Isoperimetric Inequalities for Eigenvalues of the Laplacian

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Abstract. These are extended notes based on the series of four lectures on isoperimetric inequalities for the Laplacian, given by the author at the Arizona School of Analysis with Applications, in March 2010.

Mais ce n'est pas tout; la Physique ne nous donne pas
seulement l'occasion de résoudre des problèmes;
elle nous aide à en trouver les moyens, et cela de deux manières.
Elle nous fait présentir la solution;
elle nous suggère des raisonaments.
Conference of H. Poincaré at the ICM, in Zürich, 1897, see [**63**], p. 340.

1. Introduction

Isoperimetric inequalities have played an important role in mathematics since the times of ancient Greece. The first and best known isoperimetric inequality is the *classical isoperimetric inequality*

$$A \leq \frac{L^2}{4\pi},$$

relating the area A enclosed by a planar closed curve of perimeter L. After the introduction of Calculus in the XVIIth century, many new isoperimetric inequalities have been discovered in mathematics and physics (see, e.g., the review articles [**16, 57, 59, 64**]; see also the essay [**13**] for a panorama on the subject of Isoperimetry). The eigenvalues of the Laplacian are "geometric objects" in the sense that they depend on the geometry of the underlying domain, and to some extent (see Lecture 1) the knowledge of all the eigenvalues characterizes several aspects of the geometry of the domain. Therefore it is natural to pose the problem of finding isoperimetric inequalities for the eigenvalues of the Laplacian. The first one to consider this possibility was Lord Rayleigh in his monograph *The Theory of Sound* [**65**]. In these lectures I will present some of the problems arising in the study of isoperimetric

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inequalities for the Laplacian, some of the tools needed in their proof and many bibliographic discussions about the subject. I start this review (Lecture 1) with the classical problem of Mark Kac, *Can one hear the shape of a drum?*. In the second Lecture I review some basic facts about rearrangements and the Rayleigh–Faber–Krahn inequality. In the third lecture I discuss the Szegö–Weinberger inequality, which is an isoperimetric inequality for the first nontrivial Neumann eigenvalue of the Laplacian, and the Payne–Pólya–Weinberger isoperimetric inequality for the quotient of the first two Dirichlet eigenvalues of the Laplacian, as well as several recent extensions. Finally, in the last lecture I review three different isoperimetric problems for fourth order operators. There are many interesting results that I have left out of this review. For different perspectives, selections and emphasis, please refer, for example, to the reviews [**2, 3, 4, 12, 44**], among many others. The contents of this manuscript are based on a series of four lectures given by the author at the Arizona School of Analysis with Applications, University of Arizona, Tucson, AZ, March 15-19, 2010. Other versions of this course have been given as an intensive course for graduate students in the Tunis Science City, Tunisia (May 21–22, 2010), in connection with the International Conference on the isoperimetric problem of Queen Dido and its mathematical ramifications, that was held in Carthage, Tunisia, May 24–29, 2010; and, previously, in the *IV Escuela de Verano en Análisis y Física Matemática* at the Unidad Cuernavaca del Instituto de Matemáticas de la Universidad Nacional Autónoma de México, in the summer of 2005 [**20**]. Preliminary versions of these lectures were also given in the *Short Course in Isoperimetric Inequalities for Eigenvalues of the Laplacian*, given by the author in February of 2004 as part of the *Thematic Program on Partial Differential Equations* held at the Fields Institute, in Toronto, and also during the course *Autovalores del Laplaciano y Geometría* given by the author at the Department of Mathematics of the Universidad de Pernambuco, in Recife, Brazil, in August 2003.

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2. Lecture 1: Can one hear the shape of a drum?

...but it would baffle the most skillful
mathematician to solve the Inverse Problem,
and to find out the shape of a bell by means
of the sounds which is capable of sending out.
Sir Arthur Schuster (1882).

In 1965, the Committee on Educational Media of the Mathematical Association of America produced a film on a mathematical lecture by Mark Kac (1914–1984) with the title: *Can one hear the shape of a drum?* One of the purposes of the film was to inspire undergraduates to follow a career in mathematics. An expanded version of that lecture was later published [**46**]. Consider two different smooth, bounded domains, say Ω_1 and Ω_2 in the plane. Let $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \dots$ be the sequence of eigenvalues of the Laplacian on Ω_1, with Dirichlet boundary conditions and, correspondingly, $0 < \lambda_1' < \lambda_2' \leq \lambda_3' \leq \dots$ be the sequence of Dirichlet eigenvalues for Ω_2. Assume that for each n, $\lambda_n = \lambda_n'$ (i.e., both domains are *isospectral*). Then, Mark Kac posed the following question: *Are the domains Ω_1 and Ω_2 congruent in the sense of Euclidean geometry?*. A friend of Mark Kac, the mathematician Lipman Bers (1914–1993), paraphrased this question in the famous sentence: *Can one hear the shape of a drum?*

In 1910, H. A. Lorentz, during the *Wolfskehl lecture* at the University of Göttingen, reported on his work with Jeans on the characteristic frequencies of the electromagnetic field inside a resonant cavity of volume Ω in three dimensions. According to the work of Jeans and Lorentz, the number of eigenvalues of the electromagnetic cavity whose numerical values is below λ (this is a function usually denoted by $N(\lambda)$) is given asymptotically by

$$N(\lambda) \approx \frac{|\Omega|}{6\pi^2}\lambda^{3/2}, \tag{2.1}$$

for large values of λ, for many different cavities with simple geometry (e.g., cubes, spheres, cylinders, etc.) Thus, according to the calculations of Jeans and Lorentz, to leading order in λ, the *counting function* $N(\lambda)$ seemed to depend only on the volume of the electromagnetic cavity $|\Omega|$. Apparently David Hilbert (1862–1943), who was attending the lecture, predicted that this conjecture of Lorentz would not be proved during his lifetime. This time, Hilbert was wrong, since his own student, Hermann Weyl (1885–1955) proved the conjecture less than two years after the lecture by Lorentz.

Remark: There is a nice account of the work of Hermann Weyl on the eigenvalues of a membrane in his 1948 *J. W. Gibbs Lecture* to the American Mathematical Society [**78**].

In N dimensions, (2.1) reads,

$$N(\lambda) \approx \frac{|\Omega|}{(2\pi)^N} C_N \lambda^{N/2}, \tag{2.2}$$

where $C_N = \pi^{(N/2)}/\Gamma((N/2)+1))$ denotes the volume of the unit ball in N dimensions.

Using Tauberian theorems, one can relate the behavior of the counting function $N(\lambda)$ for large values of λ with the behavior of the function

$$Z_\Omega(t) \equiv \sum_{n=1}^{\infty} \exp\{-\lambda_n t\}, \tag{2.3}$$

for small values of t. The function $Z_\Omega(t)$ is the trace of the heat kernel for the domain Ω, i.e., $Z_{\Omega(t)} = \operatorname{tr}\exp(\Delta t)$. As we mentioned above, $\lambda_n(\Omega)$ denotes the n Dirichlet eigenvalue of the domain Ω.

An example: the behavior of $Z_\Omega(t)$ for rectangles

With the help of the *Riemann Theta function* $\Theta(x)$, it is simple to compute the trace of the heat kernel when the domain is a rectangle of sides a and b, and therefore to obtain the leading asymptotic behavior for small values of t. The Riemann Theta function is defined by

$$\Theta(x) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 x}, \tag{2.4}$$

for $x > 0$. The function $\Theta(x)$ satisfies the following modular relation,

$$\Theta(x) = \frac{1}{\sqrt{x}} \Theta(\frac{1}{x}). \tag{2.5}$$

Remark: This modular form for the Theta Function already appears in the classical paper of Riemann [**67**] (manuscript where Riemann puts forward his famous *Riemann Hypothesis*). In that manuscript, the modular form (2.5) is attributed to Jacobi.

The modular form (2.5) may be obtained from a very elegant application of Fourier Analysis (see, e.g., [**32**], pp. 75–76) which I reproduce here for completeness. Define

$$\varphi_x(y) = \sum_{n=-\infty}^{\infty} e^{-\pi (n+y)^2 x}. \tag{2.6}$$

Clearly, $\Theta(x) = \varphi_x(0)$. Moreover, the function $\varphi_x(y)$ is periodic in y of period 1. Thus, we can express it as follows,

$$\varphi_x(y) = \sum_{k=-\infty}^{\infty} a_k e^{2\pi k i\, y}, \tag{2.7}$$

where the Fourier coefficients are

$$a_k = \int_0^1 \varphi_k(y) e^{-2\pi k i\, y}\, dy. \tag{2.8}$$

Replacing the expression (2.6) for $\varphi_x(y)$ in (2.8), using the fact that $e^{2\pi k i\, n} = 1$, we can write,

$$a_k = \int_0^1 \sum_{n=-\infty}^{\infty} e^{-\pi (n+y)^2 x} e^{-2\pi i k (y+n)}\, dy. \tag{2.9}$$

Interchanging the order between the integral and the sum, we get,

$$a_k = \sum_{n=-\infty}^{\infty} \int_0^1 \left(e^{-\pi (n+y)^2 x} e^{-2\pi i k (y+n)} \right) dy. \tag{2.10}$$

In the n^{th} summand we make the change of variables $y \to u = n + y$. Clearly, u runs from n to $n + 1$, in the n^{th} summand. Thus, we get,

$$a_k = \int_{-\infty}^{\infty} e^{-\pi u^2 x} e^{-2\pi i k\, u}\, du. \tag{2.11}$$

i.e., a_k is the Fourier transform of a Gaussian. Thus, we finally obtain,

$$a_k = \frac{1}{\sqrt{x}} e^{-\pi k^2/x}. \tag{2.12}$$

Since, $\Theta(x) = \varphi_x(0)$, from (2.7) and (2.12) we finally get,

$$\Theta(x) = \sum_{k=-\infty}^{\infty} a_k = \frac{1}{\sqrt{x}} \sum_{k=-\infty}^{\infty} e^{-\pi k^2/x} = \frac{1}{\sqrt{x}} \Theta(\frac{1}{x}). \tag{2.13}$$

Remarks: i) The method exhibited above is a particular case of the *Poisson Summation Formula*. See [**32**], pp. 76–77; ii) It should be clear from (2.4) that $\lim_{x\to\infty} \Theta(x) = 1$. Hence, from the modular form for $\Theta(x)$ we immediately see that

$$\lim_{x\to 0} \sqrt{x}\, \Theta(x) = 1. \tag{2.14}$$

Once we have the modular form for the Riemann Theta function, it is simple to get the leading asymptotic behavior of the trace of the heat kernel $Z_\Omega(t)$, for small values of t, when the domain Ω is a rectangle. Take Ω to be the rectangle of sides a and b. Its Dirichlet eigenvalues are given by

$$\lambda_{n,m} = \pi^2 \left[\frac{n^2}{a^2} + \frac{m^2}{b^2} \right], \tag{2.15}$$

with $n, m = 1, 2, \ldots$. In terms of the Dirichlet eigenvalues, the trace of the heat kernel, $Z_\Omega(t)$ is given by

$$Z_\Omega(t) = \sum_{n,m=1}^{\infty} e^{-\lambda_{n,m} t}. \tag{2.16}$$

and using (2.15), and the definition of $\Theta(x)$, we get,

$$Z_\Omega(t) = \frac{1}{4} \left[\Theta(\frac{\pi\, t}{a^2}) - 1 \right] \left[\Theta(\frac{\pi\, t}{b^2}) - 1 \right]. \tag{2.17}$$

Using the asymptotic behavior of the Theta function for small arguments, i.e., (2.14) above, we have

$$Z_\Omega(t) \approx \frac{1}{4} (\frac{a}{\sqrt{\pi\, t}} - 1)(\frac{b}{\sqrt{\pi\, t}} - 1) \approx \frac{1}{4\pi t} ab - \frac{1}{4\sqrt{\pi t}} (a+b) + \frac{1}{4} + O(t). \tag{2.18}$$

In terms of the area of the rectangle $A = ab$ and its perimeter $L = 2(a+b)$, the expression $Z_\Omega(t)$ for the rectangle may be written simply as,

$$Z_\Omega(t) \approx \frac{1}{4\pi t} A - \frac{1}{8\sqrt{\pi t}} L + \frac{1}{4} + O(t). \tag{2.19}$$

Remark: Using similar techniques, one can compute the small t behavior of $Z_\Omega(t)$ for various simple regions of the plane (see, e.g., [**50**]).

The fact that the leading behavior of $Z_\Omega(t)$ for t small, for any bounded, smooth domain Ω in the plane is given by

$$Z_\Omega(t) \approx \frac{1}{4\pi t} A \tag{2.20}$$

was proven by Hermann Weyl [**77**]. Here, $A = |\Omega|$ denotes the area of Ω. In fact, what Weyl proved in [**77**] is the *Weyl Asymptotics* of the Dirichlet eigenvalues, i.e., for large n, $\lambda_n \approx (4\pi\, n)/A$. Weyl's result (2.20) implies that *one can hear the area of the drum.*

In 1954, the Swedish mathematician, Åke Pleijel [**62**] obtained the improved asymptotic formula,

$$Z_\Omega(t) \approx \frac{A}{4\pi t} - \frac{L}{8\sqrt{\pi t}},$$

where L is the perimeter of Ω. In other words, one *can hear* the area and the perimeter of Ω. It follows from Pleijel's asymptotic result that if all the frequencies of a drum are equal to those of a circular drum then the drum must itself be circular. This follows from the classical isoperimetric inequality (i.e., $L^2 \geq 4\pi A$, with equality if and only if Ω is a circle). In other words, one *can hear* whether a drum is circular. It turns out that it is enough to hear the first two eigenfrequencies to determine whether the drum has the circular shape [**6**]

In 1966, Mark Kac obtained the next term in the asymptotic behavior of $Z_\Omega(t)$ [**46**]. For a smooth, bounded, multiply connected domain on the plane (with r holes)

$$Z_\Omega(t) \approx \frac{A}{4\pi t} - \frac{L}{8\sqrt{\pi t}} + \frac{1}{6}(1-r). \tag{2.21}$$

Thus, one *can hear* the *connectivity* of a drum. The last term in the above asymptotic expansion changes for domains with corners (e.g., for a rectangular membrane, using the modular formula for the Theta Function, we obtained 1/4 instead of 1/6). Kac's formula (2.21) was rigorously justified by McKean and Singer [**51**]. Moreover, for domains having corners they showed that each corner with interior angle γ makes an additional contribution to the constant term in (2.21) of $(\pi^2-\gamma^2)/(24\pi\gamma)$.

A sketch of Kac's analysis for the first term of the asymptotic expansion is as follows (here we follow [**45, 46, 50**]). If we imagine some substance concentrated at $\vec{\rho} = (x_0, y_0)$ diffusing through the domain Ω bounded by $\partial\Omega$, where the substance is absorbed at the boundary, then the concentration $P_\Omega(\vec{p} \mid \vec{r}; t)$ of matter at $\vec{r} = (x, y)$ at time t obeys the diffusion equation

$$\frac{\partial P_\Omega}{\partial t} = \Delta P_\Omega$$

with boundary condition $P_\Omega(\vec{p} \mid \vec{r}; t) \to 0$ as $\vec{r} \to \vec{a}$, $\vec{a} \in \partial\Omega$, and initial condition $P_\Omega(\vec{p} \mid \vec{r}; t) \to \delta(\vec{r} - \vec{p})$ as $t \to 0$, where $\delta(\vec{r} - \vec{p})$ is the Dirac delta function. The concentration $P_\Omega(\vec{p} \mid \vec{r}; t)$ may be expressed in terms of the Dirichlet eigenvalues of Ω, λ_n and the corresponding (normalized) eigenfunctions ϕ_n as follows,

$$P_\Omega(\vec{p} \mid \vec{r}; t) = \sum_{n=1}^{\infty} e^{-\lambda_n t} \phi_n(\vec{p}) \phi_n(\vec{r}).$$

For small t, the diffusion is slow, that is, it will not *feel* the influence of the boundary in such a short time. We may expect that

$$P_\Omega(\vec{p} \mid \vec{r}; t) \approx P_0(\vec{p} \mid \vec{r}; t),$$

ar $t \to 0$, where $\partial P_0/\partial t = \Delta P_0$, and $P_0(\vec{p} \mid \vec{r}; t) \to \delta(\vec{r} - \vec{p})$ as $t \to 0$. P_0 in fact represents the heat kernel for the whole $\mathbb{R}^2$, i.e., no boundaries present. This kernel is explicitly known. In fact,

$$P_0(\vec{p} \mid \vec{r}; t) = \frac{1}{4\pi t} \exp(-|\vec{r} - \vec{p}|^2/4t),$$

where $|\vec{r}-\vec{p}|^2$ is just the Euclidean distance between $\vec{p}$ and $\vec{r}$. Then, as $t \to 0^+$,

$$P_\Omega(\vec{p}\mid\vec{r};t) = \sum_{n=1}^{\infty} e^{-\lambda_n t}\phi_n(\vec{p})\phi_n(\vec{r}) \approx \frac{1}{4\pi t}\exp(-|\vec{r}-\vec{p}|^2/4t).$$

Thus, when set $\vec{p}=\vec{r}$ we get

$$\sum_{n=1}^{\infty} e^{-\lambda_n t}\phi_n^2(\vec{r}) \approx \frac{1}{4\pi t}.$$

Integrating both sides with respect to $\vec{r}$, using the fact that ϕ_n is normalized, we finally get,

$$\sum_{n=1}^{\infty} e^{-\lambda_n t} \approx \frac{|\Omega|}{4\pi t}, \tag{2.22}$$

which is the first term in the expansion (2.21). Further analysis gives the remaining terms (see [**46**]).

Remark: In 1951, Mark Kac proved the following universal bound on $Z_\Omega(t)$ in dimension d:

$$Z_\Omega(t) \le \frac{|\Omega|}{(4\pi t)^{d/2}}. \tag{2.23}$$

This bound is sharp, in the sense that as $t \to 0$,

$$Z_\Omega(t) \approx \frac{|\Omega|}{(4\pi t)^{d/2}}. \tag{2.24}$$

Recently, Harrell and Hermi [**43**] proved the following improvement on (2.24),

$$Z_\Omega(t) \approx \frac{|\Omega|}{(4\pi t)^{d/2}} e^{-M_d|\Omega|t/I(\Omega)}. \tag{2.25}$$

where $I_\Omega = \min_{a\in\mathbb{R}^d}\int_\Omega |x-a|^2\,dx$ and M_d is a constant depending on dimension. Moreover, they conjectured the following bound on $Z_\Omega(t)$, namely,

$$Z_\Omega(t) \approx \frac{|\Omega|}{(4\pi t)^{d/2}} e^{-t/|\Omega|^{2/d}}. \tag{2.26}$$

Recently, Geisinger and Weidl [**38**] proved the best bound up to date in this direction,

$$Z_\Omega(t) \approx \frac{|\Omega|}{(4\pi t)^{d/2}} e^{-\overline{M}_d t/|\Omega|^{2/d}}, \tag{2.27}$$

where $\overline{M}_d = [(d+2)\pi/d]\Gamma(d/2+1)^{-2/d}M_d$ (in particular $\overline{M}_2 = \pi/16$. In general $\overline{M}_d < 1$, thus the Geisinger–Weidl bound (2.27) falls short of the conjectured expression of Harrell and Hermi.

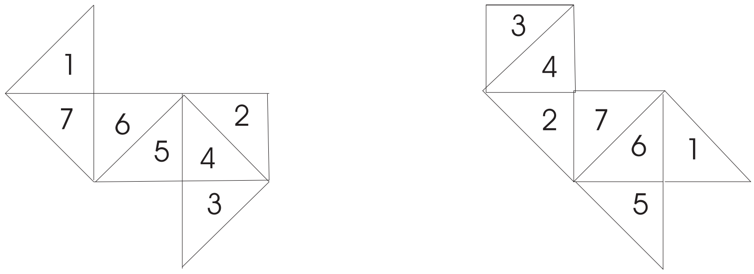

FIGURE 1. GWW Isospectral Domains D_1 and D_2

2.1. One cannot hear the shape of a drum. In the quoted paper of Mark Kac [**46**] he says that he personally believed that one cannot hear the shape of a drum. A couple of years before Mark Kac' article, John Milnor [**54**], had constructed two non-congruent sixteen dimensional tori whose Laplace–Beltrami operators have exactly the same eigenvalues. In 1985 Toshikazu Sunada [**69**], then at Nagoya University in Japan, developed an algebraic framework that provided a new, systematic approach of considering Mark Kac's question. Using Sunada's technique several mathematicians constructed isospectral manifolds (e.g., Gordon and Wilson; Brooks; Buser, etc.). See, e.g., the review article of Robert Brooks (1988) with the situation on isospectrality up to that date in [**25**] (see also, [**27**]). Finally, in 1992, Carolyn Gordon, David Webb and Scott Wolpert [**40**] gave the definite negative answer to Mark Kac's question and constructed two plane domains (henceforth called the GWW domains) with the same Dirichlet eigenvalues.

The most elementary proof of isospectrality of the GWW domains is done using the method of *transplantation.* For the method of *transplantation* see, e.g., [**21, 22**]. See also the expository article [**23**] by the same author. The method also appears briefly described in the article of Sridhar and Kudrolli [**68**], cited in the *Bibliographical Remarks, iv)* at the end of this chapter.

For the interested reader, there is a recent review article [**39**] that presents many different proofs of isospectrality, including *transplantation*, and *paper folding techniques*).

THEOREM 2.1 (C. Gordon, D. Webb, S. Wolpert). *The domains D_1 and D_2 of figures* 1 *and* 2 *are isospectral.*

Although the proof by transplantation is straightforward to follow, it does not shed light on the rich geometric, analytic and algebraic structure of the problem initiated by Mark Kac. For the interested reader it is recommendable to read the papers of Sunada [**69**] and of Gordon, Webb and Wolpert [**40**].

In the previous paragraphs we have seen how the answer to the original Kac's question is in general negative. However, if we are willing to require some analyticity of the domains, and certain symmetries, we can recover uniqueness of the domain once we know the spectrum. During the last decade there has been an important progress in this direction. In 2000, S. Zelditch, [**79**] proved that in two dimensions, simply connected domains with the symmetry of an ellipse are completely determined by either their Dirichlet or their Neumann spectrum. More recently [**80**], proved a much stronger positive result. Consider the class of planar domains with no holes and very smooth boundary and with at least one mirror symmetry. Then one can recover the shape of the domain given the Dirichlet spectrum.

2.2. Bibliographical Remarks. i) The sentence of Arthur Schuster (1851–1934) quoted at the beginning of this Lecture is cited in Reed and Simon's book, volume IV [**66**]. It is taken from the article A. Schuster, *The Genesis of Spectra*, in *Report of the fifty–second meeting of the British Association for the Advancement of Science* (held at Southampton in August 1882). Brit. Assoc. Rept., pp. 120–121, 1883. Arthur Schuster was a British physicist (he was a leader spectroscopist at the turn of the XIX century). It is interesting to point out that Arthur Schuster found the solution to the Lane–Emden equation with exponent 5, i.e., to the equation,

$$-\Delta u = u^5,$$

in $\mathbb{R}^3$, with $u > 0$ going to zero at infinity. The solution is given by

$$u = \frac{3^{1/4}}{(1 + |x|^2)^{1/2}}.$$

(A. Schuster, *On the internal constitution of the Sun*, Brit. Assoc. Rept. pp. 427–429, 1883). Since the Lane–Emden equation for exponent 5 is the Euler–Lagrange equation for the minimizer of the Sobolev quotient, this is precisely the function that, modulo translations and dilations, gives the best Sobolev constant. For a nice autobiography of Arthur Schuster see A. Schuster, *Biographical fragments*, Mc Millan & Co., London, (1932).

ii) A very nice short biography of Marc Kac was written by H. P. McKean [*Mark Kac* in *Bibliographical Memoirs*, National Academy of Science, **59**, 214–235 (1990); available on the web (page by page) at http://www.nap.edu/books/0309041988/html/214.html]. The reader may want to read his own autobiography: Mark Kac, *Enigmas of Chance*, Harper and Row, NY, 1985 [reprinted in 1987 in paperback by The University of California Press]. For his article in the American Mathematical Monthly, *op. cit.*, Mark Kac obtained the 1968 Chauvenet Prize of the Mathematical Association of America.

iii) For a beautiful account of the scientific contributions of Lipman Bers (1993–1914), who coined the famous phrase, *Can one hear the shape of a drum?*, see the article by Cathleen Morawetz and others, *Remembering Lipman Bers*, Notices of the AMS **42**, 8–25 (1995).

iv) It is interesting to remark that the values of the first Dirichlet eigenvalues of the GWW domains were obtained experimentally by S. Sridhar and A. Kudrolli, [**68**]. In this article one can find the details of the physics experiments performed by these authors using very thin electromagnetic resonant cavities with the shape of the Gordon–Webb–Wolpert (GWW) domains. This is the first time that the approximate numerical values of the first 25 eigenvalues of the two GWW were obtained. The corresponding eigenfunctions are also displayed. A quick reference to the transplantation method of Pierre

Berard is also given in this article, including the *transplantation matrix* connecting the two isospectral domains. The reader may want to check the web page of S. Sridhar's Lab (http://sagar.physics.neu.edu/) for further experiments on resonating cavities, their eigenvalues and eigenfunctions, as well as on experiments on *quantum chaos.*

v) The numerical computation of the eigenvalues and eigenfunctions of the pair of GWW isospectral domains was obtained by Tobin A. Driscoll, *Eigenmodes of isospectral domains,* SIAM Review **39**, 1–17 (1997).

vi) In their simplified forms, the Gordon–Webb–Wolpert domains (GWW domains) are made of seven congruent rectangle isosceles triangles. Certainly the GWW domains have the same area, perimeter and connectivity. The GWW domains are not convex. Hence, one may still ask the question whether one *can hear* the shape of a *convex drum.* There are examples of convex isospectral domains in higher dimension (see e.g. C. Gordon and D. Webb, *Isospectral convex domains in Euclidean Spaces*, Math. Res. Letts. **1**, 539–545 (1994), where they construct convex isospectral domains in $\mathbb{R}^n$, $n \geq 4$). *Remark:* For an update of the Sunada Method, and its applications see the article of Robert Brooks [*The Sunada Method*, in *Tel Aviv Topology Conference "Rothenberg Festschrift" 1998*, Contemprary Mathematics **231**, 25–35 (1999); electronically available at: http://www.math.technion.ac.il/ rbrooks]

vii) There is a vast literature on Kac's question, and many review lectures on it. In particular, this problem belongs to a very important branch of mathematics: Inverse Problems. In that connection, see the lectures of R. Melrose [**53**]. For a very recent review on Kac's question and its many ramifications in physics, see [**39**].

viii) There is an excellent recent article in the book **Mathematical Analysis of Evolution, Information, and Complexity**, edited by Wolfgang Arendt and Wolfgang P. Schleich, Wiley, 2009, called *Weyl's Law: Spectral Properties of the Laplacian in Mathematics and Physics*, written by Wolfgang Arendt, Robin Nittka, Wolfgang Peter, and Frank Steiner which is free available online, at
http://www.wiley-vch.de/books/sample/3527408304_c01.pdf. That article should be useful to the interested reader. It has a thorough discussion about Kac's problem, Weyl asymptotics and the historical beginnings of quantum mechanics

3. Lecture 2: Rearrangements and the Rayleigh–Faber–Krahn Inequality

For many problems of functional analysis it is useful to replace some function by an equimeasurable but more symmetric one. This method, which was first introduced by Hardy and Littlewood, is called *rearrangement* or *Schwarz symmetrization* [**42**]. Among several other applications, it plays an important role in the proofs of isoperimetric inequalities like the Rayleigh–Faber–Krahn inequality (see the end of this Lecture), the Szegö–Weinberger inequality or the Payne–Pólya–Weinberger inequality (see Lecture 3), and many others. In this lecture we present some basic definitions and theorems concerning spherically symmetric rearrangements. For a more general treatment see, e.g., [**12, 20, 26, 75**] (and also the references in the Bibliographical Remarks at the end of this lecture).

3.1. Definition and basic properties. We let Ω be a measurable subset of $\mathbb{R}^n$ and write $|\Omega|$ for its Lebesgue measure, which may be finite or infinite. If it is finite we write $\Omega^\star$ for an open ball with the same measure as Ω, otherwise we set $\Omega^\star = \mathbb{R}^n$. We consider a measurable function $u : \Omega \to \mathbb{R}$ and assume either that $|\Omega|$ is finite or that u decays at infinity, i.e., $|\{x \in \Omega : |u(x)| > t\}|$ is finite for every $t > 0$.

Definition 3.1. The function

$$\mu(t) = |\{x \in \Omega : |u(x)| > t\}|, \quad t \geq 0$$

is called *distribution function* of u.

From this definition it is straightforward to check that $\mu(t)$ is a decreasing (non–increasing), right–continuous function on $\mathbb{R}^+$ with $\mu(0) = |\mathrm{sprt}\, u|$ and $\mathrm{sprt}\,\mu = [0, \mathrm{ess\,sup}\,|u|)$.

Definition 3.2.

- The *decreasing rearrangement* $u^\sharp : \mathbb{R}^+ \to \mathbb{R}^+$ of u is the distribution function of μ.
- The *symmetric decreasing rearrangement* $u^\star : \Omega^\star \to \mathbb{R}^+$ of u is defined by $u^\star(x) = u^\sharp(C_n|x|^n)$, where $C_n = \pi^{n/2}[\Gamma(n/2+1)]^{-1}$ is the measure of the n-dimensional unit ball.

Because μ is a decreasing function, Definition 3.2 implies that $u^\sharp$ is an essentially inverse function of μ. The names for $u^\sharp$ and $u^\star$ are justified by the following two lemmas:

Lemma 3.3.

(a) *The function* $u^\sharp$ *is decreasing,* $u^\sharp(0) = \mathrm{esssup}\,|u|$ *and* $\mathrm{sprt}\, u^\sharp = [0, |\mathrm{sprt}\, u|)$
(b) $u^\sharp(s) = \min\{t \geq 0 : \mu(t) \leq s\}$
(c) $u^\sharp(s) = \int_0^\infty \chi_{[0,\mu(t))}(s)\,\mathrm{d}t$
(d) $|\{s \geq 0 : u^\sharp(s) > t\}| = |\{x \in \Omega : |u(x)| > t\}|$ *for all* $t \geq 0$.
(e) $\{s \geq 0 : u^\sharp(s) > t\} = [0, \mu(t))$ *for all* $t \geq 0$.

Proof. Part (a) is a direct consequence of the definition of $u^\sharp$, keeping in mind the general properties of distribution functions stated above. The representation formula in part (b) follows from

$$u^\sharp(s) = |\{w \geq 0 : \mu(w) > s\}| = \sup\{w \geq 0 : \mu(w) > s\} = \min\{w \geq 0 : \mu(w) \leq s\},$$

where we have used the definition of $u^\sharp$ in the first step and then the monotonicity and right-continuity of μ. Part (c) is a consequence of the 'layer-cake formula', see Theorem 6.1 in the appendix. To prove part (d) we need to show that

$$\{s \geq 0 : u^\sharp(s) > t\} = [0, \mu(t)). \tag{3.1}$$

Indeed, if s is an element of the left hand side of (3.1), then by Lemma 3.3, part (b), we have

$$\min\{w \geq 0 : \mu(w) \leq s\} > t.$$

But this means that $\mu(t) > s$, i.e., $s \in [0, \mu(t))$. On the other hand, if s is an element of the right hand side of (3.1), then $s < \mu(t)$ which implies again by part (b) that

$$u^\sharp(s) = \min\{w \geq 0 : \mu(w) \leq s\} \geq \min\{w \geq 0 : \mu(w) < \mu(t)\} > t,$$

i.e., s is also an element of the left hand side. Finally, part (e) is a direct consequence from part (d). □

It is straightforward to transfer the statements of Lemma 3.3 to the symmetric decreasing rearrangement:

LEMMA 3.4.
(a) *The function $u^\star$ is spherically symmetric and radially decreasing.*
(b) *The measure of the level set $\{x \in \Omega^\star : u^\star(x) > t\}$ is the same as the measure of $\{x \in \Omega : |u(x)| > t\}$ for any $t \geq 0$.*

From Lemma 3.3 (c) and Lemma 3.4 (b) we see that the three functions u, $u^\sharp$ and $u^\star$ have the same distribution function and therefore they are said to be *equimeasurable*. Quite analogous to the decreasing rearrangements one can also define increasing ones:

DEFINITION 3.5.
- If the measure of Ω is finite, we call $u_\sharp(s) = u^\sharp(|\Omega| - s)$ the *increasing rearrangement* of u.
- The *symmetric increasing rearrangement* $u_\star : \Omega^\star \to \mathbb{R}^+$ of u is defined by $u_\star(x) = u_\sharp(C_n|x|^n)$

In his lecture notes on rearrangements (see the reference in the Bibliographical Remarks, i) at the end of this chapter), G. Talenti, gives the following example, illustrating the meaning of the distribution and the rearrangement of a function: Consider the function $u(x) \equiv 8 + 2x^2 - x^4$, defined on the interval $-2 \leq x \leq 2$. Then, it is a simple exercise to check that the corresponding distribution function $\mu(t)$ is given by

$$\mu(t) = \begin{cases} 2\sqrt{1+\sqrt{9-t}} & \text{if } 8 \leq t \leq 8, \\ 2\sqrt{2-2\sqrt{t-8}} & \text{if } 8 < t \leq 9. \end{cases}$$

Hence,

$$u^\star(x) = \begin{cases} 9 - x^2 + x^4/4 & \text{if } x \leq \sqrt{2}, \\ u(x) & \text{if } |x| > \sqrt{2}. \end{cases}$$

This function can as well be used to illustrate the theorems below.

3.2. Main theorems. In this section I summarize the main results concerning rearrangements, which are needed in the sequel. While I omit their proof, I refer the reader to the general references cited at the beginning of this lecture. Rearrangements are a useful tool of functional analysis because they considerably simplify a function without changing certain properties or at least changing them in a controllable way. The simplest example is the fact that the integral of a function's absolute value is invariant under rearrangement. A bit more generally, we have:

THEOREM 3.6. *Let Φ be a continuous increasing map from $\mathbb{R}^+$ to $\mathbb{R}^+$ with $\Phi(0) = 0$. Then*

$$\int_{\Omega^\star} \Phi(u^\star(x))\,\mathrm{d}x = \int_\Omega \Phi(|u(x)|)\,\mathrm{d}x = \int_{\Omega^\star} \Phi(u_\star(x))\,\mathrm{d}x.$$

For later reference we state a rather specialized theorem, which is an estimate on the rearrangement of a spherically symmetric function that is defined on an asymmetric domain:

THEOREM 3.7. *Assume that* $u_\Omega : \Omega \to \mathbb{R}^+$ *is given by* $u_\Omega(x) = u(|x|)$*, where* $u : \mathbb{R}^+ \to \mathbb{R}^+$ *is a non-negative decreasing (resp. increasing) function. Then* $u^\star_\Omega(x) \le u(|x|)$ *(resp.* $u_{\Omega\star}(x) \ge u(|x|)$*) for every* $x \in \Omega^\star$*.*

The product of two functions changes in a controllable way under rearrangement:

THEOREM 3.8. *Suppose that* u *and* v *are measurable and non-negative functions defined on some* $\Omega \subset \mathbb{R}^n$ *with finite measure. Then*

$$\int_{\mathbb{R}^+} u^\sharp(s)\, v^\sharp(s)\,\mathrm{d}s \ge \int_\Omega u(x)\, v(x)\,\mathrm{d}x \ge \int_{\mathbb{R}^+} u^\sharp(s)\, v_\sharp(s)\,\mathrm{d}s \tag{3.2}$$

and

$$\int_{\Omega^\star} u^\star(x)\, v^\star(x)\,\mathrm{d}x \ge \int_\Omega u(x)\, v(x)\,\mathrm{d}x \ge \int_{\Omega^\star} u^\star(x)\, v_\star(x)\,\mathrm{d}x. \tag{3.3}$$

3.3. Gradient estimates. The integral of a function's gradient over the boundary of a level set can be estimated in terms of the distribution function:

THEOREM 3.9. *Assume that* $u : \mathbb{R}^n \to \mathbb{R}$ *is Lipschitz continuous and decays at infinity, i.e., the measure of* $\Omega_t := \{x \in \mathbb{R}^n : |u(x)| > t\}$ *is finite for every positive* t*. If* μ *is the distribution function of* u *then*

$$\int_{\partial\Omega_t} |\nabla u| H_{n-1}(\,\mathrm{d}x) \ge -n^2 C_n^{2/n} \frac{\mu(t)^{2-2/n}}{\mu'(t)}. \tag{3.4}$$

Remark: Here $H_n(A)$ denotes the n–dimensional Hausdorff measure of the set A (see, e.g., [**37**]).

Integrals that involve the norm of the gradient can be estimated using the following important theorem:

THEOREM 3.10. *Let* $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ *be a Young function, i.e.,* Φ *is increasing and convex with* $\Phi(0) = 0$*. Suppose that* $u : \mathbb{R}^n \to \mathbb{R}$ *is Lipschitz continuous and decays at infinity. Then*

$$\int_{\mathbb{R}^n} \Phi(|\nabla u^\star(x)|)\,\mathrm{d}x \le \int_{\mathbb{R}^n} \Phi(|\nabla u(x)|)\,\mathrm{d}x.$$

For the special case $\Phi(t) = t^2$ Theorem 3.10 states that the 'energy expectation value' of a function decreases under symmetric rearrangement, a fact that is key to the proof of the Rayleigh–Faber–Krahn inequality (see Section 3.4).

LEMMA 3.11. *Let* u *and* Φ *be as in Theorem 3.10. Then for almost every positive* s *holds*

$$\frac{\mathrm{d}}{\mathrm{d}s} \int_{\{x\in\mathbb{R}^n : |u(x)| > u^*(s)\}} \Phi(|\nabla u|)\,\mathrm{d}x \ge \Phi\left(-nC_n^{1/n} s^{1-1/n} \frac{\mathrm{d}u^*}{\mathrm{d}s}(s)\right). \tag{3.5}$$

3.4. The Rayleigh–Faber–Krahn Inequality. Many isoperimetric inequalities have been inspired by the question which geometrical layout of some physical system maximizes or minimizes a certain quantity. One may ask, for example, how matter of a given mass density must be distributed to minimize its gravitational energy, or which shape a conducting object must have to maximize its electrostatic capacity. The most famous question of this kind was put forward at the end of the XIXth century by Lord Rayleigh in his work on the theory of sound [**65**]: He conjectured that among all drums of the same area and the same tension the circular drum produces the lowest fundamental frequency. This statement was proven independently in the 1920s by Faber [**36**] and Krahn [**48, 49**].

To treat the problem mathematically, we consider an open bounded domain $\Omega \subset \mathbb{R}^2$ which matches the shape of the drum. Then *the oscillation frequencies of the drum* are given by the eigenvalues of the Laplace operator $-\Delta_D^\Omega$ on Ω with Dirichlet boundary conditions, up to a constant that depends on the drum's tension and mass density. In the following we will allow the more general case $\Omega \subset \mathbb{R}^n$ for $n \geq 2$, although the physical interpretation as a drum only makes sense if $n = 2$. We define the Laplacian $-\Delta_D^\Omega$ via the quadratic–form approach, i.e., it is the unique self–adjoint operator in $L^2(\Omega)$ which is associated with the closed quadratic form

$$h[\Psi] = \int_\Omega |\nabla \Psi|^2 \, \mathrm{d}x, \quad \Psi \in H_0^1(\Omega).$$

Here $H_0^1(\Omega)$, which is a subset of the Sobolev space $W^{1,2}(\Omega)$, is the closure of $C_0^\infty(\Omega)$ with respect to the form norm

$$|\cdot|_h^2 = h[\cdot] + ||\cdot||_{L^2(\Omega)}. \tag{3.6}$$

For more details about the important question of how to define the Laplace operator on arbitrary domains and subject to different boundary conditions we refer the reader to [**24, 35**].

The spectrum of $-\Delta_D^\Omega$ is purely discrete since $H_0^1(\Omega)$ is, by Rellich's theorem, compactly imbedded in $L^2(\Omega)$ (see, e.g., [**24**]). We write $\lambda_1(\Omega)$ for the lowest eigenvalue of $-\Delta_D^\Omega$.

Theorem 3.12 (Rayleigh–Faber–Krahn inequality). *Let $\Omega \subset \mathbb{R}^n$ be an open bounded domain with smooth boundary and $\Omega^\star \subset \mathbb{R}^n$ a ball with the same measure as Ω. Then*

$$\lambda_1(\Omega^*) \leq \lambda_1(\Omega)$$

with equality if and only if Ω itself is a ball.

Proof. With the help of rearrangements at hand, the proof of the Rayleigh–Faber–Krahn inequality is actually not difficult. Let Ψ be the positive normalized first eigenfunction of $-\Delta_D^\Omega$. Since the domain of a positive self-adjoint operator is a subset of its form domain, we have $\Psi \in H_0^1(\Omega)$. Then we have $\Psi^\star \in H_0^1(\Omega^\star)$. Thus we can apply first the min–max principle and then the Theorems 3.6 and 3.10 to obtain

$$\lambda_1(\Omega^\star) \leq \frac{\int_{\Omega^\star} |\nabla \Psi^\star|^2 \, \mathrm{d}^n x}{\int_{\Omega^\star} |\Psi^*|^2 \, \mathrm{d}^n x} \leq \frac{\int_\Omega |\nabla \Psi|^2 \, \mathrm{d}^n x}{\int_\Omega \Psi^2 \, \mathrm{d}^n x} = \lambda_1(\Omega).$$

□

The Rayleigh–Faber–Krahn inequality has been extended to a number of different settings, for example to Laplace operators on curved manifolds or with respect to different measures. In the following we shall give an overview of these generalizations.

3.5. Schrödinger operators. It is not difficult to extend the Rayleigh-Faber-Krahn inequality to Schrödinger operators, i.e., to operators of the form $-\Delta+V(x)$. Let $\Omega \subset \mathbb{R}^n$ be an open bounded domain and $V : \mathbb{R}^n \to \mathbb{R}^+$ a non-negative potential in $L^1(\Omega)$. Then the quadratic form

$$h_V[u] = \int_\Omega \left(|\nabla u|^2 + V(x)|u|^2\right) \mathrm{d}^n x,$$

defined on

$$\mathrm{Dom}\ h_V = H_0^1(\Omega) \cap \left\{ u \in L^2(\Omega) : \int_\Omega (1+V(x))|u(x)|^2 \, \mathrm{d}^n x < \infty \right\}$$

is closed (see, e.g., [**34, 35**]). It is associated with the positive self-adjoint Schrödinger operator $H_V = -\Delta + V(x)$. The spectrum of H_V is purely discrete and we write $\lambda_1(\Omega, V)$ for its lowest eigenvalue.

THEOREM 3.13. *Under the assumptions stated above,*

$$\lambda_1(\Omega^*, V_\star) \le \lambda_1(\Omega, V).$$

PROOF. Let $u_1 \in \mathrm{Dom}\ h_V$ be the positive normalized first eigenfunction of H_V. Then we have $u_1^\star \in H_0^1(\Omega^\star)$ and by Theorem 3.8

$$\int_{\Omega^\star} (1+V_\star) u_1^{\star 2} \, \mathrm{d}^n x \le \int_\Omega (1+V) u_1^2 \, \mathrm{d}^n x < \infty.$$

Thus $u_1^\star \in \mathrm{Dom}\ h_{V_\star}$ and we can apply first the min–max principle and then Theorems 3.6, 3.8 and 3.10 to obtain

$$\begin{aligned} \lambda_1(\Omega^\star, V_\star) &\le \frac{\int_{\Omega^\star} \left(|\nabla u_1^\star|^2 + V_\star u_1^{\star 2}\right) \mathrm{d}^n x}{\int_{\Omega^\star} |u_1^\star|^2 \, \mathrm{d}^n x} \\ &\le \frac{\int_\Omega \left(|\nabla u_1|^2 + V u_1^2\right) \mathrm{d}^n x}{\int_\Omega u_1^2 \, \mathrm{d}^n x} = \lambda_1(\Omega, V). \end{aligned}$$

□

3.6. Spaces of constant curvature. Differential operators can not only be defined for functions in Euclidean space, but also for the more general case of functions on Riemannian manifolds. It is therefore natural to ask whether the isoperimetric inequalities for the eigenvalues of the Laplacian can be generalized to such settings as well. In this section we will state Rayleigh–Faber–Krahn type theorems for the spaces of constant non-zero curvature, i.e., for the sphere and the hyperbolic space. Isoperimetric inequalities for the second Laplace eigenvalue in these curved spaces will be discussed in Lecture 3.

To start with, we define the Laplacian in hyperbolic space as a self-adjoint operator by means of the quadratic form approach. We realize $\mathbb{H}^n$ as the open unit ball $B = \{(x_1, \ldots, x_n) : \sum_{j=1}^n x_j^2 < 1\}$ endowed with the metric

$$ds^2 = \frac{4|dx|^2}{(1-|x|^2)^2} \tag{3.7}$$

and the volume element

$$dV = \frac{2^n \, \mathrm{d}^n x}{(1-|x|^2)^n}, \tag{3.8}$$

where $|\cdot|$ denotes the Euclidean norm. Let $\Omega \subset \mathbb{H}^n$ be an open domain and assume that it is bounded in the sense that Ω does not touch the boundary of B. The quadratic form of the Laplace operator in hyperbolic space is the closure of

$$h[u] = \int_\Omega g^{ij}(\partial_i u)(\partial_j u)\, \mathrm{d}V, \quad u \in C_0^\infty(\Omega). \tag{3.9}$$

It is easy to see that the form (3.9) is indeed closeable: Since Ω does not touch the boundary of B, the metric coefficients g^{ij} are bounded from above on Ω. They are also bounded from below by $g^{ij} \geq 4$. Consequently, the form norms of h and its Euclidean counterpart, which is the right hand side of (3.9) with g^{ij} replaced by δ^{ij}, are equivalent. Since the 'Euclidean' form is well known to be closeable, h must also be closeable.

By standard spectral theory, the closure of h induces an unique positive self-adjoint operator $-\Delta_{\mathbb{H}}$ which we call the Laplace operator in hyperbolic space. Equivalence between corresponding norms in Euclidean and hyperbolic space implies that the imbedding Dom $h \to L^2(\Omega, \mathrm{d}V)$ is compact and thus the spectrum of $-\Delta_{\mathbb{H}}$ is discrete. For its lowest eigenvalue the following Rayleigh–Faber–Krahn inequality holds.

THEOREM 3.14. *Let $\Omega \subset \mathbb{H}^n$ be an open bounded domain with smooth boundary and $\Omega^\star \subset \mathbb{H}^n$ an open geodesic ball of the same measure. Denote by $\lambda_1(\Omega)$ and $\lambda_1(\Omega^\star)$ the lowest eigenvalue of the Dirichlet-Laplace operator on the respective domain. Then*

$$\lambda_1(\Omega^\star) \leq \lambda_1(\Omega)$$

with equality only if Ω itself is a geodesic ball.

The Laplace operator $-\Delta_{\mathbb{S}}$ on a domain which is contained in the unit sphere $\mathbb{S}^n$ can be defined in a completely analogous fashion to $-\Delta_{\mathbb{H}}$ by just replacing the metric g^{ij} in (3.9) by the metric of $\mathbb{S}^n$.

THEOREM 3.15. *Let $\Omega \subset \mathbb{S}^n$ be an open bounded domain with smooth boundary and $\Omega^\star \subset \mathbb{S}^n$ an open geodesic ball of the same measure. Denote by $\lambda_1(\Omega)$ and $\lambda_1(\Omega^\star)$ the lowest eigenvalue of the Dirichlet-Laplace operator on the respective domain. Then*

$$\lambda_1(\Omega^\star) \leq \lambda_1(\Omega)$$

with equality only if Ω itself is a geodesic ball.

The proofs of the above theorems are similar to the proof for the Euclidean case and will be omitted here. A more general Rayleigh–Faber–Krahn theorem for the Laplace operator on Riemannian manifolds and its proof can be found in the book of Chavel [**31**].

3.7. Robin Boundary Conditions. Yet another generalization of the Rayleigh–Faber–Krahn inequality holds for the boundary value problem

$$\begin{aligned} -\sum_{j=1}^n \frac{\partial^2}{\partial x_j^2} u &= \lambda u \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} + \beta u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{3.10}$$

on a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$ with the outer unit normal ν and some constant $\beta > 0$. This so–called Robin boundary value problem can be interpreted as a mathematical model for a vibrating membrane whose edge is coupled elastically to some fixed frame. The parameter β indicates how tight this binding is and the eigenvalues of (3.10) correspond the the resonant vibration frequencies of the membrane. They form a sequence $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \dots$ (see, e.g., [**52**]).

The Robin problem (3.10) is more complicated than the corresponding Dirichlet problem for several reasons. For example, the very useful property of domain monotonicity does not hold for the eigenvalues of the Robin–Laplacian. That is, if one enlarges the domain Ω in a certain way, the eigenvalues may go up. It is known though, that a very weak form of domain monotonicity holds, namely that $\lambda_1(B) \leq \lambda_1(\Omega)$ if B is ball that contains Ω. Another difficulty of the Robin problem, compared to the Dirichlet case, is that the level sets of the eigenfunctions may touch the boundary. This makes it impossible, for example, to generalize the proof of the Rayleigh–Faber–Krahn inequality in a straightforward way. Nevertheless, such an isoperimetric inequality holds, as proven by Daners:

THEOREM 3.16. *Let $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ be a bounded Lipschitz domain, $\beta > 0$ a constant and $\lambda_1(\Omega)$ the lowest eigenvalue of (3.10). Then $\lambda_1(\Omega^\star) \leq \lambda_1(\Omega)$.*

For the proof of Theorem 3.16, which is not short, we refer the reader to [**33**].

3.8. Bibliographical Remarks. i) Rearrangements of functions were introduced by G. Hardy and J. E. Littlewood. Their results are contained in the classical book, G.H. Hardy, J. E. Littlewood, J.E., and G. Pólya, *Inequalities*, 2d ed., Cambridge University Press, 1952. The fact that the L^2 norm of the gradient of a function decreases under rearrangements was proven by Faber and Krahn [**36, 48, 49**]. A more modern proof as well as many results on rearrangements and their applications to PDE's can be found in [**75**]. The reader may want to see also the article by E.H. Lieb, *Existence and uniqueness of the minimizing solution of Choquard's nonlinear equation*, Studies in Appl. Math. **57**, 93–105 (1976/77), for an alternative proof of the fact that the L^2 norm of the gradient decreases under rearrangements using heat kernel techniques. An excellent expository review on rearrangements of functions (with a good bibliography) can be found in Talenti, G., *Inequalities in rearrangement invariant function spaces*, in *Nonlinear analysis, function spaces and applications, Vol. 5 (Prague, 1994)*, 177–230, Prometheus, Prague, 1994. (available at the website: http://www.emis.de/proceedings/Praha94/). The Riesz rearrangement inequality is the assertion that for nonnegative measurable functions f, g, h in $\mathbb{R}^n$, we have

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} f(y)g(x-y)h(x)dx\,dy \leq \int_{\mathbb{R}^n \times \mathbb{R}^n} f^\star(y)g^\star(x-y)h^\star(x)dx\,dy.$$

For $n = 1$ the inequality is due to F. Riesz, *Sur une inégalité intégrale*, Journal of the London Mathematical Society **5**, 162–168 (1930). For general n is due to S.L. Sobolev, *On a theorem of functional analysis*, Mat. Sb. (NS) **4**, 471–497 (1938) [the English translation appears in AMS Translations (2) **34**, 39–68 (1963)]. The cases of equality in the Riesz inequality were studied by A. Burchard, *Cases of equality in the Riesz rearrangement inequality*, Annals of Mathematics **143** 499–627 (1996) (this paper also has an interesting history of the problem).

ii) Rearrangements of functions have been extensively used to prove symmetry properties of positive solutions of nonlinear PDE's. See, e.g., Kawohl, Bernhard, *Rearrangements* *and convexity of level sets in PDE.* Lecture Notes in Mathematics, 1150. Springer-Verlag, Berlin (1985), and references therein.

iii) There are different types of rearrangements of functions. For an interesting approach to rearrangements see, Brock, Friedemann and Solynin, Alexander Yu. *An approach to symmetrization via polarization.* Trans. Amer. Math. Soc. **352** 1759–1796 (2000). This approach goes back through Baernstein–Taylor (Duke Math. J. 1976), who cite Ahlfors (book on "Conformal invariants", 1973), who in turn credits Hardy and Littlewood.

iv) The Rayleigh–Faber–Krahn inequality is an isoperimetric inequality concerning the lowest eigenvalue of the Laplacian, with Dirichlet boundary condition, on a bounded domain in $\mathbb{R}^n$ ($n \geq 2$). Let $0 < \lambda_1(\Omega) < \lambda_2(\Omega) \leq \lambda_3(\Omega) \leq \dots$ be the Dirichlet eigenvalues of the Laplacian in $\Omega \subset \mathbb{R}^n$, i.e.,

$$-\Delta u = \lambda u \qquad \text{in } \Omega,$$

$$u = 0 \qquad \text{on the boundary of } \Omega.$$

If $n = 2$, the Dirichlet eigenvalues are proportional to the square of the eigenfrequencies of an elastic, homogeneous, vibrating membrane with fixed boundary. The Rayleigh–Faber–Krahn inequality for the membrane (i.e., $n = 2$) states that

$$\lambda_1 \geq \frac{\pi j_{0,1}^2}{A},$$

where $j_{0,1} = 2.4048\dots$ is the first zero of the Bessel function of order zero, and A is the area of the membrane. Equality is obtained if and only if the membrane is circular. In other words, *among all membranes of given area, the circle has the lowest fundamental frequency.* This inequality was conjectured by Lord Rayleigh (see, [**65**], pp. 339–340). In 1918, Courant (see R. Courant, Math. Z. **1**, 321–328 (1918)) proved the weaker result that among all membranes of the same perimeter L the circular one yields the least lowest eigenvalue, i.e.,

$$\lambda_1 \geq \frac{4\pi^2 j_{0,1}^2}{L^2},$$

with equality if and only if the membrane is circular. Rayleigh's conjecture was proven independently by Faber [**36**] and Krahn [**48**]. The corresponding isoperimetric inequality in dimension n,

$$\lambda_1(\Omega) \geq \left(\frac{1}{|\Omega|}\right)^{2/n} C_n^{2/n}\, j_{n/2-1,1},$$

was proven by Krahn [**49**]. Here $j_{m,1}$ is the first positive zero of the Bessel function J_m, $|\Omega|$ is the volume of the domain, and $C_n = \pi^{n/2}/\Gamma(n/2+1)$ is the volume of the n–dimensional unit ball. Equality is attained if and only if Ω is a ball. For more details see, R.D. Benguria, *Rayleigh–Faber–Krahn Inequality,* in *Encyclopaedia of Mathematics,* Supplement III, Managing Editor: M. Hazewinkel, Kluwer Academic Publishers, pp. 325–327, (2001).

v) A natural question to ask concerning the Rayleigh–Faber–Krahn inequality is the question of stability. If the lowest eigenvalue of a domain Ω is within ϵ (positive and sufficiently small) of the isoperimetric value $\lambda_1(\Omega^*)$, how close is the domain Ω to being a ball? The problem of stability for (convex domains) concerning the Rayleigh–Faber–Krahn inequality was solved by Antonios Melas (Melas, A.D., *The stability of some eigenvalue estimates,* J. Differential Geom. **36**, 19–33 (1992)). In the same reference, Melas also solved the analogous stability problem for convex domains with respect to the PPW inequality (see Lecture 3, below). The work of Melas has been extended to the case of the Szegö–Weinberger inequality (for the first nontrivial Neumann eigenvalue) by Y.-Y. Xu, *The first nonzero eigenvalue of Neumann problem on Riemannian manifolds,* J. Geom. Anal. **5** 151–165 (1995), and to the case of the PPW inequality on spaces of constant curvature by A. Avila, *Stability results for the first eigenvalue of the Laplacian on domains in space forms,* J. Math. Anal. Appl. **267**, 760–774 (2002). In this connection it is worth mentioning related results on the isoperimetric inequality of R. Hall, *A quantitative isoperimetric inequality in n–dimensional space*, J. Reine Angew Math. **428** , 161–176 (1992), as well as recent results of Maggi, Pratelli and Fusco (recently reviewed by F. Maggi in Bull. Amer. Math. Soc. **45**, 367–408 (2008).

vi) The analog of the Faber–Krahn inequality for domains in the sphere $\mathbb{S}^n$ was proven by Sperner, Emanuel, Jr. *Zur Symmetrisierung von Funktionen auf Sphären,* Math. Z. **134**, 317–327 (1973).

vii) For isoperimetric inequalities for the lowest eigenvalue of the Laplace–Beltrami operator on manifolds, see, e.g., the book by Chavel, Isaac, **Eigenvalues in Riemannian geometry**. Pure and Applied Mathematics, 115. Academic Press, Inc., Orlando, FL, 1984, (in particular Chapters IV and V), and also the articles, Chavel, I. and Feldman, E. A. *Isoperimetric inequalities on curved surfaces.* Adv. in Math. **37**, 83–98 (1980), and Bandle, Catherine, *Konstruktion isoperimetrischer Ungleichungen der mathematischen Physik aus solchen der Geometrie,* Comment. Math. Helv. 46, 182–213 (1971).

viii) Recently, the analog of the Rayleigh–Faber–Krahn inequality for an elliptic operator with drift was proven by F. Hamel, N. Nadirashvili and E. Russ [**41**]. In fact, let Ω be a bounded $C^{2,\alpha}$ domain in $\mathbb{R}^n$ (with $n \geq 1$ and $0 < \alpha < 1$), and $\tau \geq 0$. Let $\vec{v} \in L^\infty(\Omega, \mathbb{R}^n)$, with $\|v\|_\infty \leq \tau$. Let $\lambda_1(\Omega, v)$ denote the principal eigenvalue of $-\Delta + \vec{v} \cdot \nabla$ with Dirichlet boundary conditions. Then, $\lambda_1(\Omega, \vec{v}) \geq \lambda_1(\Omega^*, \tau\, e_r)$, where $e_r = x/|x|$. Moreover, equality is attained up to translations, if and only if $\Omega = \Omega^*$ and $\vec{v} = \tau\, e_r$. See, F. Hamel, N. Nadirashvili, and E. Russ, *Rearrangement inequalities and applications to isoperimetric problems for eigenvalues*, to appear in Annals of Mathematics (2011) (and references therein), where the authors develop a new type of rearrangement to prove this and many other isoperimetric results for the class of elliptic operators of the form $-\mathrm{div}(A \cdot \nabla) + v \cdot \nabla + V$, with Dirichlet boundary conditions, in Ω. Here A is a positive definite matrix.

4. Lecture 3: The Szegö–Weinberger and the Payne–Polya–Weinberger inequalities

4.1. The Szegö–Weinberger inequality. In analogy to the Rayleigh–Faber–Krahn inequality for the Dirichlet–Laplacian one may ask which shape of a domain maximizes certain eigenvalues of the Laplace operator with Neumann boundary conditions. Of course, this question is trivial for the lowest Neumann eigenvalue, which is always zero. In 1952 Kornhauser and Stakgold [**47**] conjectured that the ball maximizes the first non-zero Neumann eigenvalue among all domains of the same volume. This was first proven in 1954 by Szegö [**72**] for two-dimensional simply connected domains, using conformal mappings. Two years later his result was generalized to domains in any dimension by Weinberger [**76**], who came up with a new strategy for the proof.

Although the Szegö–Weinberger inequality appears to be the analog for Neumann eigenvalues of the Rayleigh–Faber–Krahn inequality, its proof is completely different. The reason is that the first non-trivial Neumann eigenfunction must be orthogonal to the constant function, and thus it must have a change of sign. The simple symmetrization procedure that is used to establish the Rayleigh–Faber–Krahn inequality can therefore not work.

In general, when dealing with Neumann problems, one has to take into account that the spectrum of the respective Laplace operator on a bounded domain is very unstable under perturbations. One can change the spectrum arbitrarily much by only a slight modification of the domain, and if the boundary is not smooth enough, the Laplacian may even have essential spectrum. A sufficient condition for the spectrum of $-\Delta_N^\Omega$ to be purely discrete is that Ω is bounded and has a Lipschitz boundary [**35**]. We write $0 = \mu_0(\Omega) < \mu_1(\Omega) \leq \mu_2(\Omega) \leq \ldots$ for the sequence of Neumann eigenvalues on such a domain Ω.

THEOREM 4.1 (Szegö–Weinberger inequality). *Let $\Omega \subset \mathbb{R}^n$ be an open bounded domain with smooth boundary such that the Laplace operator on Ω with Neumann boundary conditions has purely discrete spectrum. Then*

$$\mu_1(\Omega) \leq \mu_1(\Omega^\star), \tag{4.1}$$

where $\Omega^\star \subset \mathbb{R}^n$ is a ball with the same n-volume as Ω. Equality holds if and only if Ω itself is a ball.

PROOF. By a standard separation of variables one shows that $\mu_1(\Omega^\star)$ is n-fold degenerate and that a basis of the corresponding eigenspace can be written in the form $\{g(r)r_j r^{-1}\}_{j=1,\ldots,n}$. The function g can be chosen to be positive and satisfies the differential equation

$$g'' + \frac{n-1}{r}g' + \left(\mu_1(\Omega^\star) - \frac{n-1}{r^2}\right)g = 0, \quad 0 < r < R_1, \tag{4.2}$$

where R_1 is the radius of $\Omega^\star$. Further, $g(r)$ vanishes at $r=0$ and its derivative has its first zero at $r = R_1$. We extend g by defining $g(r) = \lim_{r'\uparrow R_1} g(r')$ for $r \geq R_1$. Then g is differentiable on $\mathbb{R}$ and if we set $f_j(\vec{r}) := g(r)r_j r^{-1}$ then $f_j \in W^{1,2}(\Omega)$ for $j = 1\ldots,n$. To apply the min-max principle with f_j as a test function for $\mu_1(\Omega)$ we have to make sure that f_j is orthogonal to the first (trivial) eigenfunction, i.e., that

$$\int_\Omega f_j \,\mathrm{d}^n r = 0, \quad j = 1,\ldots,n. \tag{4.3}$$

We argue that this can be achieved by some shift of the domain Ω: Since Ω is bounded we can find a ball B that contains Ω. Now define the vector field $\vec{b} : \mathbb{R}^n \to \mathbb{R}^n$ by its components

$$b_j(\vec{v}) = \int_{\Omega+\vec{v}} f_j(\vec{r})\,\mathrm{d}^n r, \quad \vec{v} \in \mathbb{R}^n.$$

For $\vec{v} \in \partial B$ we have

$$\begin{aligned}
\vec{v}\cdot\vec{b}(\vec{v}) &= \int_{\Omega+\vec{v}} \frac{\vec{v}\cdot\vec{r}}{r} g(r)\,\mathrm{d}^n r \\
&= \int_\Omega \frac{\vec{v}\cdot(\vec{r}+\vec{v})}{|\vec{r}+\vec{v}|} g(|\vec{r}+\vec{v}|)\,\mathrm{d}^n r \\
&\geq \int_\Omega \frac{|\vec{v}|^2 - |v|\cdot|r|}{|\vec{r}+\vec{v}|} g(|\vec{r}+\vec{v}|)\,\mathrm{d}^n r > 0.
\end{aligned}$$

Thus $\vec{b}$ is a vector field that points outwards on every point of ∂B. By an application of the Brouwer's fixed–point theorem (see Theorem 6.3 in the Appendix) this means that $\vec{b}(\vec{v}_0) = 0$ for some $\vec{v}_0 \in B$. Thus, if we shift Ω by this vector, condition (4.3) is satisfied and we can apply the min-max principle with the f_j as test functions for the first non-zero eigenvalue:

$$\begin{aligned}\mu_1(\Omega) &\leq \frac{\int_\Omega |\nabla f_j| \, \mathrm{d}^n r}{\int_\Omega f_j^2 \, \mathrm{d}^n r} \\ &= \frac{\int_\Omega \left({g'}^2(r) r_j^2 r^{-2} + g^2(r)(1 - r_j^2 r^{-2}) r^{-2} \right) \mathrm{d}^n r}{\int_\Omega g^2 r_j^2 r^{-2} \, \mathrm{d}^n r}.\end{aligned}$$

We multiply each of these inequalities by the denominator and sum up over j to obtain

$$\mu_1(\Omega) \leq \frac{\int_\Omega B(r) \, \mathrm{d}^n r}{\int_\Omega g^2(r) \, \mathrm{d}^n r} \tag{4.4}$$

with $B(r) = {g'}^2(r) + (n-1) g^2(r) r^{-2}$. Since R_1 is the first zero of g', the function g is non-decreasing. The derivative of B is

$$B' = 2g'g'' + 2(n-1)(rgg' - g^2) r^{-3}.$$

For $r \geq R_1$ this is clearly negative since g is constant there. For $r < R_1$ we can use equation (4.2) to show that

$$B' = -2\mu_1(\Omega^\star) g g' - (n-1)(rg' - g)^2 r^{-3} < 0.$$

In the following we will use the method of rearrangements, which was described in Chapter 3. To avoid confusions, we use a more precise notation at this point: We introduce $B_\Omega : \Omega \to \mathbb{R}$, $B_\Omega(\vec{r}) = B(r)$ and analogously $g_\Omega : \Omega \to \mathbb{R}$, $g_\Omega(\vec{r}) = g(r)$. Then equation (4.4) yields, using Theorem 3.7 in the third step:

$$\mu_1(\Omega) \leq \frac{\int_\Omega B_\Omega(\vec{r}) \, \mathrm{d}^n r}{\int_\Omega g_\Omega^2(\vec{r}) \, \mathrm{d}^n r} = \frac{\int_{\Omega^\star} B_\Omega^\star(\vec{r}) \, \mathrm{d}^n r}{\int_{\Omega^\star} g_{\star\Omega}^2(\vec{r}) \, \mathrm{d}^n r} \leq \frac{\int_{\Omega^\star} B(r) \, \mathrm{d}^n r}{\int_{\Omega^\star} g^2(r) \, \mathrm{d}^n r} = \mu_1(\Omega^\star) \tag{4.5}$$

Equality holds obviously if Ω is a ball. In any other case the third step in (4.5) is a strict inequality. □

It is rather straightforward to generalize the Szegö–Weinberger inequality to domains in hyperbolic space. For domains on spheres, on the other hand, the corresponding inequality has not been established yet in full generality. At present, the most general result is due to Ashbaugh and Benguria: In [**9**] they show that an analog of the Szegö–Weinberger inequality holds for domains that are contained in a hemisphere.

4.2. The Payne–Pólya–Weinberger inequality. A further isoperimetric inequality is concerned with the second eigenvalue of the Dirichlet–Laplacian on bounded domains. In 1955 Payne, Pólya and Weinberger (PPW) showed that for any open bounded domain $\Omega \subset \mathbb{R}^2$ the bound $\lambda_2(\Omega)/\lambda_1(\Omega) \leq 3$ holds [**60, 61**]. Based on exact calculations for simple domains they also conjectured that the ratio $\lambda_2(\Omega)/\lambda_1(\Omega)$ is maximized when Ω is a circular disk, i.e., that

$$\frac{\lambda_2(\Omega)}{\lambda_1(\Omega)} \leq \frac{\lambda_2(\Omega^\star)}{\lambda_1(\Omega^\star)} = \frac{j_{1,1}^2}{j_{0,1}^2} \approx 2.539 \quad \text{for } \Omega \subset \mathbb{R}^2. \tag{4.6}$$

Here, $j_{n,m}$ denotes the m^{th} positive zero of the Bessel function $J_n(x)$. This conjecture and the corresponding inequalities in n dimensions were proven in 1991 by Ashbaugh and Benguria [**6, 7, 8**]. Since the Dirichlet eigenvalues on a ball are inversely proportional to the square of the ball's radius, the ratio $\lambda_2(\Omega^\star)/\lambda_1(\Omega^\star)$ does not depend on the size of $\Omega^\star$. Thus we can state the PPW inequality in the following form:

THEOREM 4.2 (Payne–Pólya–Weinberger inequality). *Let $\Omega \subset \mathbb{R}^n$ be an open bounded domain and $S_1 \subset \mathbb{R}^n$ a ball such that $\lambda_1(\Omega) = \lambda_1(S_1)$. Then*

$$\lambda_2(\Omega) \leq \lambda_2(S_1) \tag{4.7}$$

with equality if and only if Ω is a ball.

Here the subscript 1 on S_1 reflects the fact that the ball S_1 has the same first Dirichlet eigenvalue as the original domain Ω. The inequalities (4.6) and (4.7) are equivalent in Euclidean space in view of the mentioned scaling properties of the eigenvalues. Yet when one considers possible extensions of the PPW inequality to other settings, where λ_2/λ_1 varies with the radius of the ball, it turns out that an estimate in the form of Theorem 4.2 is the more natural result. In the case of a domain on a hemisphere, for example, λ_2/λ_1 on balls is an increasing function of the radius. But by the Rayleigh–Faber–Krahn inequality for spheres the radius of S_1 is smaller than the one of the spherical rearrangement $\Omega^\star$. This means that an estimate in the form of Theorem 4.2, interpreted as

$$\frac{\lambda_2(\Omega)}{\lambda_1(\Omega)} \leq \frac{\lambda_2(S_1)}{\lambda_1(S_1)}, \quad \Omega, S_1 \subset \mathbb{S}^n,$$

is stronger than an inequality of the type (4.6).

On the other hand, we will see that in the hyperbolic space λ_2/λ_1 on balls is a strictly decreasing function of the radius. In this case we can apply the following argument to see that an estimate of the type (4.6) cannot possibly hold true: Consider a domain Ω that is constructed by attaching very long and thin tentacles to the ball B. Then the first and second eigenvalues of the Laplacian on Ω are arbitrarily close to the ones on B. The spherical rearrangement of Ω though can be considerably larger than B. This means that

$$\frac{\lambda_2(\Omega)}{\lambda_1(\Omega)} \approx \frac{\lambda_2(B)}{\lambda_1(B)} > \frac{\lambda_2(\Omega^\star)}{\lambda_1(\Omega^\star)}, \qquad B, \Omega \subset \mathbb{H}^n,$$

clearly ruling out any inequality in the form of (4.6).

The proof of the PPW inequality (4.7) is somewhat similar to that of the Szegö–Weinberger inequality (see the previous section in this Lecture), but considerably more difficult. The additional complications mainly stem from the fact that in the Dirichlet case the first eigenfunction of the Laplacian is not known explicitly, while in the Neumann case it is just constant. We will give the full proof of the PPW inequality in the sequel. Since it is rather long, a brief outline is in order:

The proof is organized in six steps. In the first one we use the min–max principle to derive an estimate for the eigenvalue gap $\lambda_2(\Omega) - \lambda_1(\Omega)$, depending on a test function for the second eigenvalue. In the second step we define such a function and then show in the third step that it actually satisfies all requirements to be used in the gap formula. In the fourth step we put the test function into the gap inequality and then estimate the result with the help of rearrangement techniques. These depend on the monotonicity properties of two functions g and B, which are to be defined in the proof, and on a Chiti comparison argument. The later is a special comparison result which establishes a crossing property between the symmetric decreasing rearrangement of the first eigenfunction on Ω and the first eigenfunction on S_1. We end up with the inequality $\lambda_2(\Omega)-\lambda_1(\Omega) \leq \lambda_2(S_1)-\lambda_1(S_1)$, which yields (4.7). In the remaining two steps we prove the mentioned monotonicity properties and the Chiti comparison result. We remark that from the Rayleigh–Faber–Krahn inequality follows $S_1 \subset \Omega^\star$, a fact that is used in the proof of the Chiti comparison result. Although it enters in a rather subtle manner, the Rayleigh–Faber–Krahn inequality is an important ingredient of the proof of the PPW inequality.

4.3. Proof of the Payne–Pólya–Weinberger inequality. *First step:* We derive the 'gap formula' for the first two eigenvalues of the Dirichlet–Laplacian on Ω. We call $u_1 : \Omega \to \mathbb{R}^+$ the positive normalized first eigenfunction of $-\Delta_\Omega^D$. To estimate the second eigenvalue we will use the test function Pu_1, where $P : \Omega \to \mathbb{R}$ is is chosen such that Pu_1 is in the form domain of $-\Delta_\Omega^D$ and

$$\int_\Omega P u_1^2 \,\mathrm{d}r^n = 0. \tag{4.8}$$

Then we conclude from the min–max principle that

$$\begin{aligned} \lambda_2(\Omega) - \lambda_1(\Omega) &\leq \frac{\int_\Omega \left(|\nabla(Pu_1)|^2 - \lambda_1 P^2 u_1^2\right) \mathrm{d}r^n}{\int_\Omega P^2 u_1^2 \,\mathrm{d}r^n} \\ &= \frac{\int_\Omega \left(|\nabla P|^2 u_1^2 + (\nabla P^2) u_1 \nabla u_1 + P^2 |\nabla u_1|^2 - \lambda_1 P^2 u_1^2\right) \mathrm{d}r^n}{\int_\Omega P^2 u_1^2 \,\mathrm{d}r^n} \end{aligned} \tag{4.9}$$

If we perform an integration by parts on the second summand in the numerator of (4.9), we see that all summands except the first cancel. We obtain the gap inequality

$$\lambda_2(\Omega) - \lambda_1(\Omega) \leq \frac{\int_\Omega |\nabla P|^2 u_1^2 \,\mathrm{d}r^n}{\int_\Omega P^2 u_1^2 \,\mathrm{d}r^n}. \tag{4.10}$$

Second step: We need to fix the test function P. Our choice will be dictated by the requirement that equality should hold in (4.10) if Ω is a ball, i.e., if $\Omega = S_1$ up to translations. We assume that S_1 is centered at the origin of our coordinate system and call R_1 its radius. We write $z_1(r)$ for the first eigenfunction of the Dirichlet Laplacian on S_1. This function is spherically symmetric with respect to the origin and we can take it to be positive and normalized in $L^2(S_1)$. The second eigenvalue of $-\Delta_{S_1}^D$ in n dimensions is n–fold degenerate and a basis of the corresponding eigenspace can be written in the form $z_2(r) r_j r^{-1}$ with $z_2 \geq 0$ and $j = 1, \ldots, n$. This is the motivation to choose not only one test function P, but rather n functions P_j with $j = 1, \ldots, n$. We set

$$P_j = r_j r^{-1} g(r)$$

with

$$g(r) = \begin{cases} \frac{z_2(r)}{z_1(r)} & \text{for } r < R_1, \\ \lim_{r' \uparrow R_1} \frac{z_2(r')}{z_1(r')} & \text{for } r \geq R_1. \end{cases}$$

We note that $P_j u_1$ is a second eigenfunction of $-\Delta_\Omega^D$ if Ω is a ball which is centered at the origin.

Third step: It is necessary to verify that the $P_j u_1$ are admissible test functions. First, we have to make sure that condition (4.8) is satisfied. We note that P_j changes when Ω (and u_1 with it) is shifted in $\mathbb{R}^n$. Since these shifts do not change

$\lambda_1(\Omega)$ and $\lambda_2(\Omega)$, it is sufficient to show that Ω can be moved in $\mathbb{R}^n$ such that (4.8) is satisfied for all $j \in \{1, \dots, n\}$. To this end we define the function

$$\vec{b}(\vec{v}) = \int_{\Omega+\vec{v}} u_1^2(|\vec{r}-\vec{v}\,|)\frac{\vec{r}}{r}g(r)\,\mathrm{d}r^n \quad \text{for } \vec{v} \in \mathbb{R}^n.$$

Since Ω is a bounded domain, we can choose some closed ball D, centered at the origin, such that $\Omega \subset D$. Then for every $\vec{v} \in \partial D$ we have

$$\begin{aligned}\vec{v}\cdot\vec{b}(\vec{v}) &= \int_\Omega \vec{v}\cdot u_1^2(r)\frac{\vec{r}+\vec{v}}{|\vec{r}+\vec{v}\,|}g(|\vec{r}+\vec{v}\,|)\,\mathrm{d}r^n \\ &> \int_\Omega u_1^2(r)\frac{|\vec{v}|^2-|\vec{v}|\cdot|\vec{r}\,|}{|\vec{r}+\vec{v}\,|}g(|\vec{r}+\vec{v}\,|)\,\mathrm{d}r^n > 0\end{aligned}$$

Thus the continuous vector-valued function $\vec{b}(\vec{v}\,)$ points strictly outwards everywhere on ∂D. By Theorem 6.3, which is a consequence of the Brouwer fixed–point theorem, there is some $\vec{v}_0 \in D$ such that $\vec{b}(\vec{v}_0) = 0$. Now we shift Ω by this vector, i.e., we replace Ω by $\Omega - \vec{v}_0$ and u_1 by the first eigenfunction of the shifted domain. Then the test functions $P_j u_1$ satisfy the condition (4.8).

The second requirement on $P_j u_1$ is that it must be in the form domain of $-\Delta_\Omega^D$, i.e., in $H_0^1(\Omega)$: Since $u_1 \in H_0^1(\Omega)$ there is a sequence $\{v_n \in C^1(\Omega)\}_{n\in\mathbb{N}}$ of functions with compact support such that $|\cdot|_h - \lim_{n\to\infty} v_n = u_1$, using the definition (3.6) of $|\cdot|_h$. The functions $P_j v_n$ also have compact support and one can check that $P_j v_n \in C^1(\Omega)$ (P_j is continuously differentiable since $g'(R_1) = 0$). We have $|\cdot|_h - \lim_{n\to\infty} P_j v_n = P_j u_1$ and thus $P_j u_1 \in H_0^1(\Omega)$.

Fourth step: We multiply the gap inequality (4.10) by $\int P^2 u_1^2\,\mathrm{d}x$ and put in our special choice of P_j to obtain

$$\begin{aligned}(\lambda_2-\lambda_1)\int_\Omega \frac{r_j^2}{r^2}g^2(r)u_1^2(r)\,\mathrm{d}r^n &\le \int_\Omega \left|\nabla\left(\frac{r_j}{r}g(r)\right)\right|^2 u_1^2(r)\,\mathrm{d}r^n \\ &= \int_\Omega \left(\left|\nabla\frac{r_j}{r}\right|^2 g^2(r) + \frac{r_j^2}{r^2}g'(r)^2\right)u_1^2(r)\,\mathrm{d}r^n.\end{aligned}$$

Now we sum these inequalities up over $j = 1, \dots, n$ and then divide again by the integral on the left hand side to get

$$\lambda_2(\Omega) - \lambda_1(\Omega) \le \frac{\int_\Omega B(r)u_1^2(r)\,\mathrm{d}r^n}{\int_\Omega g^2(r)u_1^2(r)\,\mathrm{d}r^n} \tag{4.11}$$

with

$$B(r) = g'(r)^2 + (n-1)r^{-2}g(r)^2. \tag{4.12}$$

In the following we will use the method of rearrangements, which was described in the second Lecture. To avoid confusions, we use a more precise notation at this point: We introduce $B_\Omega : \Omega \to \mathbb{R}$, $B_\Omega(\vec{r}) = B(r)$ and analogously $g_\Omega : \Omega \to \mathbb{R}$, $g_\Omega(\vec{r}) = g(r)$. Then equation (4.11) can be written as

$$\lambda_2(\Omega) - \lambda_1(\Omega) \le \frac{\int_\Omega B_\Omega(\vec{r})u_1^2(\vec{r})\,\mathrm{d}r^n}{\int_\Omega g_\Omega^2(\vec{r})u_1^2(\vec{r})\,\mathrm{d}r^n}. \tag{4.13}$$

Then by Theorem 3.8 the following inequality is also true:

$$\lambda_2(\Omega) - \lambda_1(\Omega) \le \frac{\int_{\Omega^\star} B_\Omega^\star(\vec{r})u_1^\star(\vec{r})^2\,\mathrm{d}r^n}{\int_{\Omega^\star} g_{\Omega\star}^2(\vec{r})u_1^\star(\vec{r})^2\,\mathrm{d}r^n}. \tag{4.14}$$

Next we use the very important fact that $g(r)$ is an increasing function and $B(r)$ is a decreasing function, which we will prove in step five below. These monotonicity properties imply by Theorem 3.7 that $B_\Omega^\star(\vec{r}) \le B(r)$ and $g_{\Omega^\star}(\vec{r}) \ge g(r)$. Therefore

$$\lambda_2(\Omega) - \lambda_1(\Omega) \le \frac{\int_{\Omega^\star} B(r) u_1^\star(r)^2 \, \mathrm{d}r^n}{\int_{\Omega^\star} g^2(r) u_1^\star(r)^2 \, \mathrm{d}r^n}. \tag{4.15}$$

Finally we use the following version of Chiti's comparison theorem to estimate the right hand side of (4.15):

Lemma 4.3 (Chiti comparison result). *There is some $r_0 \in (0, R_1)$ such that*

$$\begin{array}{ll} z_1(r) \ge u_1^\star(r) & \textit{for } r \in (0, r_0) \textit{ and} \\ z_1(r) \le u_1^\star(r) & \textit{for } r \in (r_0, R_1). \end{array}$$

We remind the reader that the function z_1 denotes the first Dirichlet eigenfunction for the Laplacian defined on S_1. Applying Lemma 4.3, which will be proven below in step six, to (4.15) yields

$$\lambda_2(\Omega) - \lambda_1(\Omega) \le \frac{\int_{\Omega^\star} B(r) z_1(r)^2 \, \mathrm{d}r^n}{\int_{\Omega^\star} g^2(r) z_1(r)^2 \, \mathrm{d}r^n} = \lambda_2(S_1) - \lambda_1(S_1). \tag{4.16}$$

Since S_1 was chosen such that $\lambda_1(\Omega) = \lambda_1(S_1)$ the above relation proves that $\lambda_2(\Omega) \le \lambda_2(S_1)$. It remains the question: When does equality hold in (4.7)? It is obvious that equality does hold if Ω is a ball, since then $\Omega = S_1$ up to translations. On the other hand, if Ω is not a ball, then (for example) the step from (4.15) to (4.16) is not sharp. Thus (4.7) is a strict inequality if Ω is not a ball.

4.4. Monotonicity of B and g. *Fifth step:* We prove that $g(r)$ is an increasing function and $B(r)$ is a decreasing function. In this step we abbreviate $\lambda_i = \lambda_i(S_1)$. The functions z_1 and z_2 are solutions of the differential equations

$$\begin{aligned} -z_1'' - \frac{n-1}{r} z_1' - \lambda_1 z_1 &= 0, \\ -z_2'' - \frac{n-1}{r} z_2' + \left(\frac{n-1}{r^2} - \lambda_2 \right) z_2 &= 0 \end{aligned} \tag{4.17}$$

with the boundary conditions

$$z_1'(0) = 0, \quad z_1(R_1) = 0, \quad z_2(0) = 0, \quad z_2(R_1) = 0. \tag{4.18}$$

We define the function

$$q(r) := \begin{cases} \frac{r g'(r)}{g(r)} & \text{for } r \in (0, R_1), \\ \lim_{r' \downarrow 0} q(r') & \text{for } r = 0, \\ \lim_{r' \uparrow R_1} q(r') & \text{for } r = R_1. \end{cases} \tag{4.19}$$

Proving the monotonicity of B and g is thus reduced to showing that $0 \le q(r) \le 1$ and $q'(r) \le 0$ for $r \in [0, R_1]$. Using the definition of g and the equations (4.17), one can show that $q(r)$ is a solution of the Riccati differential equation

$$q' = (\lambda_1 - \lambda_2) r + \frac{(1-q)(q+n-1)}{r} - 2q \frac{z_1'}{z_1}. \tag{4.20}$$

It is straightforward to establish the boundary behavior

$$q(0) = 1, \quad q'(0) = 0, \quad q''(0) = \frac{2}{n} \left(\left(1 + \frac{2}{n} \right) \lambda_1 - \lambda_2 \right)$$

and

$$q(R_1) = 0.$$

LEMMA 4.4. *For $0 \leq r \leq R_1$ we have $q(r) \geq 0$.*

PROOF. Assume the contrary. Then there exist two points $0 < s_1 < s_2 \leq R_1$ such that $q(s_1) = q(s_2) = 0$ but $q'(s_1) \leq 0$ and $q'(s_2) \geq 0$. If $s_2 < R_1$ then the Riccati equation (4.20) yields

$$0 \geq q'(s_1) = (\lambda_1 - \lambda_2)s_1 + \frac{n-1}{s_1} > (\lambda_1 - \lambda_2)s_2 + \frac{n-1}{s_2} = q'(s_2) \geq 0,$$

which is a contradiction. If $s_2 = R_1$ then we get a contradiction in a similar way by

$$0 \geq q'(s_1) = (\lambda_1 - \lambda_2)s_1 + \frac{n-1}{s_1} > (\lambda_1 - \lambda_2)R_1 + \frac{n-1}{R_1} = 3q'(R_1) \geq 0.$$

□

In the following we will analyze the behavior of q' according to (4.20), considering r and q as two independent variables. For the sake of a compact notation we will make use of the following abbreviations:

$$\begin{aligned} p(r) &= z_1'(r)/z_1(r) \\ N_y &= y^2 - n + 1 \\ Q_y &= 2y\lambda_1 + (\lambda_2 - \lambda_1)N_y y^{-1} - 2(\lambda_2 - \lambda_1) \\ M_y &= N_y^2/(2y) - (n-2)^2 y/2 \end{aligned}$$

We further define the function

$$T(r, y) := -2p(r)y - \frac{(n-2)y + N_y}{r} - (\lambda_2 - \lambda_1)r. \tag{4.21}$$

Then we can write (4.20) as

$$q'(r) = T(r, q(r)).$$

The definition of $T(r, y)$ allows us to analyze the Riccati equation for q' considering r and $q(r)$ as independent variables. For r going to zero, p is $\mathcal{O}(r)$ and thus

$$T(r, y) = \frac{1}{r}\left((n - 1 + y)(1 - y)\right) + \mathcal{O}(r) \quad \text{for } y \text{ fixed}.$$

Consequently,

$$\begin{aligned} \lim_{r\to 0} T(r, y) &= +\infty && \text{for } 0 \leq y < 1 \text{ fixed}, \\ \lim_{r\to 0} T(r, y) &= 0 && \text{for } y = 1 \text{ and} \\ \lim_{r\to 0} T(r, y) &= -\infty && \text{for } y > 1 \text{ fixed}. \end{aligned}$$

The partial derivative of $T(r, y)$ with respect to r is given by

$$T' = \frac{\partial}{\partial r} T(r, y) = -2yp' + \frac{(n-2)y}{r^2} + \frac{N_y}{r^2} - (\lambda_2 - \lambda_1). \tag{4.22}$$

In the points (r, y) where $T(r, y) = 0$ we have, by (4.21),

$$p|_{T=0} = -\frac{n-2}{2r} - \frac{N_y}{2yr} - \frac{(\lambda_2 - \lambda_1)r}{2y}. \tag{4.23}$$

From (4.17) we get the Riccati equation

$$p' + p^2 + \frac{n-1}{r}p + \lambda_1 = 0. \tag{4.24}$$

Putting (4.23) into (4.24) and the result into (4.22) yields

$$T'|_{T=0} = \frac{M_y}{r^2} + \frac{(\lambda_2 - \lambda_1)^2}{2y} r^2 + Q_y. \tag{4.25}$$

LEMMA 4.5. *There is some $r_0 > 0$ such that $q(r) \leq 1$ for all $r \in (0, r_0)$ and $q(r_0) < 1$.*

PROOF. Suppose the contrary, i.e., $q(r)$ first increases away from $r = 0$. Then, because $q(0) = 1$ and $q(R_1) = 0$ and because q is continuous and differentiable, we can find two points $s_1 < s_2$ such that $\hat{q} := q(s_1) = q(s_2) > 1$ and $q'(s_1) > 0 > q'(s_2)$. Even more, we can chose s_1 and s_2 such that $\hat{q}$ is arbitrarily close to one. Writing $\hat{q} = 1 + \epsilon$ with $\epsilon > 0$, we can calculate from the definition of Q_y that

$$Q_{1+\epsilon} = Q_1 + \epsilon n \left(\lambda_2 - (1 - 2/n)\,\lambda_1\right) + \mathcal{O}(\epsilon^2).$$

The term in brackets can be estimated by

$$\lambda_2 - (1 - 2/n)\lambda_1 > \lambda_2 - \lambda_1 > 0.$$

We can also assume that $Q_1 \geq 0$, because otherwise $q''(0) = \frac{2}{n^2} Q_1 < 0$ and Lemma 4.5 is immediately true. Thus, choosing R_1 and r_2 such that ϵ is sufficiently small, we can make sure that $Q_{\hat{q}} > 0$.

Now consider $T(r, \hat{q})$ as a function of r for our fixed $\hat{q}$. We have $T(s_1, \hat{q}) > 0 > T(s_2, \hat{q})$ and the boundary behavior $T(0, \hat{q}) = -\infty$. Consequently, $T(r, \hat{q})$ changes its sign at least twice on $[0, R_1]$ and thus we can find two zeros $0 < \hat{s}_1 < \hat{s}_2 < R_1$ of $T(r, \hat{q})$ such that

$$T'(\hat{s}_1, \hat{q}) \geq 0 \quad \text{and} \quad T'(\hat{s}_2, \hat{q}) \leq 0. \tag{4.26}$$

But from (4.25), together with $Q_{\hat{q}} > 0$, one can see easily that this is impossible, because the right hand side of (4.25) is either positive or increasing (depending on $M_{\hat{q}}$). This is a contradiction to our assumption that q first increases away from $r = 0$, proving Lemma 4.5. □

LEMMA 4.6. *For all $0 \leq r \leq R_1$ the inequality $q'(r) \leq 0$ holds.*

PROOF. Assume the contrary. Then, because of $q(0) = 1$ and $q(R_1) = 0$, there are three points $s_1 < s_2 < s_3$ in $(0, R_1)$ with $0 < \hat{q} := q(s_1) = q(s_2) = q(s_3) < 1$ and $q'(s_1) < 0$, $q'(s_2) > 0$, $q'(s_3) < 0$. Consider the function $T(r, \hat{q})$, which coincides with $q'(r)$ at s_1, s_2, s_3. Taking into account its boundary behavior at $r = 0$, it is clear that $T(r, \hat{q})$ must have at least the sign changes positive-negative-positive-negative. Thus $T(r, \hat{q})$ has at least three zeros $\hat{s}_1 < \hat{s}_2 < \hat{s}_3$ with the properties

$$T'(\hat{s}_1, \hat{q}) \leq 0, \quad T'(\hat{s}_2, \hat{q}) \geq 0, \quad T'(\hat{s}_3, \hat{q}) \leq 0.$$

Again one can see from (4.25) that this is impossible, because the term on the right hand side is either a strictly convex or a strictly increasing function of r. We conclude that Lemma 4.6 is true. □

Altogether we have shown that $0 \leq q(r) \leq 1$ and $q'(r) \leq 0$ for all $r \in (0, R_1)$, which proves that g is increasing and B is decreasing.

4.5. The Chiti comparison result. *Sixth step:* We prove Lemma 4.3: Here and in the sequel we write short-hand $\lambda_1 = \lambda_1(\Omega) = \lambda_1(S_1)$. We introduce a change of variables via $s = C_n r^n$, where C_n is the volume of the n–dimensional unit ball. Then by Definition 3.2 we have $u_1^\sharp(s) = u_1^\star(r)$ and $z_1^\sharp(s) = z_1(r)$.

LEMMA 4.7. *For the functions $u_1^\sharp(s)$ and $z_1^\sharp(s)$ we have*

$$-\frac{\mathrm{d}u_1^\sharp}{\mathrm{d}s} \leq \lambda_1 n^{-2} C_n^{-2/n} s^{n/2-2} \int_0^s u_1^\sharp(w)\,\mathrm{d}w, \tag{4.27}$$

$$-\frac{\mathrm{d}z_1^\sharp}{\mathrm{d}s} = \lambda_1 n^{-2} C_n^{-2/n} s^{n/2-2} \int_0^s z_1^\sharp(w)\,\mathrm{d}w. \tag{4.28}$$

PROOF. We integrate both sides of $-\Delta u_1 = \lambda_1 u_1$ over the level set $\Omega_t := \{\vec{r} \in \Omega : u_1(\vec{r}) > t\}$ and use Gauss' Divergence Theorem to obtain

$$\int_{\partial\Omega_t} |\nabla u_1| H_{n-1}(\,\mathrm{d}r) = \int_{\Omega_t} \lambda_1\, u_1(\vec{r})\,\mathrm{d}^n r, \tag{4.29}$$

where $\partial\Omega_t = \{\vec{r} \in \Omega : u_1(\vec{r}) = t\}$. Now we define the distribution function $\mu(t) = |\Omega_t|$. Then by Theorem 3.9 we have

$$\int_{\partial\Omega_t} |\nabla u_1| H_{n-1}(\,\mathrm{d}r) \geq -n^2 C_n^{2/n} \frac{\mu(t)^{2-2/n}}{\mu'(t)}. \tag{4.30}$$

The left sides of (4.29) and (4.30) are the same, thus

$$\begin{aligned} -n^2 C_n^{2/n} \frac{\mu(t)^{2-2/n}}{\mu'(t)} &\leq \int_{\Omega_t} \lambda_1\, u_1(\vec{r})\,\mathrm{d}^n r \\ &= \int_0^{(\mu(t)/C_n)^{1/n}} n C_n r^{n-1} \lambda_1 u_1^\star(r)\,\mathrm{d}r. \end{aligned}$$

Now we perform the change of variables $r \to s$ on the right hand side of the above chain of inequalities. We also chose t to be $u_1^\sharp(s)$. Using the fact that $u_1^\sharp$ and μ are essentially inverse functions to one another, this means that $\mu(t) = s$ and $\mu'(t)^{-1} = (u_1^\sharp)'(s)$. The result is (4.27). Equation (4.28) is proven analogously, with equality in each step. □

Lemma 4.7 enables us to prove Lemma 4.3. The function $z_1^\sharp$ is continuous on $(0, |S_1|)$ and $u_1^\sharp$ is continuous on $(0, |\Omega^\star|)$. By the normalization of $u_1^\sharp$ and $z_1^\sharp$ and because $S_1 \subset \Omega^\star$ it is clear that either $z_1^\sharp \geq u_1^\sharp$ on $(0, |S_1|)$ or $u_1^\sharp$ and $z_1^\sharp$ have at least one intersection on this interval. In the first case there is nothing to prove, simply setting $r_0 = R_1$ in Lemma 4.3. In the second case we have to show that there is no intersection of $u_1^\sharp$ and $z_1^\sharp$ such that $u_1^\sharp$ is greater than $z_1^\sharp$ on the left and smaller on the right. So we assume the contrary, i.e., that there are two points $0 \leq s_1 < s_2 < |S_1|$ such that $u_1^\sharp(s) > z_1^\sharp(s)$ for $s \in (s_1, s_2)$, $u_1^\sharp(s_2) = z_1^\sharp(s_2)$ and either $u_1^\sharp(s_1) = z_1^\sharp(s_1)$ or $s_1 = 0$. We set

$$v^\sharp(s) = \begin{cases} u_1^\sharp(s) & \text{on } [0, s_1] \text{ if } \int_0^{s_1} u_1^\sharp(s)\,\mathrm{d}s > \int_0^{s_1} z_1^\sharp(s)\,\mathrm{d}s, \\ z_1^\sharp(s) & \text{on } [0, s_1] \text{ if } \int_0^{s_1} u_1^\sharp(s)\,\mathrm{d}s \leq \int_0^{s_1} z_1^\sharp(s)\,\mathrm{d}s, \\ u_1^\sharp(s) & \text{on } [s_1, s_2], \\ z_1^\sharp(s) & \text{on } [s_2, |S_1|]. \end{cases} \tag{4.31}$$

Then one can convince oneself that because of (4.27) and (4.28)

$$-\frac{\mathrm{d}v^\sharp}{\mathrm{d}s} \leq \lambda_1 n^{-2} C_n^{-2/n} s^{n/2-2} \int_0^s v^\sharp(s')\,\mathrm{d}s' \tag{4.32}$$

for all $s \in [0, |S_1|]$. Now define the test function $v(r) = v^\sharp(C_n r^n)$. Using the Rayleigh–Ritz characterization of λ_1, then (4.32) and finally an integration by parts, we get (if z_1 and u_1 are not identical)

$$\begin{aligned}
\lambda_1 \int_{S_1} v^2(r)\,\mathrm{d}^n x &< \int_{S_1} |\nabla v|^2\,\mathrm{d}^n x = \int_0^{|S_1|} \left(nC_n r^{n-1}\, {v^\sharp}'(s)\right)^2 \mathrm{d}s \\
&\leq -\int_0^{|S_1|} {v^\sharp}'(s)\lambda_1 \int_0^s v^\sharp(s')\,\mathrm{d}s'\,\mathrm{d}s \\
&= \lambda_1 \int_0^{|S_1|} v^\sharp(s)^2\,\mathrm{d}s - \lambda_1 \left[v^\sharp(s)\int_0^s v^\sharp(s')\,\mathrm{d}s'\right]\Big|_0^{S_1} \\
&\leq \lambda_1 \int_{S_1} v^2(r)\,\mathrm{d}^n x
\end{aligned}$$

Comparing the first and the last term in the above chain of (in)equalities reveals a contradiction to our assumption that the intersection point s_2 exists, thus proving Lemma 4.3.

4.6. Schrödinger operators. Theorem 4.2 can be extended in several directions. One generalization, which has been considered by Benguria and Linde in [**18**], is to replace the Laplace operator on the domain $\Omega \subset \mathbb{R}^n$ by a Schrödinger operator $H = -\Delta + V$. In this case the question arises which is the most suitable comparison operator for H. In analogy to the PPW inequality for the Laplacian, it seems natural to compare the eigenvalues of H to those of another Schrödinger operator $\tilde{H} = -\Delta + \tilde{V}$, which is defined on a ball and has the same lowest eigenvalue as H. The potential $\tilde{V}$ should be spherically symmetric and it should reflect some properties of V, but it will also have to satisfy certain requirements in order for the PPW type estimate to hold. The precise result is stated in Theorem 4.8 below, which can be considered as a natural generalization of Theorem 4.2 to Schrödinger operators.

We assume that Ω is open and bounded and that $V : \Omega \to \mathbb{R}^+$ is a non-negative potential from $L^1(\Omega)$. Then we can define the Schrödinger operator $H_V = -\Delta + V$ on Ω in the same way as we did in Section 3.5, i.e., H_V is positive and self-adjoint in $L^2(\Omega)$ and has purely discrete spectrum. We call $\lambda_i(\Omega, V)$ its i-th eigenvalue and, as usual, we write $V_\star$ for the symmetric increasing rearrangement of V.

THEOREM 4.8. *Let $S_1 \subset \mathbb{R}^n$ be a ball centered at the origin and of radius R_1 and let $\tilde{V} : S_1 \to \mathbb{R}^+$ be a radially symmetric non-negative potential such that $\tilde{V}(r) \leq V_\star(r)$ for all $0 \leq r \leq R_1$ and $\lambda_1(\Omega, V) = \lambda_1(S_1, \tilde{V})$. If $\tilde{V}(r)$ satisfies the conditions*

a) $\tilde{V}(0) = \tilde{V}'(0) = 0$ *and*
b) $\tilde{V}'(r)$ *exists and is increasing and convex,*

then

$$\lambda_2(\Omega, V) \leq \lambda_2(S_1, \tilde{V}). \tag{4.33}$$

If V is such that $V_\star$ itself satisfies the conditions a) and b) of the theorem, the best bound is obtained by choosing $\tilde{V} = V_\star$ and then adjusting the size of S_1 such that $\lambda_1(\Omega, V) = \lambda_1(S_1, V_\star)$ holds. (Note that $S_1 \subset \Omega^\star$ by Theorem 3.13). In this case Theorem 4.8 is a typical PPW result and optimal in the sense that equality holds in (4.33) if Ω is a ball and $V = V_\star$. For a general potential V we still get a non-trivial bound on $\lambda_2(\Omega, V)$ though it is not sharp anymore.

For further reference we state the following theorem, which is a direct consequence of Theorem 4.8 and Theorem 3.7:

THEOREM 4.9. *Let $\tilde{V} : \mathbb{R}^n \to \mathbb{R}^+$ be a radially symmetric positive potential that satisfies the conditions a) and b) of Theorem 4.2. Further, assume that $\Omega \subset \mathbb{R}^n$ is an open bounded domain and that $S_1 \subset \mathbb{R}^n$ be the open ball (centered at the origin) such that $\lambda_1(\Omega, \tilde{V}) = \lambda_1(S_1, \tilde{V})$. Then*

$$\lambda_2(\Omega, \tilde{V}) \leq \lambda_2(S_1, \tilde{V}).$$

The proof of Theorem 4.8 is similar to the one of Theorem 4.2 and can be found in [**18**]. One of the main differences occurs in step five (see Section 4.4), since the potential $\tilde{V}(r)$ now appears in the Riccati equation for p. It turns out that the conditions a) and b) in Theorem 4.8 are required to establish the monotonicity properties of q. A second important difference is that a second eigenfunction of a Schrödinger operator with a spherically symmetric potential can not necessarily be written in the form $u_2(r)r_j r^{-1}$. It has been shown by Ashbaugh and Benguria [**5**] that it can be written in this form if $rV(r)$ is convex. On the other hand, the second eigenfunction is radially symmetric (with a spherical nodal surface) if $rV(r)$ is concave. This fact, which is also known as the Baumgartner–Grosse–Martin Inequality [**17**], is another reason why the conditions a) and b) of Theorem 4.8 are needed.

4.7. Spaces of constant curvature. There are generalizations of the Payne-Pólya-Weinberger inequality to spaces of constant curvature. Ashbaugh and Benguria showed in [**11**] that Theorem 4.2 remains valid if one replaces the Euclidean space $\mathbb{R}^n$ by a hemisphere of $\mathbb{S}^n$ and 'ball' by 'geodesic ball'. Similar to the Szegö–Weinberger inequality, it is still an open problem to prove a Payne–Pólya–Weinberger result for the whole sphere. Although there seem to be no counterexamples known that rule out such a generalization, the original scheme of proving the PPW inequality is not likely to work. One reason is that numerical studies show the function g to be not monotone on the whole sphere.

For the hyperbolic space, on the other hand, things are settled. Following the general lines of the original proof, Benguria and Linde established in [**19**] a PPW type inequality that holds in any space of constant negative curvature.

4.8. Bibliographical Remarks. i) In 1952, Kornhauser and Stakgold [**47**] conjectured that the lowest nontrivial Neumann eigenvalue for a smooth bounded domain Ω in $\mathbb{R}^2$ satisfies the isoperimetric inequality

$$\mu_1(\Omega) \leq \mu_1(\Omega^*) = \frac{\pi p^2}{A},$$

where Ω^* is a disk with the same area as Ω, and $p = 1.8412\ldots$ is the first positive zero of the derivative of the Bessel function J_1. This conjecture was proven by G. Szegö in 1954, using conformal maps [**72**]. The extension to n dimensions was proven by H. Weinberger [**76**].

ii) For the case of mixed boundary conditions, Marie–Helene Bossel [*Membranes élastiquement liées inhomogénes ou sur une surface: une nouvelle extension du théoreme isopérimétrique de Rayleigh–Faber–Krahn*, Z. Angew. Math. Phys. **39**, 733–742 (1988)] proved the analog of the Rayleigh–Faber–Krahn inequality.

iii) Very recently, A. Girouard, N. Nadirashvili and I. Polterovich proved that the second positive eigenvalue of a bounded simply connected planar domain of a given area does not exceed the first positive Neumann eigenvalue on a disk of a twice smaller area (see, *Maximization of the second positive Neumann eigenvalue for planar domains*, preprint (2008)). For a review of optimization of eigenvalues with respect to the geometry of the domain, see the recent monograph of A. Henrot [**44**].

iv) In the Bibliographical Remarks of Section 4 (see Section 4.6, ii)) we discussed the stability results of A. Melas for the Rayleigh–Faber–Krahn inequality. In the same vein, recently L. Brasco and A. Pratelli, *Sharp Stability of some Spectral Inequalities*, preprint (2011), have proven related stability results for the Szegö–Weinberger inequality. Moreover, these authors have also proven stability results for the E. Krahn–P. Szego inequality, which says that among all sets of a given measure (in Euclidean Space) the disjoint union of two balls with the same radius minimizes the second eigenvalue of the Dirichlet Laplacian.

5. Lecture 4: Fourth order differential operators

In continuum mechanics, the vibrations of more rigid objects, like plates, rods, etc., are governed by wave equations involving higher order operators on the spatial variables. The normal modes of oscillations of these equations give rise to an eigenvalue problem associated to fourth order operators. There are isoperimetric inequalities for these eigenvalues, which are analogous to the ones that we have been discussing for vibrating membranes. In this section we will briefly review three of these isoperimetric inequalities, arising in connection to i) the vibrations of the clamped plate, ii) the buckling problem, and iii) the vibrations of the free vibrating plate. In connection to fourth order operators, there is also a vast literature (in particular in the last few years) involving universal inequalities for the eigenvalues of these spectral problems. We will not discuss these universal inequalities here (see, however, the note iv) in the Bibliographical Remarks to this lecture).

5.1. The clamped plate.

Consider a bounded, smooth domain $\Omega \subset \mathbb{R}^2$. The eigenvalue problem that determines the eigenfrequencies of a clamped plate is given by

$$\Delta^2 u = \Gamma u, \qquad \text{in } \Omega \tag{5.1}$$

together with the *clamped* boundary conditions,

$$u = |\nabla u| = 0, \qquad \text{in } \partial\Omega \tag{5.2}$$

(the eigenfrequencies are proportional to the square root of the eigenvalues). The boundary value problem (5.1), (5.2) has a countable sequence of eigenvalues

$$0 < \Gamma_1(\Omega) \leq \Gamma_2(\Omega) \leq \dots$$

and $\Gamma_n(\Omega) \to \infty$ as $n \to \infty$. Because we are dealing with the operator Δ^2 (and not just the Laplacian) two *nasty* things may occur:

i) The principal eigenfunction u_1 of the boundary value problem defined by (5.1) and (5.2) is not necessarily of one sign (say positive).

ii) The lowest eigenvalue Γ_1 for the clamped plate may be degenerate.

There is an extensive literature on these two facts (see, in particular the Bibliographical Remarks i), ii) and iii) at the end of this Lecture). These two facts make it impossible to use the standard techniques that we have discussed in previous sections in connection with the proof of isoperimetric inequalities for the Laplacian and Schrödinger operators.

5.2. Rayleigh's conjecture for the clamped plate. In the first edition of his book *The Theory of Sound*, Lord Rayleigh conjectured [**65**] that

$$\Gamma_1(\Omega) \geq \Gamma_1(\Omega^*) \tag{5.3}$$

where $\Gamma_1(\Omega)$ denotes the first eigenvalue for the vibrations of a clamped plate, and Ω^* is a disk with the same area as Ω.

In 1950, G. Szegö [**70**] (see also [**71**], and the Erratum [**73**]) proved Rayleigh's conjecture for the clamped plate under the assumption that u_1 is of one sign (an assumption which we now know does not always hold).

A significant step towards the proof of Rayleigh's conjecture was done by G. Talenti [**74**] in 1981. Finally, in 1995, the conjecture was proven by N.S. Nadirashvili [**56**]. The analog of Rayleigh's conjecture for the *clamped plate* in three dimensions was proven by M.S. Ashbaugh and R.D. Benguria [**10**] (see also, [**14**]). The analog of Rayleigh's conjecture for the clamped plate in dimensions larger than 3 (i.e., $n \geq 4$) is still an open problem. Although not sharp, the best results to date for $n \geq 4$ have been obtained by M. S. Ashbaugh and R. S. Laugesen, [**15**]

The proof of Rayleigh's conjecture for the clamped plate, i.e., the proof of (5.3) is based on several steps. The first step, as usual, is the variational characterization of $\Gamma_1(\Omega)$. In the second step, taking into account that the ground state, say u, of (5.1) and (5.2) is not necessarily positive, one defines the sets $\Omega_+ = \{x \mid u_+ > 0\}$ and $\Omega_- = \{x \mid u_- > 0\}$, where $u_+ = \max(u, 0)$ and $u_- = \max(-u, 0)$ are the positive and negative parts of u, respectively. The third step is to consider the positive and negative parts of Δu, in other words we write $\Delta = (\Delta u)_+ - (\Delta u)_-$. Then, one considers the rearrangement,

$$g(s) = (\Delta u)_+^* (s) - (\Delta u)_-^* (\mu(\Omega) - s),$$

where $s = C_n |x|^n$, and C_n is the volume of the unit ball in n dimensions (here $n = 2$ or 3, as we mentioned above), and $\mu(\Omega)$ is the volume of Ω. The next step is to consider the solutions, v and w of some Dirichlet problem in the balls Ω_+^* and Ω_-^* respectively. Then, one uses a comparison theorem of Talenti [**74**], namely

$u_+^* \leq v$ in Ω_+^* and $u_-^* \leq w$ in Ω_-^*, respectively. The functions v and w can be found explicitly in terms of modified Bessel functions. The final step is to prove the necessary monotonicity properties of these functions v, and w. We refer the reader to [**10**], for details.

5.3. Rayleigh's conjecture for the buckling of a clamped plate. The *buckling* eigenvalues for the clamped plate for a domain Ω in $\mathbb{R}^2$, correspond to the eigenvalues of the following boundary value problem,

$$-\Delta^2 u = \Lambda \Delta u, \qquad \text{in } \Omega \tag{5.4}$$

together with the *clamped* boundary conditions,

$$u = |\nabla u| = 0, \qquad \text{in } \partial\Omega \tag{5.5}$$

The lowest eigenvalue, Λ_1 say, is related to the minimum uniform load applied in the boundary of the plate necessary to *buckle* it. There is a conjecture of L. Payne [**58**], regarding the isoperimetric behavior of Λ_1, namely,

$$\Lambda_1(\Omega) \geq \Lambda_1(\Omega^*).$$

To prove this conjecture is still an open problem. For details see, e.g., [**15, 58**], and the review articles [**2, 59**]. Recently, Antunes [**1**] has checked numerically Payne's conjecture for a large class of domains (mainly families of triangles or other simple polygons). Also, in [**1**] Antunes has studied the validity of other eigenvalue inequalities (mainly relating Λ_1 with different Dirichlet eigenvalues for the same domain Ω).

5.4. The fundamental tones of free plates. The analog of the Szegö–Weinberger problem for the "free vibrating plate" has been recently considered by L. Chasman, in her Ph.D. thesis [**28**] (see also [**29, 30**]). As discussed in [**28**], the *fundamental tone* for that problem, say $\omega_1(\Omega)$, corresponds to the first nontrivial eigenvalue for the boundary value problem

$$\Delta\Delta u - \tau \Delta u = \omega u,$$

in a bounded region Ω in the d dimensional Euclidean Space, with some natural boundary conditions, where τ is a positive constant. In fact $\omega_1(\Omega)$ is the fundamental tone of a free vibrating plate with tension (physically τ represents the ratio of the lateral tension to the flexural rigidity). In [**28, 29**], Chasman proves the following isoperimetric inequality,

$$\omega_1(\Omega) \leq \omega_1(\Omega^*),$$

where Ω^* is a ball of the same volume as Ω (here, equality is attained if and only if Ω is a ball). This result is the natural generalization of the corresponding Szegö–Weinberger result for the fundamental tone of the free vibrating membrane. Here, we will not discuss the exact boundary conditions appropriate for this problem (we refer the reader to [**28**] for details). In fact, the appropriate "free" boundary conditions are essentially obtained as some *transversality conditions* of the Direct Calculus of Variations for this problem. In [**28**], Chasman first derives the equivalent of the classical result of F. Pöckels for the membrane problem in this case, i.e., the existence of a discrete sequence of positive eigenvalues accumulating at infinity.

Then, she carefully discusses the *free* boundary conditions both, for smooth domains, and for domains with corners (in fact she considers as specific examples the rectangle and the ball, and, moreover the analog one dimensional problem). Then, she finds universal upper and lower bounds are derived for ω_1 in terms of τ. As for the proof of the isoperimetric inequality for ω_1, she uses a similar path as the one used in the proof of the Szegö–Weinberger inequality. Since in the present situation the operator is more involved, this task is not easy. She starts by carefully analyzing the necessary monotonicity properties of the Bessel (and modified) Bessel functions that naturally appear in the solution for the (d–degenerate) eigenfunctions corresponding to the fundamental tone, ω_1, of the ball. Then, the Weinberger strategy takes us through the standard road: use the variational characterization of $\omega_1(\Omega)$ in terms of d different trial functions (given as usual as a radial function g times the angular part x_i/r, for $i = 1, \ldots, d$) and averaging, to get a rotational invariant, variational upper bound on ω_1. As usual, a Brower's fixed point theorem is needed to insure the orthogonality of this trial functions to the constants. Then, one chooses the right expression for the variational function g guided by the expressions of the eigenfunctions associated to the fundamental tone of the ball. As in the proof of many of the previous isoperimetric inequalities, Chasman has to prove monotonicity properties of g (chosen as above) and of the expressions involving g and higher derivatives that appear in the bound obtained after the averaging procedure in the previous section. Finally, rearrangements and symmetrization arguments are used to conclude the proof of the isoperimetric result (see [**28**] for details).

5.5. Bibliographical Remarks. i) There is a recent, very interesting article on the sign of the principal eigenfunction of the *clamped plate* by Guido Sweers, *When is the first eigenfunction for the clamped plate equation of fixed sign?*, USA–Chile Workshop on Nonlinear Analysis, Electronic J. Diff. Eqns., Conf. 06, 2001, pp. 285–296, [available on the web at http://ejde.math.swt.edu/conf-proc/06/s3/sweers.pdf], where the author reviews the status of this problem and the literature up to 2001.

ii) For general properties of the spectral properties of fourth order operators the reader may want to see: Mark P. Owen, *Topics in the Spectral Theory of 4th order Elliptic Differential Equations*, Ph.D. Thesis, University of London, 1996. Available on the Web at http://www.ma.hw.ac.uk/∼mowen/research/thesis/thesis.ps .

iii) Concerning the two *problems* mentioned in the introduction of this Lecture, the reader may want to check the following references: R. J. Duffin, *On a question of Hadamard concerning super–biharmonic functions*, J. Math. Phys. **27**, 253–258 (1949); R. J. Duffin, D. H. Shaffer, *On the modes of vibration of a ring–shaped plate*, Bull. AMS **58**, 652 (1952); C.V. Coffman, R. J. Duffin, D. H. Shaffer, *The fundamental mode of vibration of a clamped annular plate is not of one sign*, in Constructive approaches to mathematical models (Proc. Conf. in honor of R. Duffin, Pittsburgh, PA, 1978), pp. 267–277, Academic Press, NY (1979); C.V. Coffman, R. J. Duffin, *On the fundamental eigenfunctions of a clamped punctured disk*, Adv. in Appl. Math. **13**, 142–151 (1992).

iv) Many authors, in recent years, have obtained universal inequalities among eigenvalues of fourth (and higher) order operators. In particular, see: J. Jost, X. Li–Jost, Q. Wang, and C. Xia, *Universal bounds for eigenvalues of the polyharmonic operators*, Trans. Amer. Math. Soc. **363**, 1821–1854 (2011), and references therein.

6. Appendix

6.1. The layer-cake formula.

THEOREM 6.1. *Let ν be a measure on the Borel sets of $\mathbb{R}^+$ such that $\Phi(t) := \nu([0,t))$ is finite for every $t > 0$. Let further (Ω, Σ, m) be a measure space and v a non-negative measurable function on Ω. Then*

$$\int_\Omega \Phi(v(x)) m(\,\mathrm{d}x) = \int_0^\infty m(\{x \in \Omega : v(x) > t\}) \nu(\,\mathrm{d}t). \tag{6.1}$$

In particular, if m is the Dirac measure at some point $x \in \mathbb{R}^n$ and $\nu(\,\mathrm{d}t) = \mathrm{d}t$ then (6.1) takes the form

$$v(x) = \int_0^\infty \chi_{\{y \in \Omega : v(y) > t\}}(x)\,\mathrm{d}t. \tag{6.2}$$

PROOF. Since $m(\{x \in \Omega : v(x) > t\}) = \int_\Omega \chi_{\{v>t\}}(x) m(\,\mathrm{d}x)$ we have, using Fubini's theorem,

$$\int_0^\infty m(\{x \in \Omega : v(x) > t\}) \nu(\,\mathrm{d}t) = \int_\Omega \left(\int_0^\infty \chi_{\{v>t\}}(x) \nu(\,\mathrm{d}t) \right) m(\,\mathrm{d}x).$$

Theorem 6.1 follows from observing that

$$\int_0^\infty \chi_{\{v>t\}}(x) \nu(\,\mathrm{d}t) = \int_0^{v(x)} \nu(\,\mathrm{d}t) = \Phi(v(x)).$$

□

6.2. A consequence of the Brouwer fixed-point theorem.

THEOREM 6.2 (Brouwer's fixed-point theorem). *Let $B \subset \mathbb{R}^n$ be the unit ball for $n \geq 0$. If $f : B \to B$ is continuous then f has a fixed point, i.e., there is some $x \in B$ such that $f(x) = x$.*

The proof appears in many books on topology, e.g., in [**55**]. Brouwer's theorem can be applied to establish the following result:

THEOREM 6.3. *Let $B \subset \mathbb{R}^n$ ($n \geq 2$) be a closed ball and $\vec{b}(\vec{r})$ a continuous map from B to $\mathbb{R}^n$. If $\vec{b}$ points strictly outwards at every point of ∂B, i.e., if $\vec{b}(\vec{r}) \cdot \vec{r} > 0$ for every $\vec{r} \in \partial B$, then $\vec{b}$ has a zero in B.*

PROOF. Without losing generality we can assume that B is the unit ball centered at the origin. Since $\vec{b}$ is continuous and $\vec{b}(\vec{r}) \cdot \vec{r} > 0$ on ∂B, there are two constants $0 < r_0 < 1$ and $p > 0$ such that $\vec{b}(\vec{r}) \cdot \vec{r} > p$ for every $\vec{r}$ with $r_0 < |\vec{r}\,| \leq 1$. We show that there is a constant $c > 0$ such that

$$|-c\vec{b}(\vec{r}\,) + \vec{r}| < 1$$

for all $\vec{r} \in B$: In fact, for all $\vec{r}$ with $|\vec{r}\,| \leq r_0$ the constant c can be any positive number below $(\sup_{\vec{r} \in B} |\vec{b}(\vec{r}\,)|)^{-1}(1 - r_0)$. The supremum exists because $|\vec{b}|$ is continuously defined on a compact set and therefore bounded. On the other hand, for all $\vec{r} \in B$ with $|\vec{r}| > r_0$ we have

$$\begin{aligned} |-c\vec{b}(\vec{r}\,) + \vec{r}\,|^2 &= c^2 |\vec{b}(\vec{r}\,)|^2 - 2c\vec{b}(\vec{r}\,) \cdot \vec{r} + |\vec{r}\,|^2 \\ &\leq c^2 \sup_{\vec{r} \in B} |\vec{b}\,|^2 - 2cp + 1, \end{aligned}$$

which is also smaller than one if one chooses $c > 0$ sufficiently small. Now set

$$\vec{g}(\vec{r}\,) = -c\vec{b}(\vec{r}\,) + \vec{r} \quad \text{for } \vec{r} \in B.$$

Then $\vec{g}$ is a continuous mapping from B to B and by Theorem 6.2 it has some fixed point $\vec{r}_1 \in B$, i.e., $\vec{g}(\vec{r}_1) = \vec{r}_1$ and $\vec{b}(\vec{r}_1) = 0$. □

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