taste for working on problems related to statistical independence. Using Kac’s own words [31] (see Chapter 3, p. 48) to describe his interaction with Steinhaus: *“My Mathematical life began with my collaboration with Hugo Steinhaus. That three–year period (from the Spring of 1935 to the end of November 1938)... was decisive in my development as a mathematician”.* In the period 1936–1937 Kac and Steinhaus wrote a series of four papers under the general title “Sur les fonctions indépendantes”, which were published in the Polish mathematical journal “Studia Mathematica” founded by Banach and Steinhaus in 1929. Mark Kac completed his Ph.D. thesis [22] precisely on the subject of statistical independence in 1937. A bit more than twenty years later, Kac was asked to deliver a series of lectures at Haverford College (Spring of 1958), and he retook the subject of statistical independence. As an outgrowth of these lectures, Mark Kac wrote the book "Statistical Independence in Probability Analysis and Number Theory" [28], in the Carus Series of the Mathematical Association of America. Kac dedicated this book to his teacher Hugo Steinhaus. As described by Henry McKean, *"This is a splendid book. It ranges from the primitive idea of statistical independence to applications of the most diverse sort: coin–tossing, anharmonic oscillators, prime numbers and continued fractions. And it does all that with Kac's customary clarity and charm..."*. In particular, in this book one can find the proof of the Kac's theorem stated as the entry 161 of "Scottish Book of Problems" quoted above.

In 1938, Kac obtained a Polish fellowship from the Parnas Foundation to visit Johns Hopkins University. He left behind his whole family, most of whom perished in Krzemieniec in the mass executions of 1942-43. He arrived at Baltimore in December 1938 [1]. Soon after his arrival in the US, Kac met Norbert Wiener and Paul Erdös, which were influential in his mathematical career. The same year Kac arrived in the US, Paul Erdös arrived at the Institute for Advanced Study (IAS) for a one year appointment. In 1939, Mark Kac was invited to give a lecture at the IAS and met Erdös. Both discovered that they could apply their respective background in Number Theory (Erdös) and Probability to solve a problem in Number Theory, namely that for any natural number n, the number of prime divisors of the integers less than n has a normal distribution. If $\nu(n)$ is the number of prime divisors of n, loosely speaking, the probability distribution of

$$\frac{\nu(n) - \log\log n}{\sqrt{\log\log n}}$$

is the standard normal distribution.

Their joint work published in 1940 in the Journal of the American Mathematical Society [14] is one of the pillars of the then new field of Probabilistic Number Theory. In an interview with Mitchell Feigenbaum [15], Kac shows his pride on this particular work: *"...In retrospect the thing which I am happiest about, and it was done in cooperation with Erdös... was the introduction of probabilistic methods in number theory. To put it poetically, primes play a game of chance"*.

With the recommendation of Norbert Wiener, Mark Kac obtained an instructorship at Cornell University in 1939. It was in Ithaca, for his 27th birthday that Mark Kac met his future wife Katherine Mayberry to whom he married in 1942. They had two children, Michael and Deborah. Kac was firs promoted to Assistant Professor in 1943 and to Full Professor in 1947. He stayed at Cornell University until 1961, when he moved to Rockefeller. During the Second World War, Mark Kac worked also at the Radiation Laboratory at MIT, in Cambridge, Massachusetts. The wartime research at the Radiation Lab was mainly devoted to waveguide theory and to the study of problems of noise in radar systems. It was in Cambridge that Mark Kac met George Uhlenbeck [17], who had left Ann Arbor, MI, to direct the Radiation Lab. Mark Kac returned to MIT in the academic year 1946–47, on leave from Cornell, supported by a Guggenheim fellowship. Kac and Uhlenbeck developed a close friendship and collaboration that lasted until Kac's death in 1984.

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[1] Kac sailed from Poland to the US in the M/S Piłsudski, which served in the Gdynia–Amerika Shipping Lines Ltd. from Gydnia to Hoboken from 1935 to the beginning of the Second World War. The M/S Piłsudski sank on November 26, 1939 after hitting two mines off the coast of Yorkshire, on war service.

Uhlenbeck introduced Kac to problems in physics, in particular in Statistical Mechanics. It was Uhlenbeck who introduced Kac to the *dog–flea* problem, a problem formulated by Paul Ehrenfest (Uhlenbeck's advisor) and his wife Tatiana in 1907 [12] to illustrate the second law of thermodynamics. The model (see, e.g.,[13]) considers N particles in two containers. The particles independently change containers at a rate λ. If $X(t) = i$ is set to be the number of particles in one container at time t, then it is a birth–death process with transition rates,

$$q_{i,i-1} = i\lambda, \qquad \text{for } 1 \leq i \leq N$$

$$q_{i,i+1} = (N-i)\lambda, \qquad \text{for } 0 \leq i \leq N-1$$

and equilibrium distribution $\pi_i = 2^{-N}\binom{N}{i}$. In 1947, Mark Kac [25] proved that if the initial state is not an equilibrium state, then the Boltzmann entropy, i.e.,

$$H(t) = -\sum_{i=1}^{N} P(X(t) = i) \log\left(\frac{P(X(t) = i)}{\pi_i}\right)$$

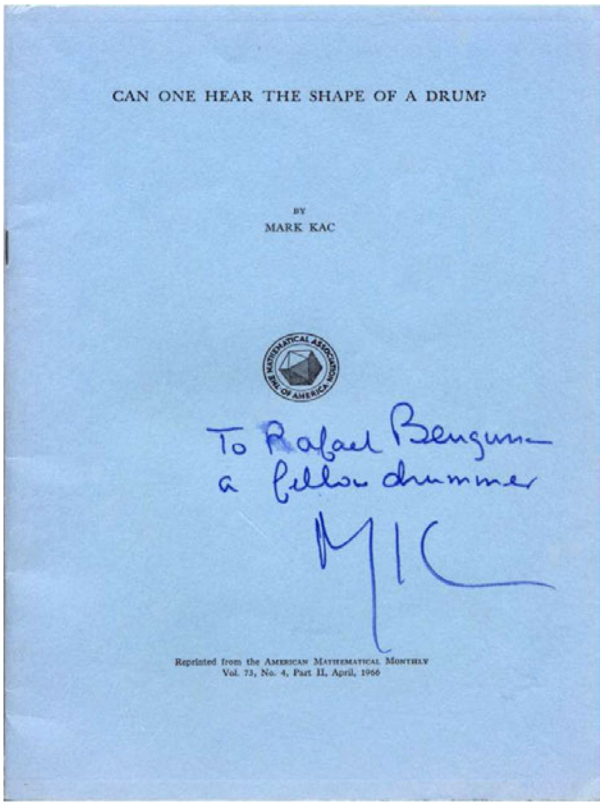
is monotonically increasing. The manuscript of Mark Kac [25] with the solution of the approach to equilibrium of the Ehrenfests' model was awarded the 1950 Chauvenet Prize (for expository writing) of the Mathematical Association of America. Kac got a second Chauvenet Prize in 1968 for his paper *Can one hear the shape of a drum* that I discuss later. Kac spent his sabbatical year 1951–1952 at the IAS in Princeton, where he met and collaborated with John Ward on a new combinatorial solution of the 2–dimensional Ising model, which had been solved by Onsager in 1944. Although the solution of Kac and Ward had a gap that took time and effort of various people to fill, it gave new insight into the problem. The same year, Ted Berlin and Mark Kac solved [4] another model of a ferromagnet, the so called "spherical model", which is somewhat a simplification of the Ising Model. Consider a square (2–d) or a cubic (3–d) lattice containing N spin sites. But instead of allowing the spins σ_i (here i denotes a lattice site) to take only the ± 1 values, allow them to be independently distributed Gaussian variables, with the additional constraint $\sum \sigma_i^2 = N$ (constraint which is obviously satisfied in the Ising model). Berlin and Kac proved that this model exhibits a phase transition in the three dimensional case (there is no phase transition at finite temperature in lower dimension), and computed the critical temperature and the critical exponents for the model.

After solving the dog–flea problem, Mark Kac made further contributions in trying to solve the paradox raised by Loschmidt to the Boltzmann equation [5]. The Boltzmann equation is very successful, through the H-theorem, in establishing the approach to equilibrium in Statistical Mechanics and the derivation of the second law of Thermodynamics. However, as Loschmidt pointed out in 1876, it is not the final picture because it is not compatible as it stands with the reversibility of the equations of motion between the interacting particles. In order to understand the problems of the Boltzmann equation, Kac [27] introduced the concept of "propagation of chaos" in connection with a specific stochastic process modelling binary collisions in a gas of a large number N of identical particles (the "Kac walk", in the space of velocities). In particular Kac was interested in the approach to equilibrium of the gas. Kac kept his interest in the "propagation of chaos" until the end of his life (see, e.g., [20]). There have been many recent developments on the properties of the "Kac Model" (see, e.g., [8, 9, 7]).

The subject of integration in function space s was introduced by Norbert Wiener in the 1920's. According to [45], Wiener was inspired by the experimental observations on Brownian Motion, in particular by the quotation in *Les Atomes* of Perrin [38]: *"...the very irregular curves followed by particles in Brownian Motion led one think of the supposed continuous non differential curves of the mathematicians"*. To justify the remark of Perrin, Wiener introduced a theory based on "the statistics of paths", constructing a measure in the space of continuous functions (see, e.g., [30, 40]). Using Wiener's measure one can prove the connection between potential theory and Brownian Motion. In particular, if $T_\Omega(y)$ is the total time a Brownian particle starting at $y \in \Omega$ in $t = 0$ spends inside the bounded domain $\Omega \subset \mathbb{R}^3$, then,

$$\mathbb{E}(T_\Omega(y)) = \frac{1}{2\pi} \int_\Omega \frac{1}{|x-y|}\, dx,$$

(the potential at the interior point y produced by a uniform density supported at the boundary of Ω). Other quantities like the capacity of a set, or the scattering length of a set can by similarly characterised in terms of properties of Brownian Motion. Influenced by Feynman's Ph.D. thesis [16], Mark Kac [26, 30, 32] established a rigorous connection between Schrödinger's equation and Wiener's theory, connection which is known as the Feynman–Kac formula.



A signed copy of Kac's paper: "Can one hear the shape of a drum?"

One of my favourite papers of Mark Kac is [29], dedicated to his friend and colleague George Uhlenbeck on the occasion of his 65th birthday. In 1965, the Committee on Educational Media of the Mathematical Association of America produced a film on a mathematical lecture by Mark Kac (1914–1984) with the title: *Can one hear the shape of a drum?* One of the purposes of the film was to inspire undergraduates to follow a career in mathematics. The article [29] consists of an expanded version of that lecture. Consider two different smooth, bounded domains, say Ω_1 and Ω_2 in the plane. Let $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \ldots$ be the sequence of eigenvalues of the Laplacian on Ω_1, with Dirichlet boundary conditions and, correspondingly, $0 < \lambda_1' < \lambda_2' \leq \lambda_3' \leq \ldots$ be the sequence of Dirichlet eigenvalues for Ω_2. Assume that for each n, $\lambda_n = \lambda_n'$ (i.e., both domains are *isospectral*). Then, Mark Kac posed the following question: *Are the domains Ω_1 and Ω_2 congruent in the sense of Euclidean geometry?*. A friend of Mark Kac, the mathematician Lipman Bers (1914–1993), paraphrased this question in the famous sentence: *Can one hear the shape of a drum?*

In 1910, H. A. Lorentz, at the *Wolfskehl lecture* at the University of Göttingen, reported on his work with Jeans on the characteristic frequencies of the electromagnetic field inside a resonant cavity of volume Ω in three dimensions. According to the work of Jeans and Lorentz, the number of eigenvalues of the electromagnetic cavity whose numerical values is below λ (this is a function usually denoted by $N(\lambda)$) is given asymptotically by

$$N(\lambda) \approx \frac{|\Omega|}{6\pi^2}\lambda^{3/2}, \tag{1}$$

for large values of λ, for many different cavities with simple geometry (e.g., cubes, spheres, cylinders, etc.) Thus, according to the calculations of Jeans and Lorentz, to leading order in λ, the *counting function* $N(\lambda)$ seemed to depend only on the volume of the electromagnetic cavity $|\Omega|$. Apparently David Hilbert (1862–1943), who was attending the lecture, predicted that this conjecture of Lorentz would not be proved during his lifetime. This time, Hilbert was wrong, since his own student, Hermann Weyl (1885–1955) proved the conjecture less than two years after the Lorentz' lecture. An account of the work of Hermann Weyl on the eigenvalues of a membrane is given in his 1948 *J. W. Gibbs Lecture* to the American Mathematical Society [44].

In N dimensions, (1) reads,

$$N(\lambda) \approx \frac{|\Omega|}{(2\pi)^N} C_N \lambda^{N/2}, \tag{2}$$

where $C_N = \pi^{(N/2)}/\Gamma((N/2)+1))$ denotes the volume of the unit ball in N dimensions.

Using Tauberian theorems, one can relate the behaviour of the counting function $N(\lambda)$ for large values of λ with the behaviour of the function

$$Z_\Omega(t) \equiv \sum_{n=1}^{\infty} \exp\{-\lambda_n t\}, \tag{3}$$

for small values of t. The function $Z_\Omega(t)$ is the trace of the heat kernel for the domain Ω, i.e., $Z_{\Omega(t)} = \operatorname{tr}\exp(\Delta t)$. As I mention above, $\lambda_n(\Omega)$ denotes the n Dirichlet eigenvalue of the domain Ω.

The fact that the leading behaviour of $Z_\Omega(t)$ for t small, for any bounded, smooth domain Ω in the plane is given by

$$Z_\Omega(t) \approx \frac{1}{4\pi t} A \tag{4}$$

was proven by Hermann Weyl [43]. Here, $A = |\Omega|$ denotes the area of Ω. In fact, what Weyl proved in [43] is the *Weyl Asymptotics* of the Dirichlet eigenvalues, i.e., for large n, $\lambda_n \approx (4\pi n)/A$. Weyl's result (4) implies that *one can hear the area of the drum.*

In 1954, the Swedish mathematician Åke Pleijel [39] obtained the improved asymptotic formula,

$$Z(t) \approx \frac{A}{4\pi t} - \frac{L}{8\sqrt{\pi t}},$$

where L is the perimeter of Ω. In other words, one *can hear* the area and the perimeter of Ω. It follows from Pleijel's asymptotic result that if all the frequencies of a drum are equal to those of a circular drum then the drum must itself be circular. This follows from the classical isoperimetric inequality (i.e., $L^2 \geq 4\pi A$, with equality if and only if Ω is a circle). In other words, one *can hear* whether a drum is circular. It turns out that it is enough to hear the first two eigenfrequencies to determine whether the drum has the circular shape [1].

In 1966, Mark Kac obtained the next term in the asymptotic behaviour of $Z(t)$ [29]. For a smooth, bounded, multiply connected domain on the plane (with r holes)

$$Z(t) \approx \frac{A}{4\pi t} - \frac{L}{8\sqrt{\pi t}} + \frac{1}{6}(1 - r). \tag{5}$$

Thus, one *can hear* the *connectivity* of a drum. Kac's formula (5) was rigorously justified by McKean and Singer [35].

A sketch of Kac's analysis for the first term of the asymptotic expansion is as follows [29]. If we imagine some substance concentrated at $\vec{\rho} = (x_0, y_0)$ diffusing through the domain Ω bounded by $\partial\Omega$, where the substance is absorbed at the boundary, then the concentration $P_\Omega(\vec{p} \mid \vec{r}; t)$ of matter at $\vec{r} = (x, y)$ at time t obeys the diffusion equation

$$\frac{\partial P_\Omega}{\partial t} = \Delta P_\Omega$$

with boundary condition $P_\Omega(\vec{p} \mid \vec{r}; t) \to 0$ as $\vec{r} \to \vec{a}$, $\vec{a} \in \partial\Omega$, and initial condition $P_\Omega(\vec{p} \mid \vec{r}; t) \to \delta(\vec{r} - \vec{p})$ as $t \to 0$, where $\delta(\vec{r} - \vec{p})$ is the Dirac delta function. The concentration $P_\Omega(\vec{p} \mid \vec{r}; t)$ may be expressed in terms of the Dirichlet eigenvalues of Ω, λ_n and the corresponding (normalized) eigenfunctions ϕ_n as follows,

$$P_\Omega(\vec{p} \mid \vec{r}; t) = \sum_{n=1}^{\infty} e^{-\lambda_n t} \phi_n(\vec{p}) \phi_n(\vec{r}).$$

For small t, the diffusion is slow, that is, it will not *feel* the influence of the boundary in such a short time. We may expect that

$$P_\Omega(\vec{p} \mid \vec{r}; t) \approx P_0(\vec{p} \mid \vec{r}; t),$$

ar $t \to 0$, where $\partial P_0/\partial t = \Delta P_0$, and $P_0(\vec{p} \mid \vec{r}; t) \to \delta(\vec{r} - \vec{p})$ as $t \to 0$. P_0 in fact represents the heat kernel for the whole $\mathbb{R}^2$, i.e., no boundaries present. This kernel is explicitly known. In fact,

$$P_0(\vec{p} \mid \vec{r}; t) = \frac{1}{4\pi t} \exp(-|\vec{r} - \vec{p}|^2/4t),$$

where $|\vec{r} - \vec{p}|^2$ is just the Euclidean distance between $\vec{p}$ and $\vec{r}$. Then, as $t \to 0^+$,

$$P_\Omega(\vec{p} \mid \vec{r}; t) = \sum_{n=1}^{\infty} e^{-\lambda_n t} \phi_n(\vec{p}) \phi_n(\vec{r}) \approx \frac{1}{4\pi t} \exp(-|\vec{r} - \vec{p}|^2/4t).$$

Thus, when set $\vec{p} = \vec{r}$ we get

$$\sum_{n=1}^{\infty} e^{-\lambda_n t} \phi_n^2(\vec{r}) \approx \frac{1}{4\pi t}.$$

Integrating both sides with respect to $\vec{r}$, using the fact that ϕ_n is normalized, we finally get,

$$\sum_{n=1}^{\infty} e^{-\lambda_n\, t} \approx \frac{|\Omega|}{4\pi t}, \tag{6}$$

which is the first term in the expansion (5). Further analysis gives the remaining terms (see [29]).

Remark: In 1951, Mark Kac proved the following universal bound on $Z(t)$ in dimension d:

$$Z(t) \leq \frac{|\Omega|}{(4\pi t)^{d/2}}. \tag{7}$$

This bound is sharp, in the sense that as $t \to 0$,

$$Z(t) \approx \frac{|\Omega|}{(4\pi t)^{d/2}}. \tag{8}$$

Recently, Harrell and Hermi [21] proved the following improvement on (8),

$$Z(t) \approx \frac{|\Omega|}{(4\pi t)^{d/2}} e^{-M_d |\Omega| t / I(\Omega)}. \tag{9}$$

where $I_\Omega = \min_{a \in \mathbb{R}^d} \int_\Omega |x - a|^2\, dx$ and M_d is a constant depending on dimension. Moreover, they conjectured the following bound on $Z(t)$, namely,

$$Z(t) \approx \frac{|\Omega|}{(4\pi t)^{d/2}} e^{-t/|\Omega|^{2/d}}. \tag{10}$$

Recently, Geisinger and Weidl [18]proved the best bound up to date in this direction,

$$Z(t) \approx \frac{|\Omega|}{(4\pi t)^{d/2}} e^{-\overline{M}_d t/|\Omega|^{2/d}}, \tag{11}$$

where $\overline{M}_d = [(d+2)\pi/d]\Gamma(d/2+1)^{-2/d} M_d$ (in particular $\overline{M}_2 = \pi/16$. In general $\overline{M}_d < 1$, thus the Geisinger–Weidl bound (11) falls short of the conjectured expression of Harrell and Hermi.

In the quoted paper of Mark Kac [29] he says that he personally believed that one cannot hear the shape of a drum. A couple of years before Mark Kac' article, John Milnor [36], had constructed two non-congruent sixteen dimensional tori whose Laplace–Beltrami operators have exactly the same eigenvalues. In 1985 Toshikazu Sunada [41] developed an algebraic framework that provided a new, systematic approach of considering Mark Kac's question. Using Sunada's technique several mathematicians constructed isospectral manifolds (e.g., Gordon and Wilson; Brooks; Buser, etc.). See, e.g., the review article of Robert Brooks (1988) with the situation on isospectrality up to that date in [6]. Finally, in 1992, Carolyn Gordon, David Webb and Scott Wolpert [19] gave the definite negative answer to Mark Kac's question and constructed two plane domains (henceforth called the GWW domains) with the same Dirichlet eigenvalues.

After twenty years at the Rockefeller University (1961–1981), Mark Kac joined the University of Southern California, where he served as the Chair of the Mathematics Department. Mark Kac had many distinguished Ph.D. students and postdocs, including Daniel Stroock, Harry Kesten, Murray Rosenblatt, Henry McKean and many others. Kac earned several distinctions and awards. Apart from the two Chauvenet Prizes and the Guggenheim fellowship awarded to him, which I have already discussed, he was the John von Neumann Lecturer (SIAM) in 1961, the Josiah Williard Gibbs lecturer in the joint AMS-MAA meeting in 1967. In 1978 he was awarded the George Birkhoff prize of Applied Mathematics (AMS–SIAM). He was elected a member of the National Academy of the United States. For a long time Mark Kac served as co–chair of the "Committee of Concerned Scientists" and association which monitors and document violations of the human rights and scientific freedom of scientists all over the world. Mark Kac died on October 25, 1984 after a long battle with cancer. To conclude I would like to recall some thoughts of Henry McKean [34], which I certainly share:

"...I am sure I speak for all of Kac's friends when I remember him for his wit, his personal kindness, and his scientific style. In a summer at MIT, I had the luck to have Kac as my instructor. I was enchanted not only by the content of the lectures but by the person of the lecturer. I had never seen mathematics like that, nor anybody who could impart such (to me) difficult material with such a charm."

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