Scale Setting for Self-consistent Quantum Backgrounds

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Outline

I Introduction Γ_k

II Improving solutions

III Self-consistent scale setting

IV Example: “inoffensive”

V Examples: “offensive”

VI Summary and outlook

Based on arXiv:1409.4443 (accepted by PRD) and arXiv:1501.00904

I
Introduction

I Introduction Γ_k

Ideal case

$$\Gamma_k(\phi) \tag{1}$$

In "Pleasantville"

Generating functional

$$Z[J] = \int D\varphi \exp\left(-i\int dx(L(\varphi) - \varphi J)\right) \tag{2}$$

Connected diagrams

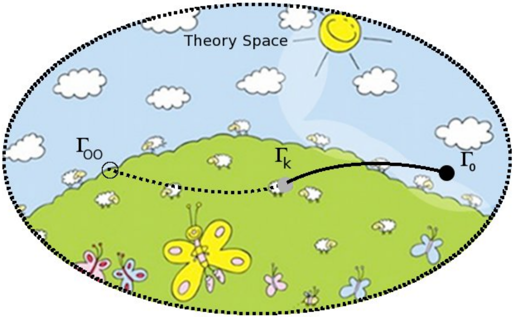
$$W[J] = \ln Z[J] \tag{3}$$

Effective action (1PI)

$$\Gamma[\phi] = sup_J(\int J\phi - W[J]) \tag{4}$$

What happens if one integrates out only certain Momentum shell ?

$$k \to k + \delta k?$$
$$\Rightarrow \text{RG-Flow}$$



$$\partial_k \Gamma_k = \ldots \tag{5}$$

Solve:
Running couplings λ_k and effective action

$$\Gamma_k = \Gamma_k(\lambda_k, \phi) \tag{6}$$

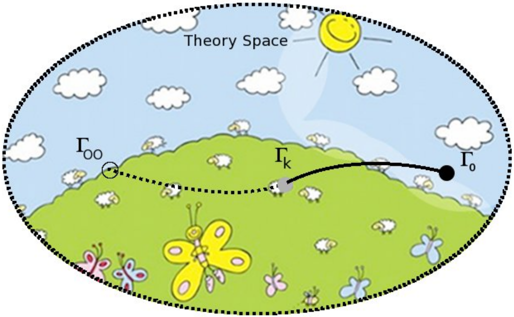
I Introduction Γ_k

Wilsons flow dangers

gauge redundance

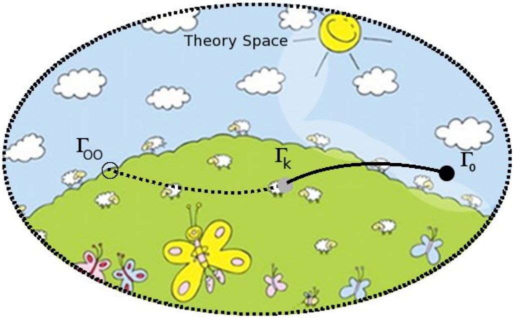
$$\phi \to e^{i\alpha}\phi$$

Fadeev Popov



Infrared divergencies

$$k \to 0,\ \Gamma_0 =?$$



Infrared divergencies

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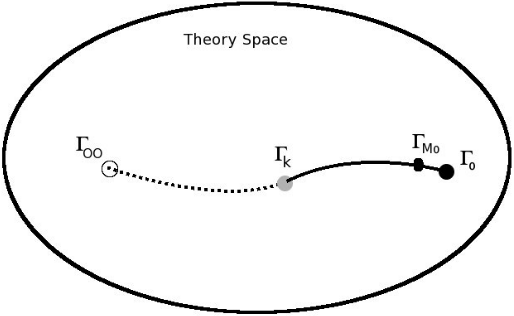
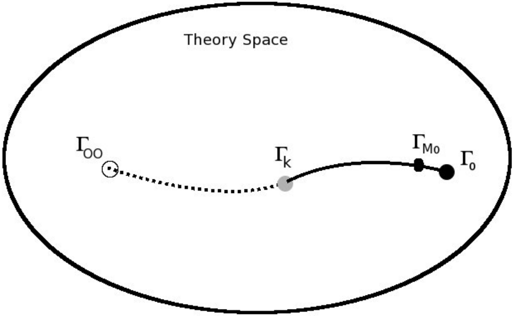


Ultra violet divergencies

$$k \to \infty, \Gamma_\infty = ?$$



Ultra violet divergencies

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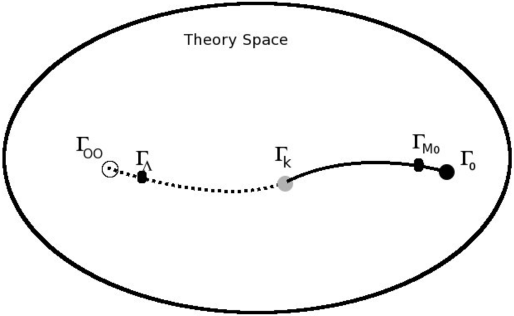


I Introduction Γ_k

Wilsons flow dangers

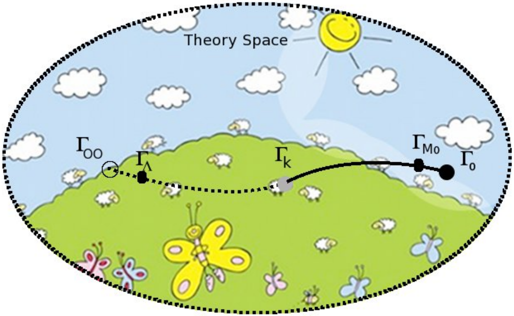
Renormalization:

Live with the problems

Absorb Infinities $\sim \Lambda$ in definition of couplings λ at scale M_0

$$\lambda_{i,0} = \lambda_i(M_0) \qquad (7)$$

AFTER renormalization can
pretend to be back in
"Pleasantville"



Coupling flow $\lambda_i(k)$ and effective action Γ_k

At the end effective quantum action:

$$\Gamma_k(\phi_i(x), \lambda_n(k)) = \int d^4x \sqrt{-g} \mathcal{L}_k(\phi_i(x), \lambda_n(k)) \tag{8}$$

Quantum background?

$$\frac{\delta \Gamma_k}{\delta \phi_i} = 0 \quad . \tag{9}$$

"GAP EQUATION"

solutions?

quantum solitons?

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Quantum background?

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"GAP EQUATION"
solutions?
quantum solitons?

II
Improving solutions

$\frac{\delta\Gamma_k}{\delta\phi_i} = 0$ is too hard

$\Rightarrow$ first solve classical eom

$$\frac{\delta S}{\delta \phi_i} = 0 \quad . \tag{10}$$

and take $\lambda_0 \to \lambda_k$ as small correction of those solutions^{i,ii}

i) Uehling potential: in QED textbooks

ii) Improved black holes: B.K. and Frank Saueressig; Int.J.Mod.Phys. A29 (2014) 8, 1430011 ...

Improving solutions II

Improving solutions II

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Improving solutions II

More general consideration

"Improved solution" – fully consistent?

No

Problem: Scale setting

$$k \to^{?} k(r) \quad (\text{typically} \quad k \sim 1/r) \tag{11}$$

Implies

- Improved classical solution does not solve "gap equations"
- Typically not even $T^{\mu\nu}$ conserved

$\Rightarrow$ Propose different scale setting

III
Self-consistent scale setting

Self-consistent scale setting III
Proposal

Effective action and running couplings

$$\Gamma_k(\phi_i(x),\lambda_n(k)) = \int d^4x\sqrt{-g}\mathcal{L}_k(\phi_i(x),\lambda_n(k)) \quad , \tag{12}$$

"Gap equations" for quantum background

$$\frac{\delta\Gamma_k}{\delta\phi_i} = 0 \quad . \tag{13}$$

Scale setting for this background?

Self-consistent scale setting III

Proposal

Scale setting for this background $k = k(r)$?

Idea: Minimal k sensitivity

(just like Callan-Symanzik equations $\frac{d}{dk}\langle T\phi_1(x_1)\phi_2(x_2)\ldots\rangle_k\Big|_{k=k_{opt}} \equiv 0$)

Realization: Promote k to field in Γ_k

$$\Gamma_k(\phi_i(x), \lambda_n(k)) \rightarrow \Gamma(\phi_i(x), k(x), \lambda_n(k)) \quad . \tag{14}$$

Self-consistent scale setting III

Proposal

Realization:

"Scale field" $k(x)$

coupled equations of motion

$$\frac{\delta\Gamma(\phi_i(x),k(x))}{\delta\phi_i}=0 \quad , \quad \frac{d}{dk}\mathcal{L}(\phi_i(x),k(x),\lambda_n(k))\Big|_{k=k_{opt}}=0 \quad . \qquad (15)$$

The simultaneous solution of (15) assures optimal scale setting

First questions on proposal

- Consistent set of equations?
- Solves $\nabla_\mu T^{\mu\nu} \leftrightarrow k(r)$ problem?
- Examples?

IV
Examples: “inoffensive”

Inoffensive:

Examples IV
ϕ^4

Effective action

$$\Gamma(k,\phi) = \int d^4x \left(\frac{\alpha_k}{2}(\partial\phi)^2 - \frac{\tilde{m}_k^2}{2}\phi^2 - \frac{\tilde{g}_k}{4!}\phi^4 \right) \tag{16}$$

Eom $\delta\phi$:

$$\partial_\mu(\alpha_k \partial^\mu \phi) + \tilde{m}_k^2 \phi + \frac{\tilde{g}_k}{6}\phi^3 = 0 \tag{17}$$

eom k:

$$\alpha_k'(\partial\phi)^2 - (\tilde{m}_k^2)'\phi^2 - \frac{1}{12}\tilde{g}_k'\phi^4 = 0 \quad , \tag{18}$$

implies conservation

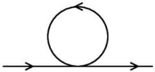
$$\nabla_\mu T^{\mu\nu} = 0 \tag{19}$$

Examples IV

ϕ^4 loops

Example: Self-consistent scale setting for ϕ^4 at 1 loop:

$$\begin{aligned}
\gamma_{\mathbf{Z}} &= \frac{d\ln Z_k}{d\ln k^2} = 0 \quad , \\
\beta_{\mathbf{g}} &= \frac{dg_k}{d\ln k} = \frac{3}{16\pi^2} g_k^2 \quad , \\
\beta_{\mathbf{m^2}} &= \frac{dm_k^2}{d\ln k} = (-2 + \frac{g_k}{16\pi^2}) m_k^2 \quad .
\end{aligned}$$



....

Run proposed machine for spherical symmetry ...

$$k(r) = k_i + \exp\left(-2\sqrt{\frac{14}{3}\frac{k_0}{k_i}} m_0 r\right) \cdot \frac{c_1}{r} \quad . \tag{20}$$

Examples IV

ϕ^4 loops

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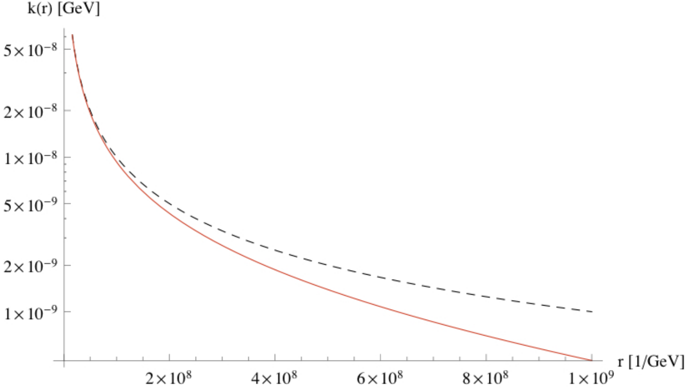


- Self-consistent
- Like classical for small r

V
Examples: “offensive”

Offensive:

Effective action

$$\Gamma_k[g_{\mu\nu}, A_\alpha] \quad = \quad \int_M d^4x\sqrt{-g}\left(\frac{R-2\Lambda_k}{16\pi G_k} - \frac{1}{4e_k^2}F_{\mu\nu}F^{\mu\nu}\right) \quad , \qquad (21)$$

Einstein-Hilbert-Maxwell

Equations of motion

eom $\delta g_{\mu\nu}$:

$$G_{\mu\nu} = -g_{\mu\nu}\Lambda_k - \Delta t_{\mu\nu} + 8\pi G_k T_{\mu\nu} \quad , \tag{22}$$

with

$$\Delta t_{\mu\nu} = G_k \left(g_{\mu\nu}\Box - \nabla_\mu \nabla_\nu \right) \frac{1}{G_k} \quad . \tag{23}$$

and

$$T_{\mu\nu} = F_\nu^{\ \alpha} F_{\mu\alpha} - \frac{1}{4} g_{\mu\nu} F_{\mu\nu} F^{\mu\nu} \quad . \tag{24}$$

eom δA_μ:

$$D_\mu \left(\frac{1}{e_k^2} F^{\mu\nu} \right) = 0 \quad , \tag{25}$$

Examples V

Gravity and gauge

Conservation and symmetry

Symmetry coordintates:

$$\nabla^{\mu} G_{\mu\nu} = 0 \tag{26}$$

Symmetry $U(1)$:

$$\nabla_{[\mu} F_{\alpha\beta]} = 0 \quad . \tag{27}$$

Consistency:

Using everybody ... one can actually show that

Scale setting eom δk:

$$\left[R\nabla_\mu\left(\frac{1}{G_k}\right)-2\nabla_\mu\left(\frac{\Lambda(k)}{G_k}\right)-\nabla_\mu\left(\frac{4\pi}{e_k^2}\right)F_{\alpha\beta}F^{\alpha\beta}\right]\cdot(\partial^\mu k)=0 \quad . \tag{28}$$

is actually self-consistent consequence of eoms and conservation laws

Background solutions?

Actually, yes:

Background solutions?

Actually, yes:

De Sitter case:

eom $g_{\mu\nu}$:

$$G_{\mu\nu} = -g_{\mu\nu}\Lambda_k - \Delta t_{\mu\nu} \quad , \tag{29}$$

eom k:

$$R\nabla_\mu \left(\frac{1}{G_k} \right) - 2\nabla_\mu \left(\frac{\Lambda_k}{G_k} \right) = 0 \quad . \tag{30}$$

unknown functions: $g_{00}(r)$, $g_{11}(r)$, and $k(r)$

known (with caveat): Λ_k, and G_k

Don't like caveat ...

Trade (trick):

$$g_{00}(r),\ g_{11}(r),\ k(r),\ \Lambda_k,\ \text{and}\ G_k$$
$$\Rightarrow$$
$$g_{00}(r),\ g_{11}(r),\ \Lambda(r),\ \text{and}\ G(r)$$

Too many unknowns: Schwarzschild ansatz: $g_{00} = 1/g_{11} \equiv f$

$$\Rightarrow$$
$$f(r),\ \Lambda(r),\ \text{and}\ G(r)$$

Can be solved

generalized de Sitter solution:

$$G(r) = \frac{G_0}{\epsilon r + 1} \tag{31}$$

$$f(r) = 1 + 3G_0 M_0 \epsilon - \frac{2G_0 M_0}{r} - (1 + 6\epsilon G_0 M_0)\epsilon r - \frac{\Lambda_0 r^2}{3} + r^2\epsilon^2(6\epsilon G_0 M_0 + 1)\ln\left(\frac{c_4(\epsilon r + 1)}{r}\right) \tag{32}$$

$$\Lambda(r) = \frac{-72\epsilon^2 r(\epsilon r + 1)\left(\epsilon r + \frac{1}{2}\right)\left(G_0 M_0 \epsilon + \frac{1}{6}\right)\ln\left(\frac{c_4(\epsilon r+1)}{r}\right) + 4r^3\Lambda_0\epsilon^2 + \left(12\epsilon^3 + 6\Lambda_0\epsilon + 72\epsilon^4 G_0 M_0\right)r^2}{2r(\epsilon r + 1)^2} + \frac{\left(72\epsilon^3 G_0 M_0 + 11\epsilon^2 + 2\Lambda_0\right)r + 6\epsilon^2 G_0 M_0}{2r(\epsilon r + 1)^2} \quad . \tag{33}$$

Constants of integration: G_0, M_0, Λ_0, ϵ, c_4

Classical limit:

$$G(r)|_{\epsilon\to 0} = G_0 \tag{34}$$

$$\Lambda(r)|_{\epsilon\to 0} = \Lambda_0 \quad . \tag{35}$$

$$f(r)|_{\epsilon\to 0} = -\frac{\Lambda_0 r^2}{3} - \frac{2G_0 M_0}{r} + 1 \quad , \tag{36}$$

ϵ parametrizes scale dependence of the couplings

Examples V

de Sitter black hole

Asymptotics and horizons:

$+\ r \to 0$

$$R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} = \frac{48 G_0^2 M_0^2}{r^6} - \frac{48 G_0^2 M_0^2 \epsilon}{r^5} + \mathcal{O}(r^{-4}) \quad . \tag{37}$$

$\Rightarrow$ Singularity persists

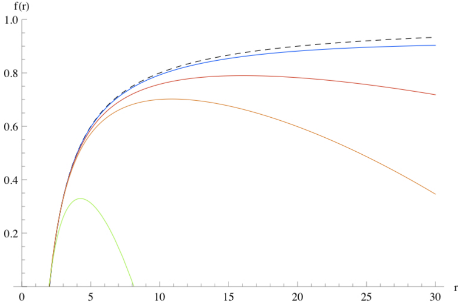
$+\ r \to \infty$

$$f(r) = -r^2 \frac{1}{3}\left(\Lambda_0 - 3\epsilon^2(6\epsilon G_0 M_0 + 1)\ln\left(c_4\epsilon\right)\right) + \mathcal{O}(r) \quad , \tag{38}$$

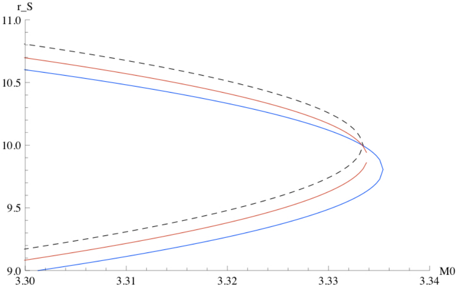
$\Rightarrow$ shift of effective cosmological constant

Examples V

de Sitter black hole



$f(r)$ for var. ϵ



Horizons for var. ϵ

Reissner Nordstrom case:

eom $g_{\mu\nu}$:

$$G_{\mu\nu} = -\Delta t_{\mu\nu} + 8\pi \frac{G_k}{{e_k}^2} T_{\mu\nu} \quad , \tag{39}$$

eom A^μ

$$D_\mu \left(\frac{1}{e_k^2} F^{\mu\nu} \right) = 0 \quad . \tag{40}$$

eom k:

$$\left[R \nabla_\mu \left(\frac{1}{G_k} \right) - \nabla_\mu \left(\frac{4\pi}{e_k^2} \right) F_{\alpha\beta} F^{\alpha\beta} \right] \cdot (\partial^\mu k) = 0 \quad . \tag{41}$$

Examples V

Reissner Nordstrom black hole

Unknown functions:

$$g_{00}(r) \equiv f(r),\ g_{11}(r),\ k(r),\ F_{01} = q(r),$$

Known

$$G_k,\ e_k$$

Same trade as before $g_{11}(r) \equiv 1/f(r)$ and $k(r)$ versus $G(r)$, $e(r)$,
$\Rightarrow$

$$f(r),\ q(r),\ G(r),\ e(r)$$

Can be solved ...

Examples V

Reissner Nordstrom black hole

Solution:

$$
\begin{aligned}
G(r) &= \frac{G_0}{\epsilon r + 1} \qquad (42)\\
f(r) &= \frac{r^4\epsilon^2 {e_0}^2 + 4\epsilon r^3 {e_0}^2 + 4(1 - G_0 M_0 \epsilon){e_0}^2 r^2 - 8rG_0 M_0 {e_0}^2 + 16\pi G_0 {Q_0}^2}{4r^2(\epsilon r + 1)^2 {e_0}^2}\\
e^2(r) &= \frac{\left(r^6\epsilon^4 {e_0}^2 + 3r^5\epsilon^3 {e_0}^2 + \left(3r^4 {e_0}^2 - 4r^3 {e_0}^2 G_0 M_0 + 48r^2\pi G_0 {Q_0}^2\right)\epsilon^2 + 48\epsilon r\pi G_0 {Q_0}^2 + 16\pi G_0 {Q_0}^2\right){e_0}^2\pi}{{Q_0}^2 G_0(\epsilon r + 1)^3}\\
q(r) &= \frac{Q_0}{4\pi e_0^2}\frac{e^2(r)}{r^2} \quad .
\end{aligned}
$$

Integration constants: G_0, Q_0, e_0, and ϵ

Classical limit:

$$f(r)|_{\epsilon\to 0} = 1 - \frac{2G_0 M_0}{r} + \frac{4\pi G_0 Q_0^2}{r^2 {e_0}^2} \quad , \tag{43}$$

$$e^2(r)|_{\epsilon\to 0} = (4\pi)^2 e_0^2 \tag{44}$$

$$q(r)|_{\epsilon\to 0} = \frac{4\pi Q_0}{r^2} \quad . \tag{45}$$

ϵ parametrizes scale dependence of the couplings (again)

Examples V

Reissner Nordstrom black hole

Asymptotics:

$+ \; r \to 0$

$$R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} = 2^7 7 \frac{G_0^2 \pi^2 Q_0^4}{e_0^4 r^8} + \mathcal{O}\left(1/r^7\right) \quad . \tag{46}$$

$\Rightarrow$ Singularity persists

$+ \; r \to \infty$

line element

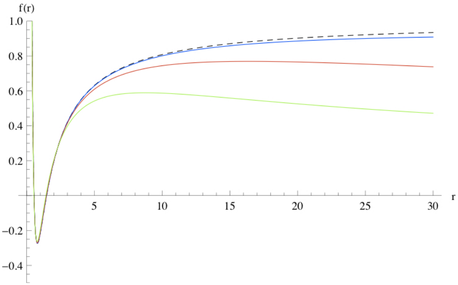
$$ds_\infty^2 = -\frac{1}{4}dt^2 + 4dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \quad . \tag{47}$$

$\Rightarrow$ like global monopole (still charge is unchanged)

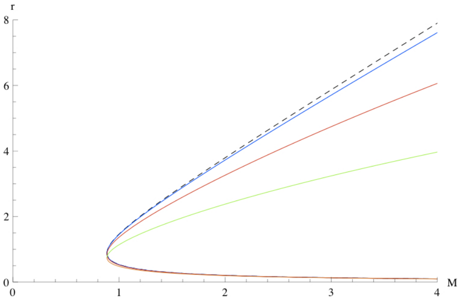
$$Q = \int_{\partial\Sigma} d^2 z \sqrt{\gamma^{S_2}} n_\mu \sigma_\nu \frac{F^{\mu\nu}}{e^2} = \frac{Q_0}{e_0^2} \quad , \tag{48}$$

Examples V

Reissner Nordstrom black hole



$f(r)$ for var. ϵ



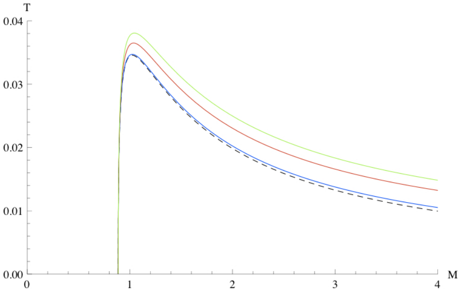
Horizons $r_{\pm}(M_0)$ for var. ϵ

Observe that cosmic censureship unchanged

$$M_0 = 2\sqrt{\pi}\frac{Q_0}{e_0 G_0} \quad , \tag{49}$$

Examples V

Reissner Nordstrom black hole



Temperature slightly increased for $\epsilon \neq 0 \sim \epsilon^2$

$$T = T_0 + \epsilon^2 \frac{G_0 \left(G_0 M_0{}^2 + M_0 \sqrt{\frac{G_0 \left(e_0{}^2 G_0 M_0{}^2 - 4\pi Q_0{}^2 \right)}{e_0{}^2}} - 4\pi \frac{Q_0{}^2}{e_0^2} \right)}{8\pi \left(\sqrt{\frac{G_0 \left(e_0{}^2 G_0 M_0{}^2 - 4\pi Q_0{}^2 \right)}{e_0{}^2}} + G_0 M_0 \right)} + O(\epsilon^3) \quad . \tag{50}$$

V
Summary

Summary

- **Context:** Self-consistent quantum backgrounds $\delta\Gamma_k/\delta\phi_i = 0$
- Problem: Scale setting and consistency
- Proposal: Self-consistent scale setting $k \to k(x)$
  $\delta\Gamma_k/\delta\phi_i = 0$ and $\delta\Gamma_k/\delta k = 0$
- Examples: Studied ϕ^4, and de Sitter-, RN black holes
- Outlook: Hopefully to be applied in many more contexts ;-)

Summary

Thank you