A Variational Principle for the Asymptotic Speed of Fronts of the Density Dependent Diffusion-Reaction Equation

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Abstract

We show that the minimal speed for the existence of monotonic fronts of the equation $u_t = (u^m)_{xx} + f(u)$ with $f(0) = f(1) = 0$, $m > 1$ and $f > 0$ in $(0, 1)$, derives from a variational principle. The variational principle allows to calculate, in principle, the exact speed for general f. The case $m = 1$ when $f'(0) = 0$ is included as an extension of the results.

Several problems arising in population growth [1, 2], combustion theory [3, 4], chemical kinetics [5], and others [6], lead to an equation of the form

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{\jmath} = F(\rho),$$

where the source term $F(\rho)$ represents net growth and saturation processes. The flux $\vec{\jmath}$ is given by Fick's law

$$\vec{\jmath} = -D(\rho)\vec{\nabla}\rho,$$

where the diffusion coefficient $D(\rho)$ may depend on the density or in simple cases be taken as a constant. In one dimension this leads to the equation

$$\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial x}\left(D(\rho)\frac{\partial \rho}{\partial x}\right) + F(\rho). \tag{1a}$$

In what follows we shall assume that

$$F(\rho) > 0 \quad \text{in} \quad (0,1), \quad \text{and} \quad F(0) = F(1) = 0, \tag{1b}$$

restrictions which are satisfied by several models. When the diffusion coefficient is constant and the additional requirement $F'(0) > 0$ is satisfied, the asymptotic speed of propagation of localized small perturbations to the unstable state $u = 0$ is bounded below and in some cases coincides [7] with the value $c_L = 2\sqrt{F'(0)}$ which is obtained from considerations on the linearized equation [8]. However, when either $F'(0) = 0$ or $D(\rho)$ is not a constant, no hint for the speed of propagation of disturbances can be obtained from linear theory alone. A common choice for the diffusion coefficient is a power law, case with which we shall be concerned here. Therefore the equation that we study is

$$\frac{\partial \rho}{\partial t} = (\rho^m)_{xx} + F(\rho) \tag{2a}$$

with

$$F(0) = F(1) = 0, \quad \text{and} \quad F > 0 \quad \text{in} \quad (0,1). \tag{2b}$$

Aronson and Weinberger [7, 2] have shown that the asymptotic speed of propagation of disturbances from rest is the minimal speed $c^*(m)$ for which there exist monotonic travelling fronts $\rho(x,t) = q(x-ct)$ joining $q=1$ to $q=0$. The equation satisfied by the travelling fronts is

$$(q^m)_{zz} + cq_z + F(q) = 0 \tag{3a}$$

with

$$q(-\infty) = 1, \quad q > 0, \quad q' < 0 \quad \text{in} \quad (-\infty, \omega), \quad q(\omega) = 0 \tag{3b}$$

where $z = x - ct$. The wave of minimal speed is sharp, that is, $\omega < \infty$ when $m > 1$ [2].

An explicit solution is known [1, 2] for the case $F(q) = q(1-q)$ and $m = 2$, the waveform is given by

$$q(z) = \left[1 - \frac{1}{2}\mathrm{e}^{z/2}\right]_+$$

and it travels with speed $c^*(2) = 1$ (here $[x]_+ \equiv \max(x, 0)$). Recently the derivative dc/dm at $m = 2$ has been calculated by two different methods. Its value is -7/24 [9, 10]. Other exact solutions for different choices for m and F have been given in [11].

The purpose of this work is to give a variational characterization of the minimal speed $c^*(m)$ for Eq.(3) when $m > 1$, and as a byproduct for the case $m = 1$ when $F'(0) = 0$, both, cases for which no information is obtained from linear theory. The case $m = 1$ with $F'(0) > 0$ has been studied elsewhere [13]. Lower bounds have been obtained on the minimal speed $c^*(m)$ [12]; the present results allow its exact calculation for arbitrary f.

Since the selected speed corresponds to that of a decreasing monotonic front, we may consider the dependence of its derivative dq/dz on q. Calling $p(q) = -q^{m-1}dq/dz$, where the minus sign is included so that p is positive, we find that the monotonic fronts are solutions of

$$p\frac{dp}{dq} - \frac{c^*}{m}p + \frac{1}{m}q^{m-1}F(q) = 0 \tag{4a}$$

with

$$p(0) = p(1) = 0, \quad p > 0 \quad \text{in} \quad (0, 1). \tag{4b}$$

Although the wave of minimal speed is sharp and therefore $q'(0) < 0$, by its definition $p(0) = 0$ is true. We now show that the minimal speed $c^*(m)$ follows from a variational principle whose Euler equation is Eq.(4a).

Let g be a positive function such that $h = -g' > 0$. Multiplying Eq.(4a) by g/p and integrating we obtain after integration by parts,

$$\frac{c}{m} = \frac{\int_0^1 \left[\frac{1}{m} q^{m-1} F(q) \frac{g(q)}{p(q)} + h(q) p(q)\right] dq}{\int_0^1 g(q)\, dq}. \tag{5}$$

By Schwarz's inequality, since, q, F, g and h are positive we know

$$\frac{1}{m} q^{m-1} F \frac{g}{p} + hp \geq 2\sqrt{\frac{1}{m} q^{m-1} F g h} \tag{6}$$

and therefore, replacing in Eq.(5) we have

$$c \geq 2 \frac{\int_0^1 \sqrt{m q^{m-1} F g h} dq}{\int_0^1 g dq}. \tag{7}$$

This bound has been already given by us [12]. We now show that it is always possible to find a $g(q)$ such that the equality in Eq.(6) and therefore also in Eq.(7) holds. We do so by explicit construction of such a function g. The equality in Eq.(6) holds if

$$\frac{1}{m} q^{m-1} F \frac{g}{p} = hp \tag{8}$$

Let $v(q)$ be the positive solution of

$$\frac{v'}{v} = \frac{c}{mp} \tag{9a}$$

and choose

$$g = \frac{1}{v'}. \tag{9b}$$

We have then

$$\frac{v''}{v} = \frac{(v')^2}{v^2} - \frac{c}{mp^2} p' = \frac{c}{m^2 p^3} q^{m-1} F(q)$$

where we have used Eq.(9a) to eliminate v' and Eq.(4a) to eliminate p'. Therefore,

$$h = -g' = \frac{v''}{(v')^2} = \frac{1}{mp^2} q^{m-1} F g > 0 \tag{9c}$$

where we have made use of Eqs.(9a) and (9b). With this expression for h, we can see that Eq.(8) is satisfied. In addition we must check that g as we have defined it is such that its integral exists. In fact as it exists and moreover one can always normalize g so that $g(0) = 1$ and $g(1) = 0$. From the definition of g we obtain

$$g(q) = \frac{mp(q)}{c} \exp\left[-\int_{q_0}^{q} \frac{c}{mp}\, dq'\right]$$

where $0 < q_0 < 1$. Since $p(1) = 0$ and p is positive between 0 and 1 it follows that $g(1) = 0$. At zero no divergence occurs, as we now show. Call $\hat{c} = c/m$ and $f(q) = q^{m-1}F(q)/m$. Then Eq.(4a) reads

$$pp' - \hat{c}p + f = 0 \tag{10a}$$

with

$$f(0) = f(1) = 0 \quad \text{and} \quad f'(0) = 0. \tag{10b}$$

For this case Aronson and Weinberger [7] have shown that $p(q)$ approaches the fixed point $q = 0$ as $p = \hat{c}q = cq/m$. Then, near 0, $v'/v \approx 1/q$ or $v \approx q$ and from its definition $g(0) = 1$. Then the integral of g exists. We have shown then

$$c^*(m) = \max 2 \frac{\int_0^1 \sqrt{mq^{m-1}Fgh}\, dq}{\int_0^1 g\, dq}. \tag{11}$$

where the maximum is taken over all functions g such that

$$g(0) = 1, \quad g(1) = 0 \quad \text{and} \quad h = -g' > 0.$$

It is perhaps of some interest to verify explicitly that the Euler equation for the maximizing g is indeed Eq.(4a). Let us study the maximization of the functional

$$J_m(g) = 2\int_0^1 \sqrt{mq^{m-1}Fgh}\, dq$$

where $h = -g' > 0$ subject to

$$\int_0^1 g(q)\, dq = 1.$$

The Euler equation for this problem is

$$\lambda + \sqrt{\frac{mq^{m-1}Fh}{g}} + \frac{d}{dq}\left(\sqrt{\frac{mq^{m-1}Fg}{h}}\right) = 0$$

where λ is the Lagrange multiplier. Using the expression given in Eq.(9c) for h we see that this is exactly Eq.(4a) with the Lagrange multiplier $\lambda = -c$.

As an application we shall consider the case $F(q) = q(1-q)$ and $m = 2$ for which the exact solution is known. Take as the trial function $g(q) = (1-q)^2$. Then we obtain

$$c \geq 4\frac{\int_0^1 q(1-q)^2 dq}{\int_0^1 (1-q)^2 dq} = 1$$

the exact value, which shows that this is the function g for which the maximum is attained. In addition, due to the existence of the variational principle we may use the Feynman-Hellman formula to calculate the dependence of $c(m)$ on parameters of F. We illustrate this by applying it to the calculation of dc/dm at $m = 2$. Taking the derivative of Eq.(10) with respect to m we obtain

$$\frac{dc}{dm} = \frac{1}{\int_0^1 gdq} \int_0^1 \frac{ghF}{\sqrt{mFq^{m-1}gh}} [q^{m-1}(1 + m \log q)] dq.$$

Evaluating at $m = 2$, with $g(q) = (1 - q)^2$ we obtain

$$\frac{dc}{dm}(2) = 3 \int_0^1 q(1 - q)^2 (1 + 2 \log q) dq = -\frac{7}{24}$$

the value previously obtained by other methods.

A fast estimation of the speed for other values of m can be obtained with simple trial functions. In Fig. 1 we show lower bounds for $F = q(1-q)$ using as trial functions $g_1 = (1-q)^2$ and $g_2 = (1 - q)$. With the first trial function we have the exact value at $m = 2$. The dotted line is the line of slope -7/24 that coincides with the tangent at $m = 2$. For larger m a better estimate is obtained using g_2. The dashed line is the curve $\sqrt{2/m}$ which has been suggested by Newman [1] as the best fit to his numerical results. With better choice of trial functions the exact value can be approached arbitrarily close.

Finally we observe that the case $m = 1$ when $F'(0) = 0$ follows directly here. Repeating the procedure starting now from equation (10), one obtains,

$$c = \max 2 \, \frac{\int_0^1 \sqrt{Fgh} \, dq}{\int_0^1 g \, dq}$$

where the maximum is taken over all functions g such that

$$g(0) = 1, \quad g(1) = 0 \quad \text{and} \quad h = -g' > 0.$$

To show this we have used $v'/v = c/p$ and $g = 1/v'$ and the asymptotic behavior described above.

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