The Path Integral for Relativistic Worldlines

B. Koch

with E. Muñoz and I. Reyes

based on:

Phys.Rev. D96 (2017) no.8, 085011

and arXiv:1706.05388.

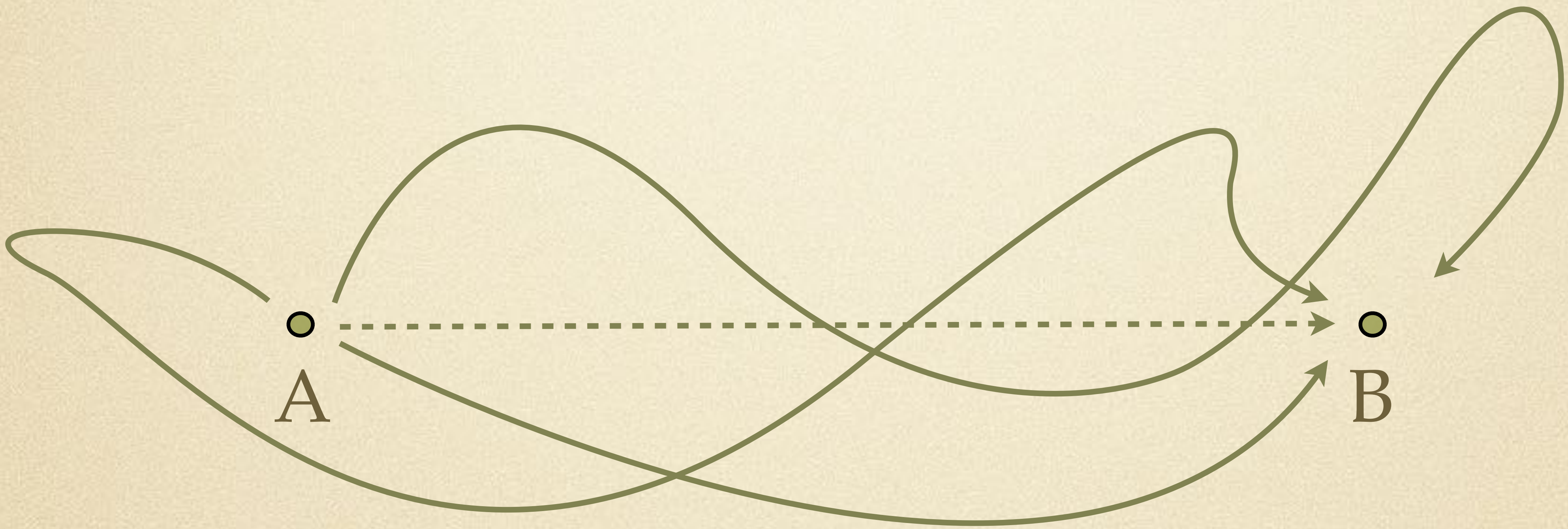
Afunalhue, La parte y el todo, 2018

Content

- PI of the RPP, Status
- Local Symmetry: Velocity Rotations
- Constructing the PI of the RPP
- Conclusion

The Path Integral

Propagator



$$\langle A|B\rangle \sim \int \mathcal{D}x \, e^{-S_{A,B}}$$

(after Wick rotation)

Non Relativistic Propagator

$$\langle A|B\rangle \sim \int \mathcal{D}x \, e^{-S_{A,B}}$$

$$S = \int_{t_A,A}^{t_B,B} dt \frac{m}{2}(\dot{\vec{x}})^2 = \sum_i \Delta \frac{m}{2} \frac{(\vec{x}_{i+1} - \vec{x}_i)^2}{\Delta^2}$$

Two Nice Features

- Quadratic in field variable
- Can be connected (Chapman, Kolmogorov)

Non Relativistic Propagator

$$\langle A|B\rangle = \Pi_i \left\{ \mathcal{N}_i \cdot \int d^d x_i \; e^{-\left(\sum_i \Delta \frac{m}{2} \frac{(\vec{x}_{i+1}-\vec{x}_i)^2}{\Delta^2}\right)} \right\}$$

- Feature 1: Quadratic

Simple Gaussian integrals

- Feature 2: Chapman Kolmogorov

$$\langle A|C\rangle = \int d^d x_b \langle A|B\rangle\langle B|C\rangle$$

Probability conservation

&

stepwise construction of PI

Relativistic Propagator

$$\langle A|B\rangle \sim \int \mathcal{D}x \; e^{-S_{A,B}}$$

$$S = \int d\lambda \sqrt{\left(\frac{dx^\mu}{d\lambda}\right)^2}$$

Why intersting, why here

- Unsolved NON PERTURBATIVE problem
- Simplest system with general covariance

Two Problems!

Relativistic Propagator

$$\langle A|B\rangle =^? \Pi_i \left\{ \mathcal{N}_i \cdot \int d^d x_i \, e^{-\left(\sum_i \Delta \sqrt{\frac{(x^\mu_{i+1} - x^\mu_i)^2}{\Delta^2}}\right)} \right\}$$

- Problem 1: Square root

Horrible integrals & still wrong result

- Problem 2: No Chapman Kolmogorov

$$\langle A|C\rangle \neq \int d^d x_B \langle A|B\rangle\langle B|C\rangle$$

No probability conservation

&

no stepwise construction of PI

Relativistic Propagator

„Solutions“ in the Literature

- Hamiltonian formalism (classically equivalent)[*1]

  Evades P1

- Redefine probability [*2]

  Solves P1 & P2, but high price

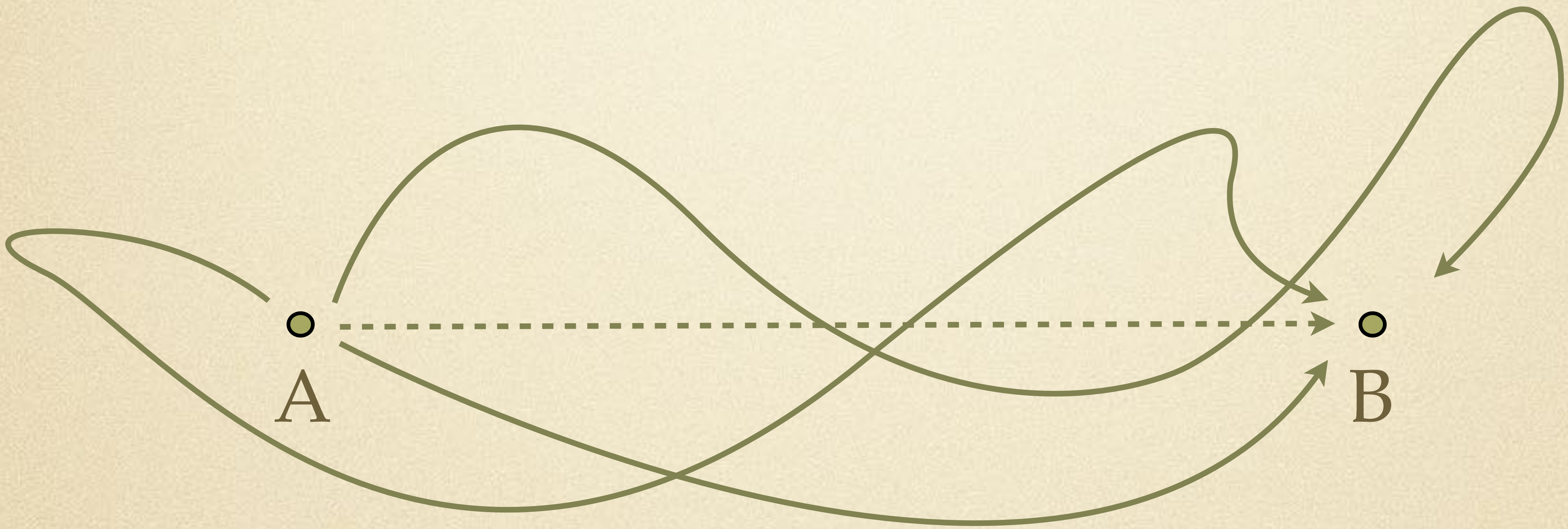
- Restrict PI to spheres, or other approx.[*3]

  Interesting, neither P1 nor P2 are solved

- Ignore the problems and do QFT right away

  Thats what we mostly do …

Relativistic PI: our Proposal



$$\langle A|B\rangle \sim \int \mathcal{D}x\, e^{-S_{A,B}}$$

Relativistic Propagator

$$\langle A|B\rangle \sim \int \mathcal{D}x \; e^{-S_{A,B}}$$

with $$S = \int d\lambda \sqrt{\left(\frac{dx^\mu}{d\lambda}\right)^2}$$

can be done, if one considers

Three issues:

- Issue 1: Local reparametrizations (known)
- Issue 2: Local velocity rotations (trivial?)
- Issue 3: Measure without anomalies

Relativistic Propagator

$$\langle A|B\rangle \sim \int \mathcal{D}x \; e^{-S_{A,B}}$$

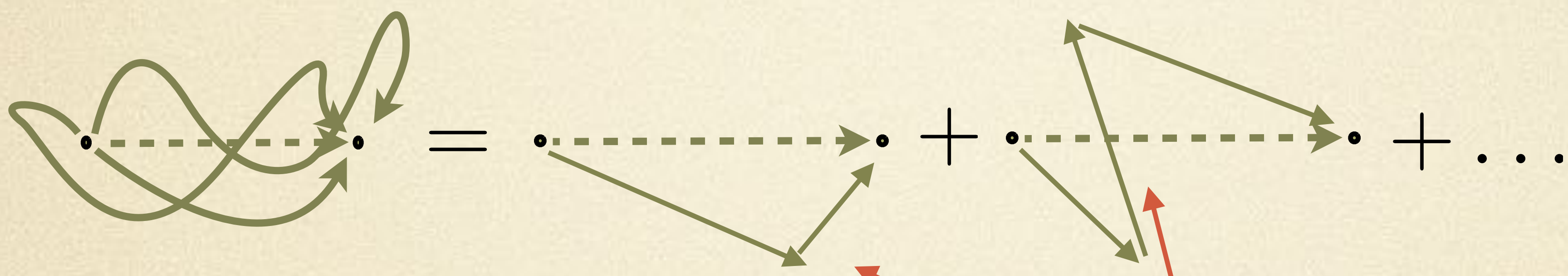
$$S = \int d\lambda \sqrt{\left(\frac{dx^\mu}{d\lambda}\right)^2}$$

using I1,I2,I3 works

Functional *0
Fadeev Popov method

Geometric *00
stepwise proof

Stepwise proof



$$\langle A|B\rangle = \langle A|B\rangle_1 + \langle A|B\rangle_2 + \dots$$

Don't count again!

Strategy

- Clarify geometry meaning of I1,2,3
- Calculate $\langle A|B\rangle_1$ using I2,3
- Show with I1,2 $\langle A|B\rangle_1$ contains $\langle A|B\rangle_2 \dots$

- Issue 1: Local reparametrizations (known)

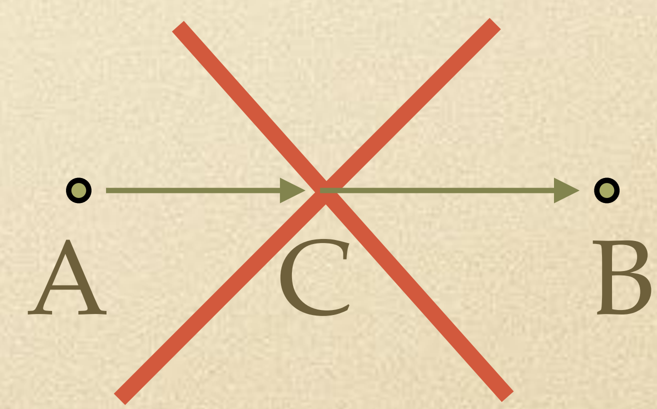
$$S = \int d\lambda \sqrt{\left(\frac{dx^\mu}{d\lambda}\right)^2}$$

Invariant under $\lambda \to \lambda'(\lambda)$

We fix proper time such that

$\tau = \tau(\lambda)$ with $\left(\frac{dx^\mu}{d\tau}\right)^2 = 1$

Geometric over counting:



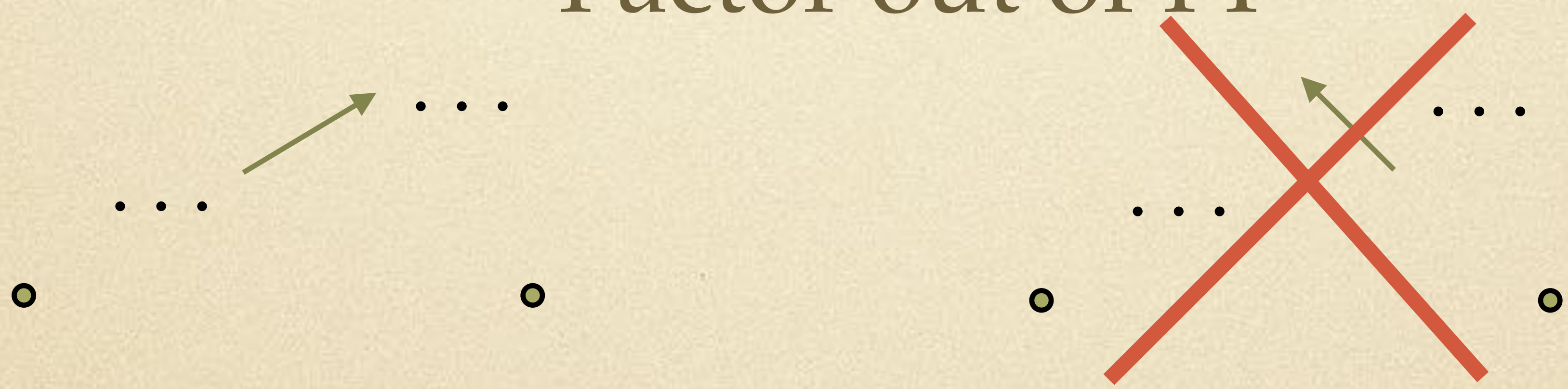
- Issue 2: Local velocity rotations (trivial?)

$$S = \int d\lambda \sqrt{\left(\frac{dx^\mu}{d\lambda}\right)^2} = \int d\lambda \sqrt{v^\mu v_\mu}$$

Invariant under $v^\mu \to v'^\mu = \Lambda^\mu{}_\nu(\lambda) v^\nu$

with $v'^\mu v'_\mu = v^\mu v_\mu$

Factor out of PI



if $S = S'$ and $\mathcal{L} = \mathcal{L}'$!

- Issue 3: Measure without anomalies

When performing transformation

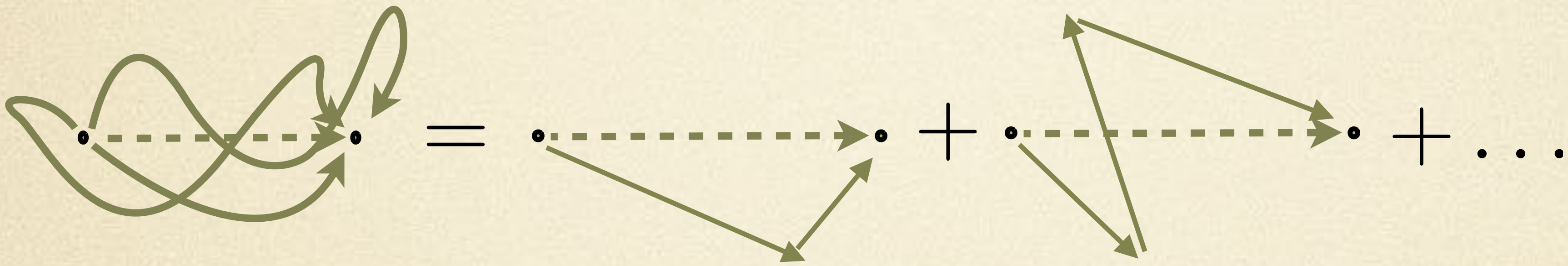
$$v^\mu \to v'^\mu = \Lambda^\mu{}_\nu(\lambda) v^\nu$$

define right measure invariant under this symmetry:

$$\mathcal{D}x \to \mathcal{D}x' = \mathcal{D}x$$

Geometric example for two step propagator

Stepwise proof



$$\langle A|B\rangle = \langle A|B\rangle_1 + \langle A|B\rangle_2 + \ldots$$

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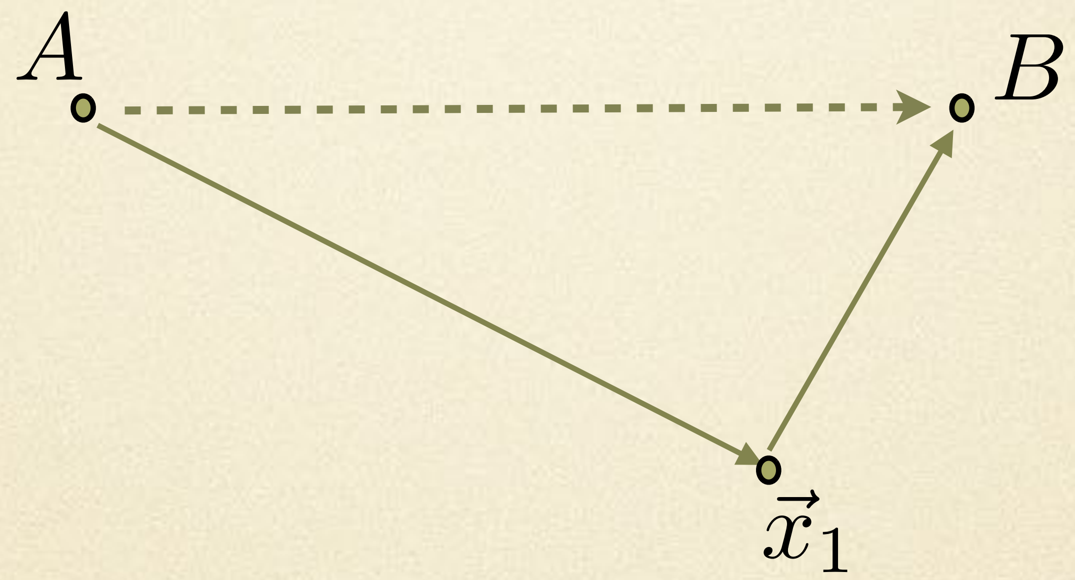
- Calculate $\langle A|B\rangle_1$ using I2,3



$$\langle A|B\rangle_1 = N \int d^2x_1 \, \Delta_1 \, e^{-(S_{A,\vec{x}_1}+S_{\vec{x}_1,B})}$$

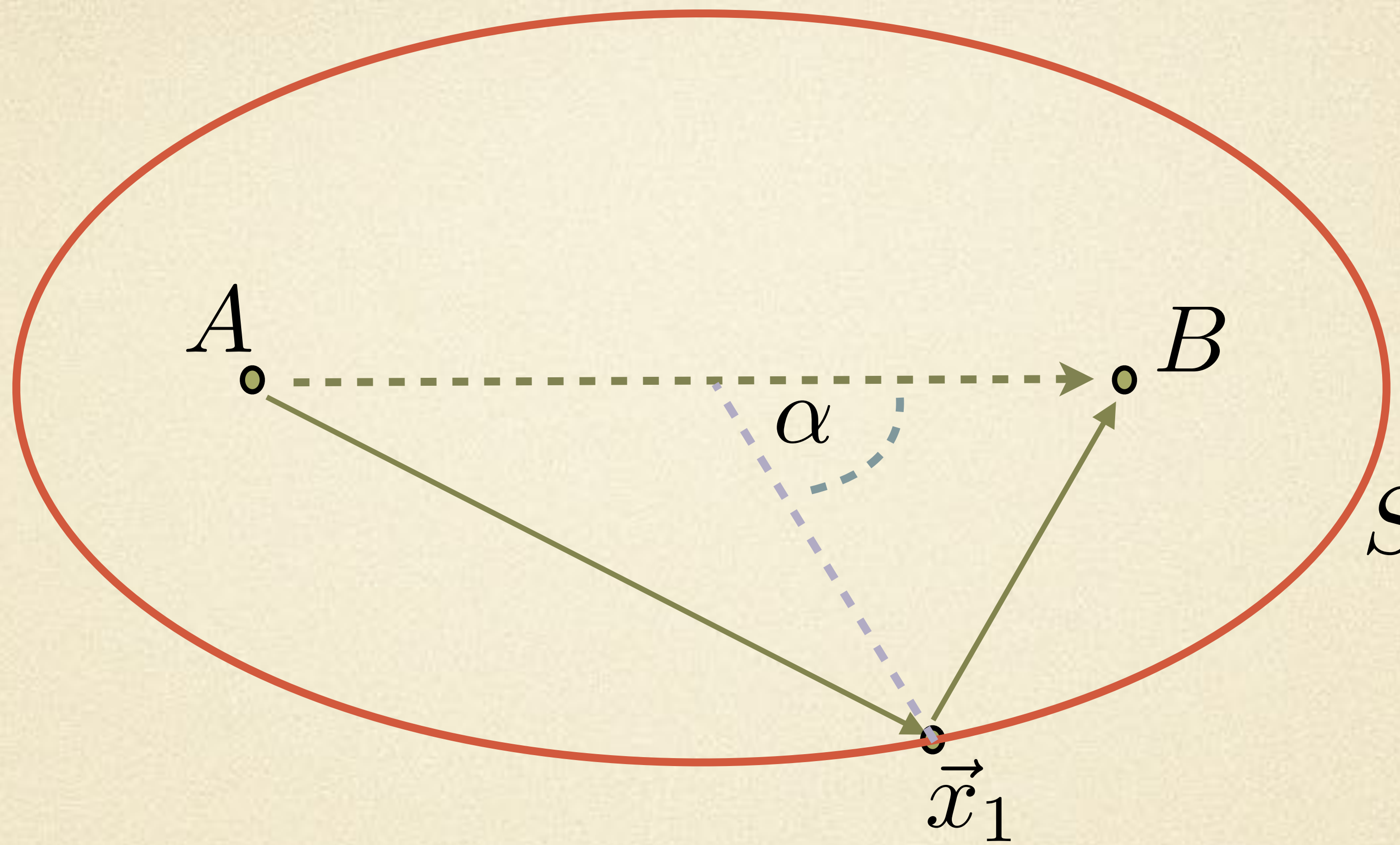
Change of integration coordinates

$$(x_1, y_1) \to (S, \alpha)$$

$$x_1 = \frac{S}{2M}\cos(\alpha) \qquad y_1 = \frac{x_f}{2}\sqrt{\left(\frac{S}{x_f M}\right)^2 - 1} \cdot \sin(\alpha)$$

$$x_f = |\vec{B} - \vec{A}|$$

- Calculate $\langle A|B\rangle_1$ using I2,3



$$\langle 0, \vec{x}_f\rangle_1 = \mathcal{N}_{2,1}(t_{i,f}) \int_{x_f M}^{\infty} dS \int_0^{2\pi} d\alpha \cdot \Delta_1 \frac{2\,(S/M)^2 - x_f^2(1+\cos(2\alpha))}{8\sqrt{(S)^2 - (x_f M)^2}} \exp\left[-S\right]$$

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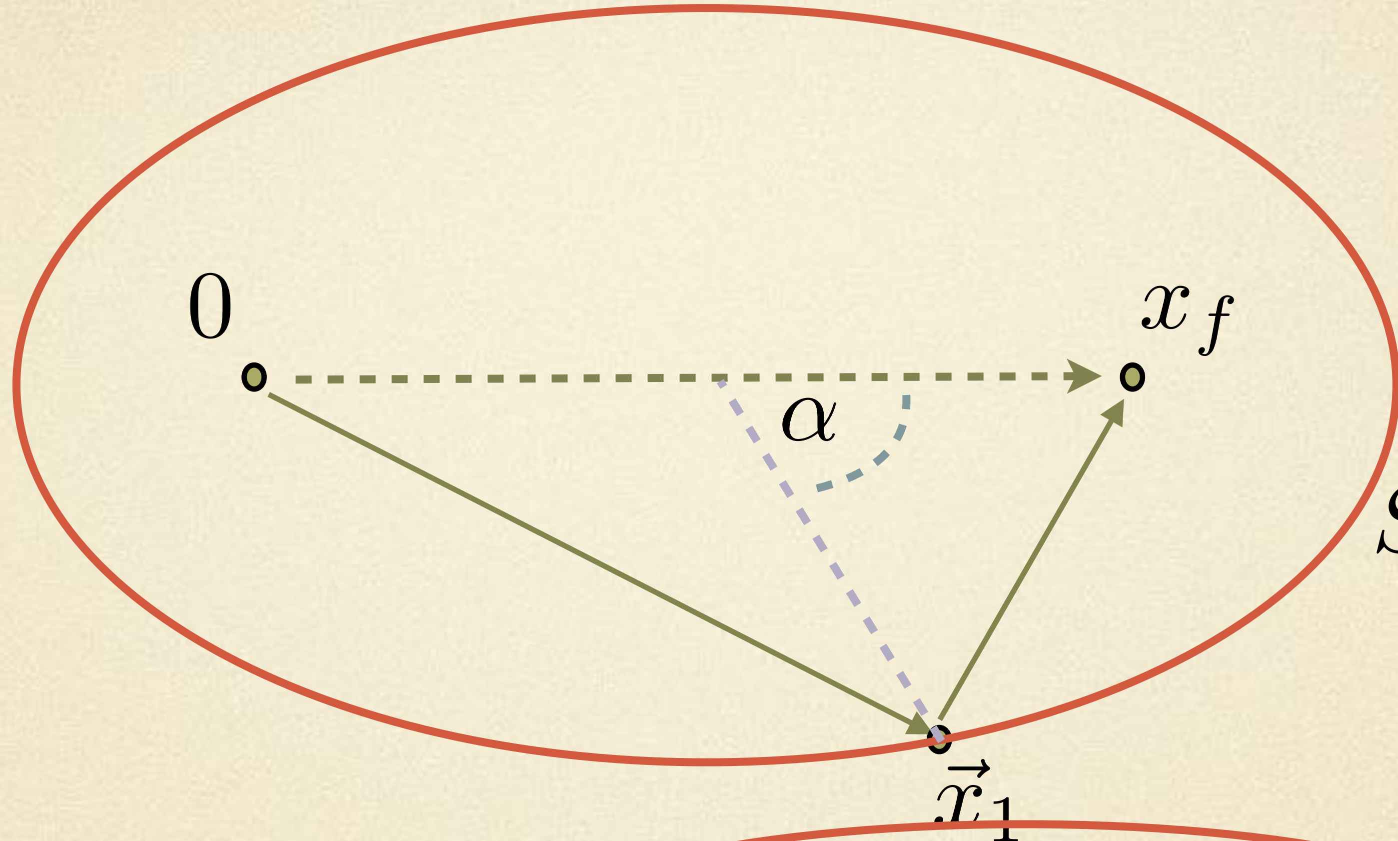

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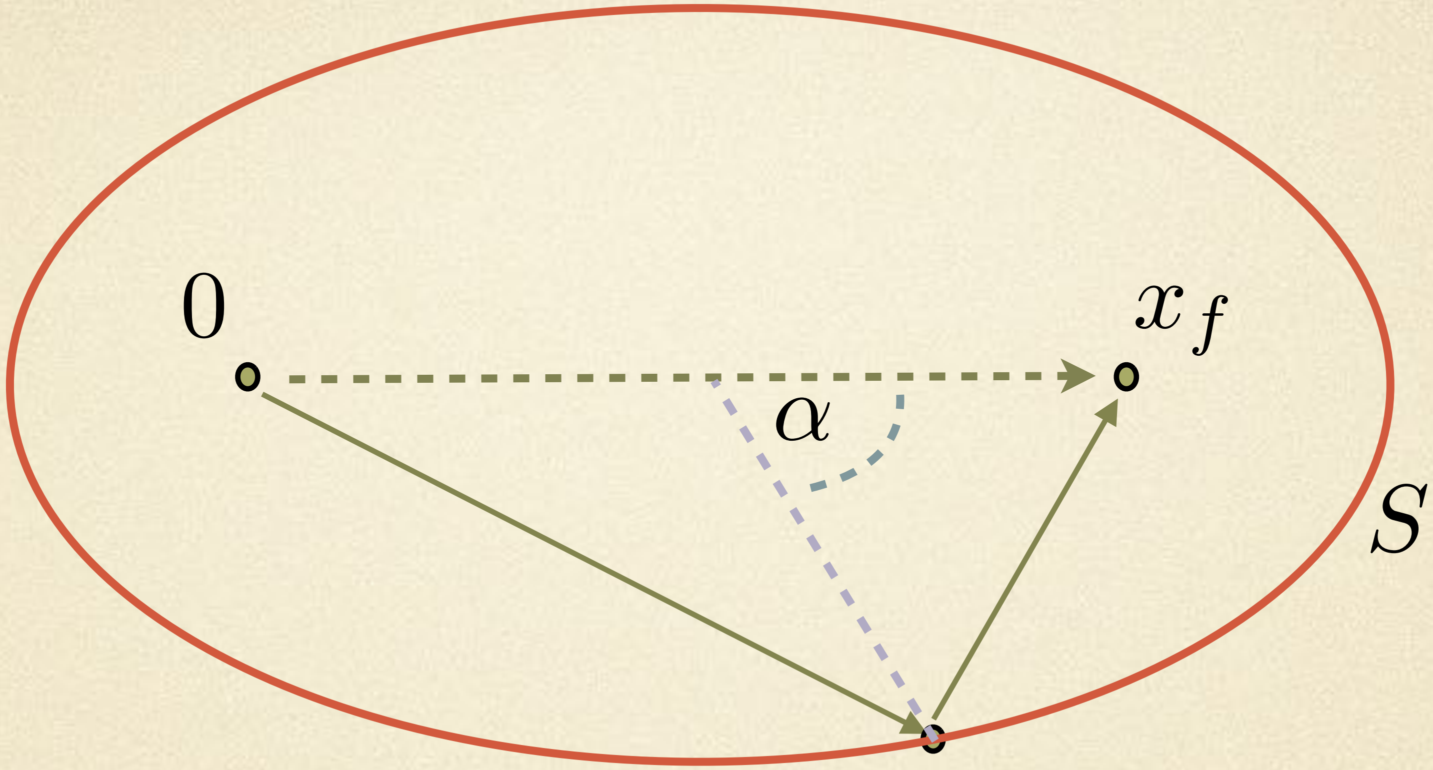
- For $\alpha \rightarrow \alpha'$ one sees $S = S'$ and $\mathcal{L} = \mathcal{L}'$

I2 velocity rotation!

- Naive, measure depends on α : anomaly I3

$\Delta_1^{-1} \equiv |\vec{x}_1 - 0| \cdot |\vec{x}_f - \vec{x}_1|$ cancels anomaly

- Calculate $\langle A|B\rangle_1$ using I2,3

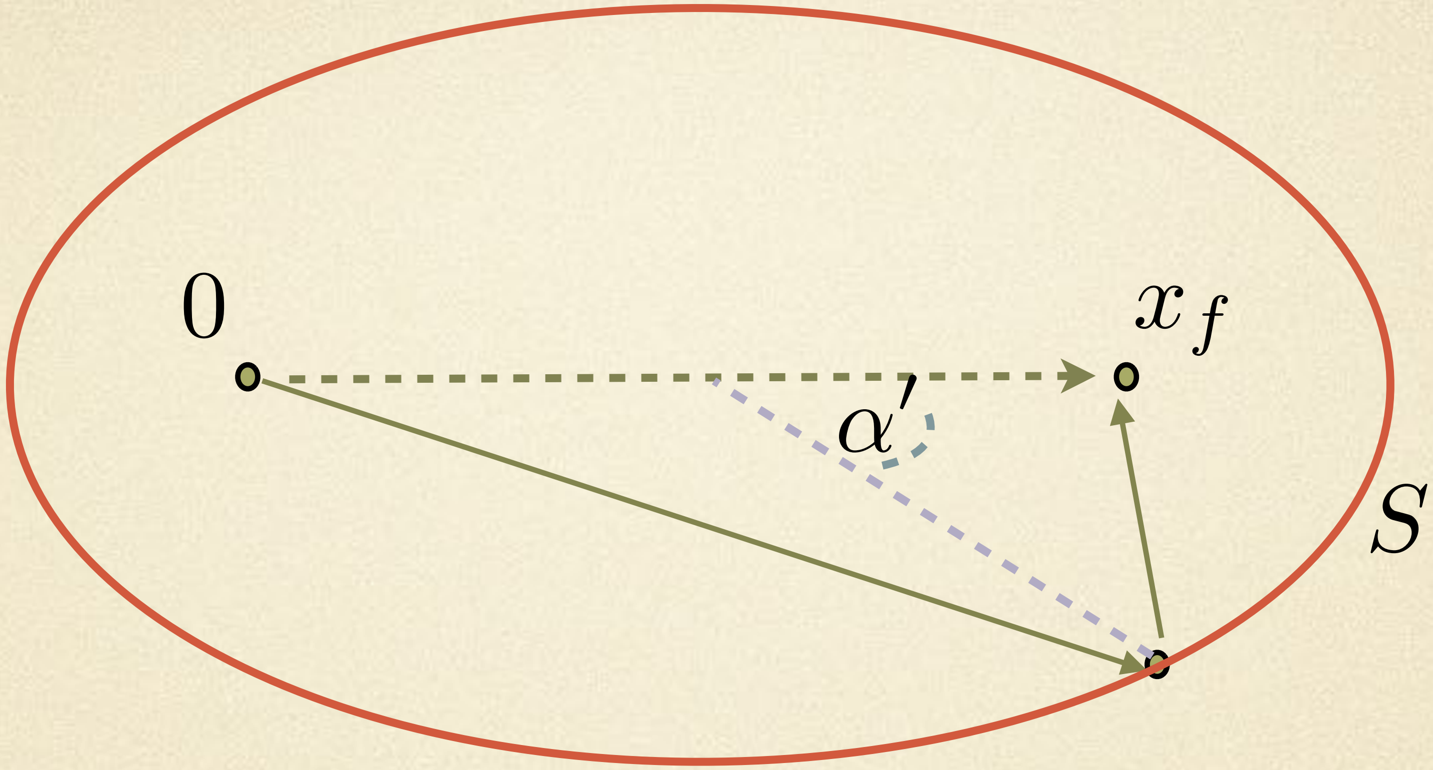


$$\langle 0,\vec{x}_f\rangle_1 = \mathcal{N}_{2,1}\left(\int_0^{2\pi} d\alpha\right)\cdot\int_{x_f M}^{\infty} dS\,\frac{1}{\sqrt{(S)^2-(x_f M)^2}}\exp\left[-S\right]$$

- For $\alpha \to \alpha'$ nothing changes

$$\langle 0,\vec{x}_f\rangle_1 = \mathcal{N}\cdot K_0(x_f M)$$

- Calculate $\langle A|B\rangle_1$ using I2,3



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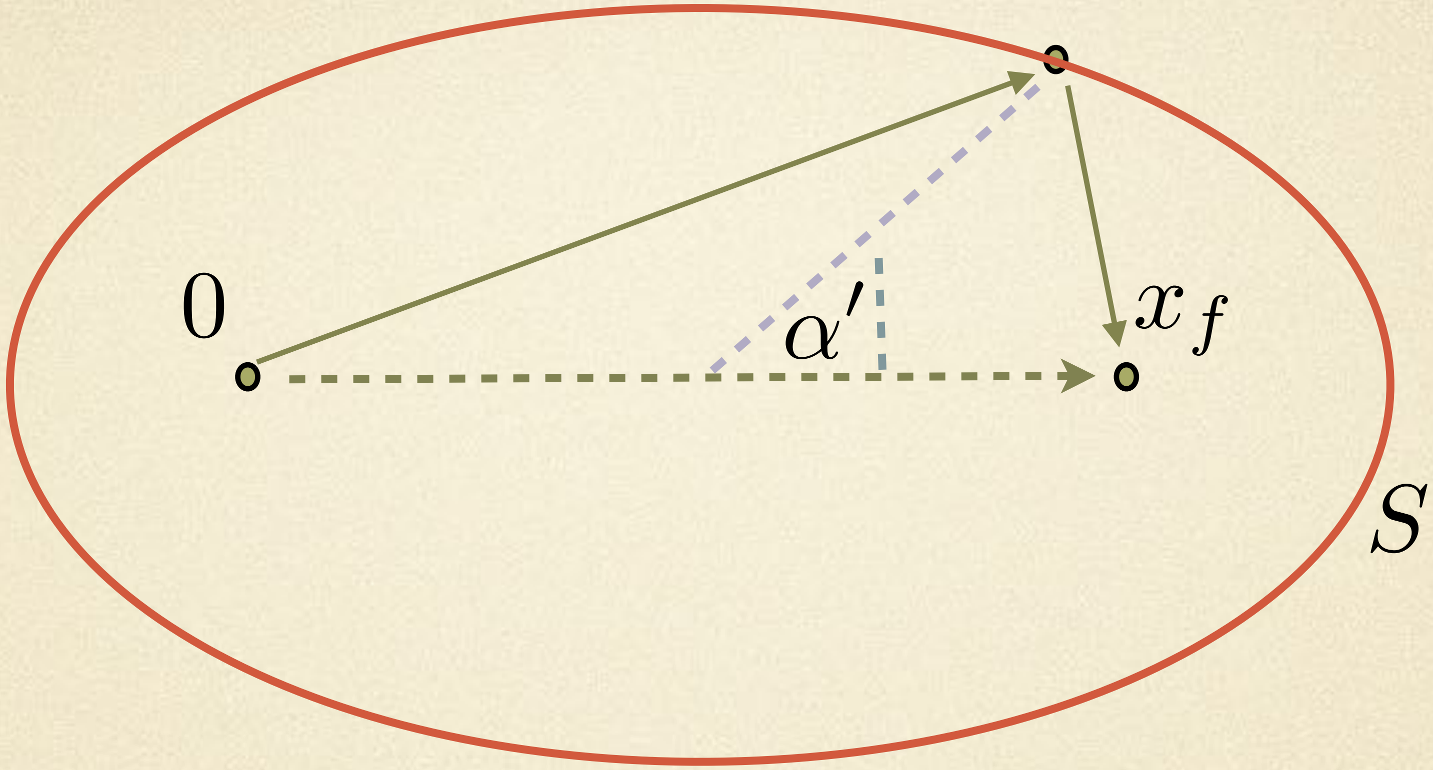

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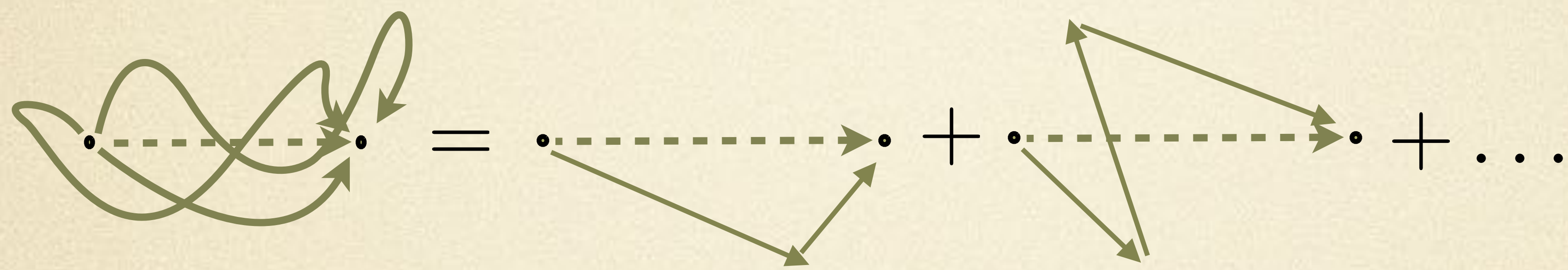
$$\boxed{\langle 0, \vec{x}_f \rangle_1 = \mathcal{N} \cdot K_0(x_f M)}$$

Stepwise proof



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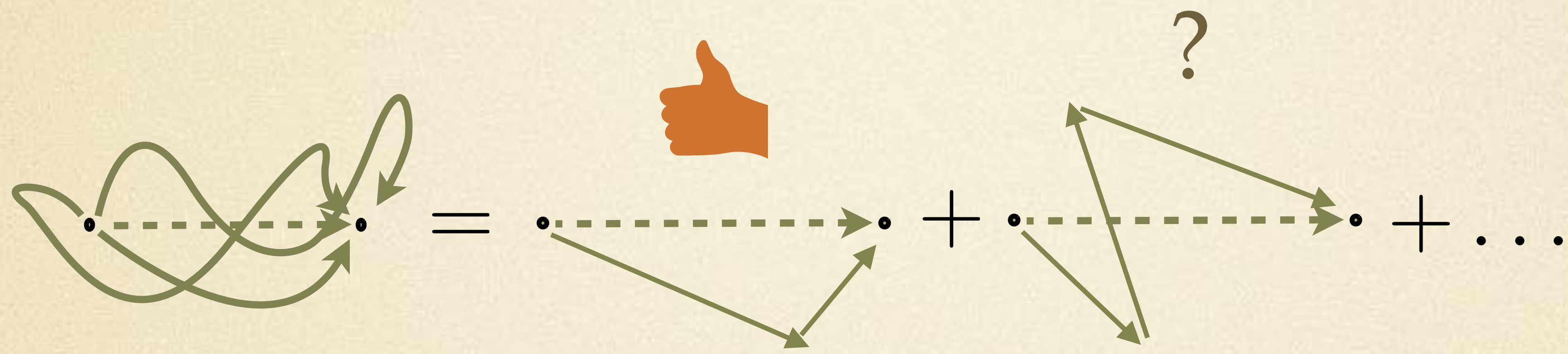

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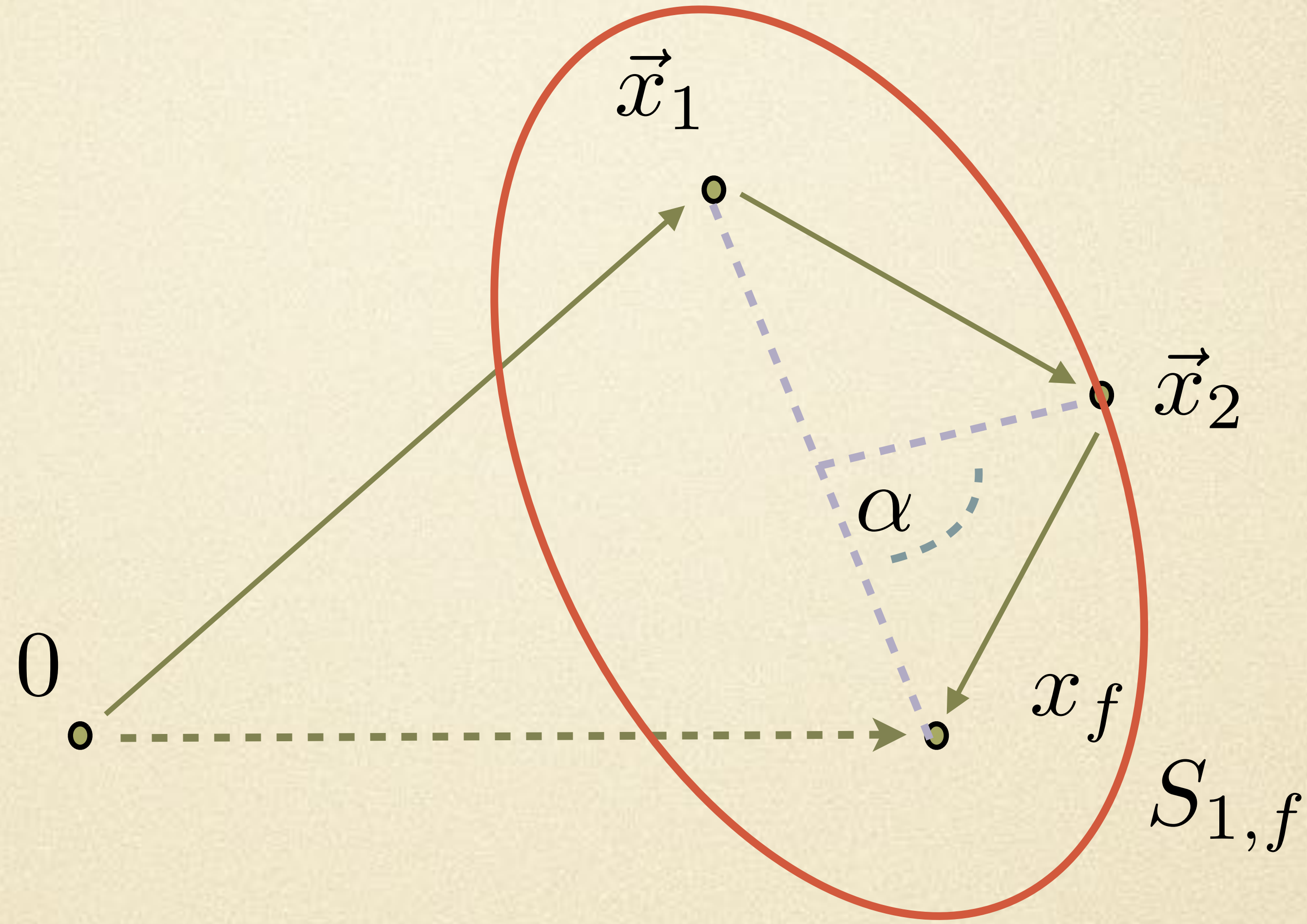
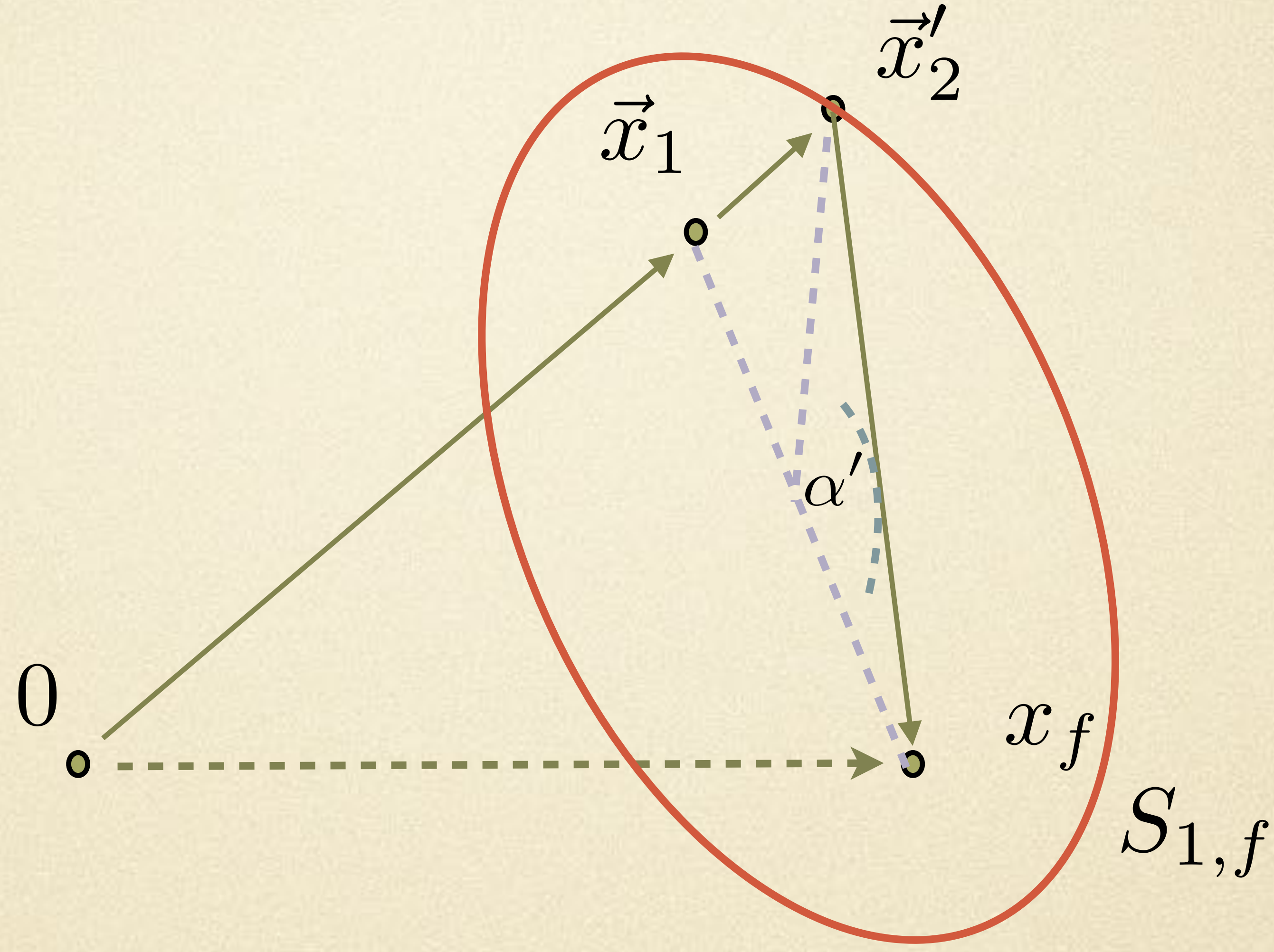
$\langle 0|x_f\rangle_2$:



- For $\alpha \to \alpha'$ nothing changes $\vec{x}_2 \to \vec{x}_2'$

- Show with I1,2 $\langle A|B\rangle_1$ contains $\langle A|B\rangle_2 \ldots$

$\langle 0|x_f\rangle_2$:



- For $\alpha \to \alpha'$ nothing changes (I2) $\vec{x}_2 \to \vec{x}'_2$
- $0 \to \vec{x}_1 \to \vec{x}'_2$ straight line: I1 reparametrizations

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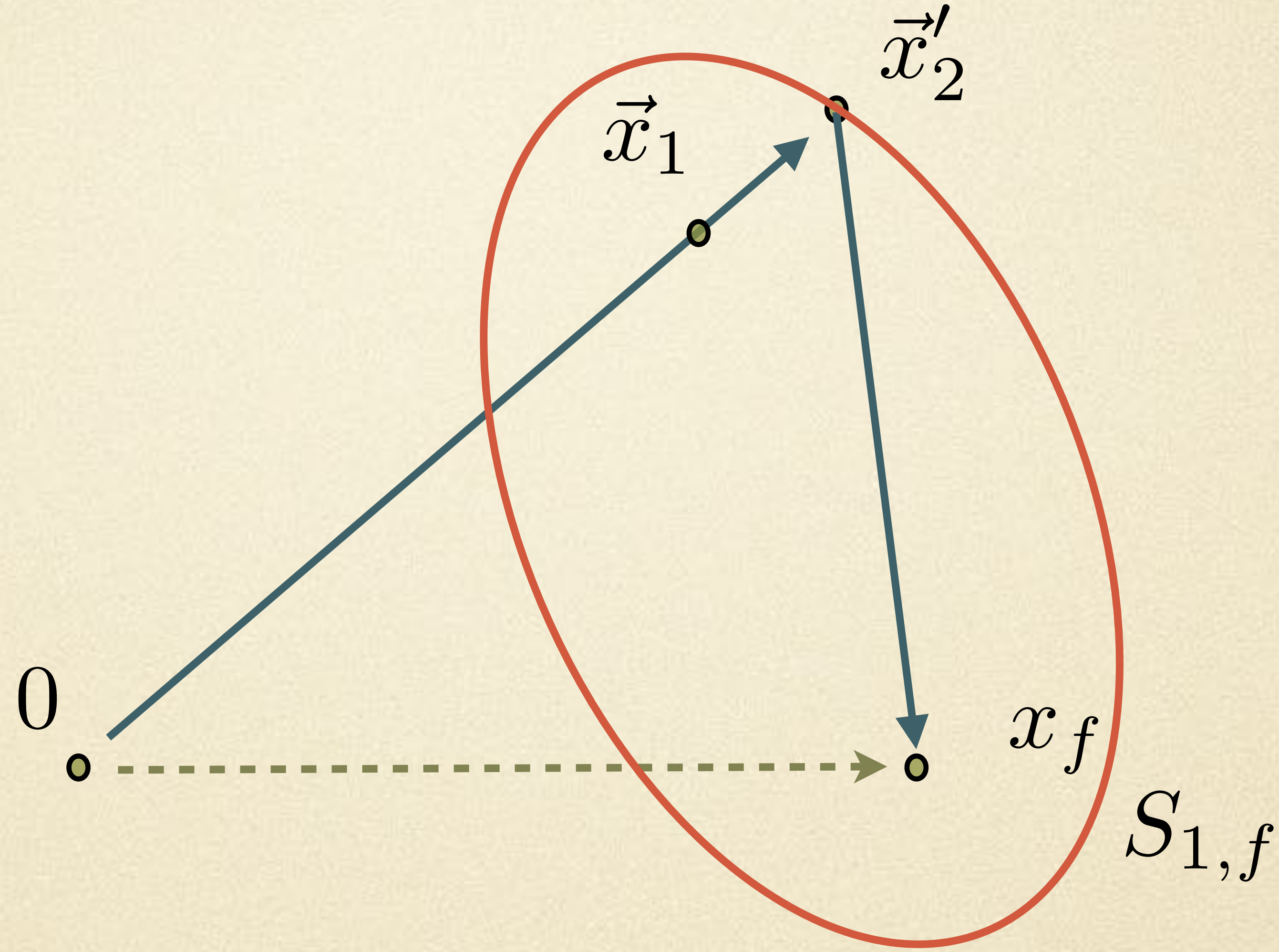
$\langle 0|x_f\rangle_2$:



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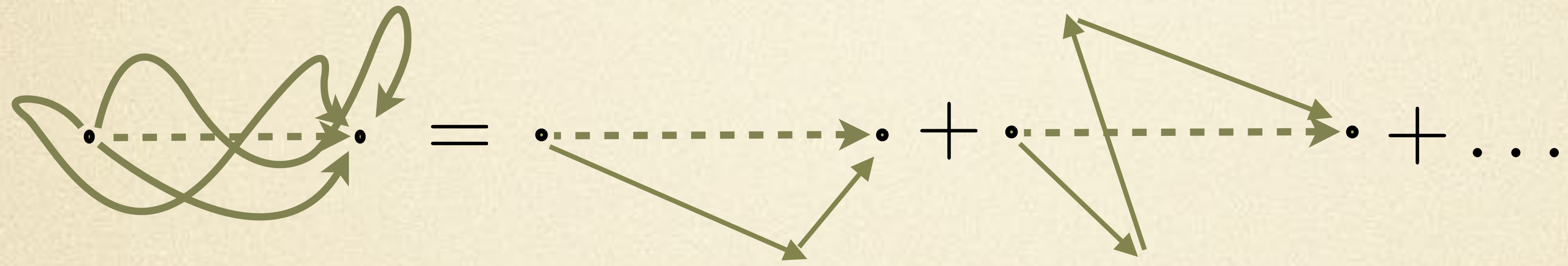
$$\langle 0|x_f\rangle_2 \subset_{1,2} \langle 0|x_f\rangle_1$$

Stepwise proof



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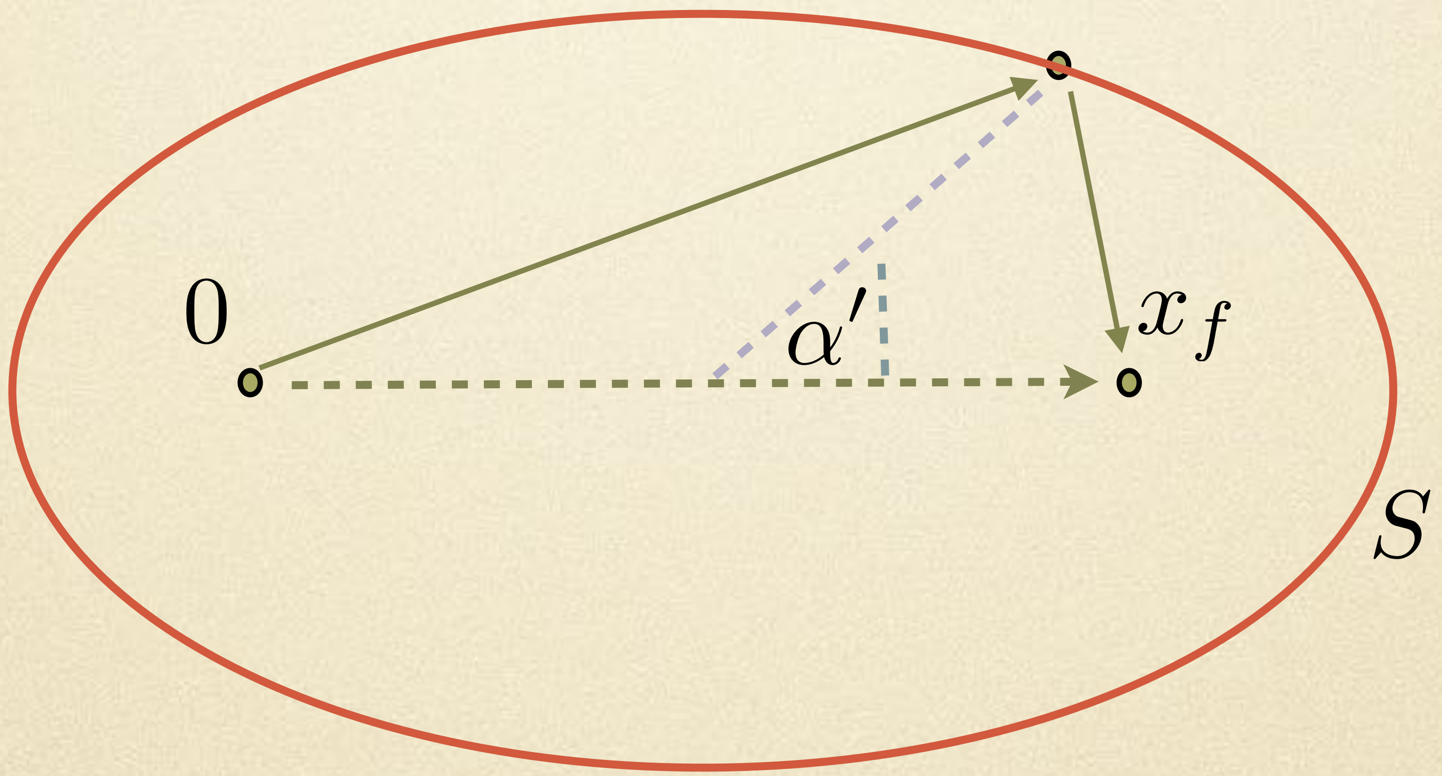
$$\langle 0|x_f\rangle = \mathcal{N}\cdot\langle 0|x_f\rangle_1 = \mathcal{N}\cdot K_0(Mx_f)$$

Concluding Comments

- Generalization to D dimensions
- PI of RPP action can be done, considering I1,I2,I3
- Chapman Kolmogorov becomes „trivial“
- Future work …

Thank You



Literature

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