Black holes and running couplings

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collaboration

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Outline

I
Functional Renormalization Group for gravity

The quantization problem of gravity

What is the quantization problem?

"Gravity is not renormalizable"

What is renormalizable?

"Well ... "

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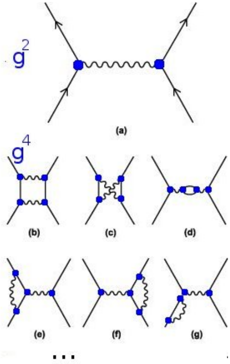
What is renormalizable?

Feynman method:
Power expansion in coupling g

$$\text{Result} = c_1 \cdot g^2 + c_2 \cdot g^4 \cdot \infty + \dots \qquad (1)$$

Problem ∞ canceled by N adjustments
(N=small for any order g^m)

$$\text{Result}' = c_1 \cdot g^2 + c_2' \cdot g^4 + \dots \qquad (2)$$



Gravity: $N_G \to \infty$ for $m \to \infty$

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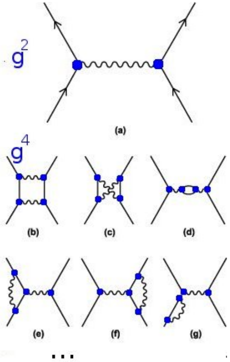

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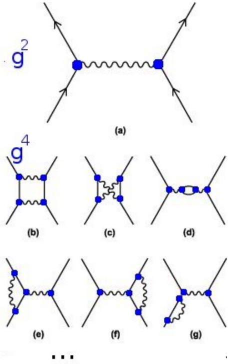

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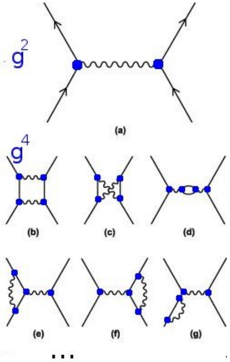


Gravity: $N_G \to \infty$ for $m \to \infty$

Weinbergs Idea, "Asymptotic Safety" [*]

Maybe expansion wrong!
$\rightarrow$ needs the whole functional $\Gamma[\psi]$?
Important: Existence of non-trivial UV-fixed points

Wetterichs realization [**]

$$\partial_t \Gamma[\psi] = \frac{1}{2} Tr \left[\partial_t R_k \cdot (\Gamma^{(2)}[\psi] + R_k)^{-1} \right] \tag{3}$$

Flow equation where ψ are fields, $\Gamma^{(2)} = \delta^2\Gamma/\delta\psi^2$), $t = \ln(k)$, and R_k cut-off function.

$\Rightarrow$ running couplings

[*] S. Weinberg, "General Relativity" Cambridge University Press
[**] M. Reuter, C. Wetterich, Nucl.Phys. B417, 181 (1994)

Define dimensionless couplings

$$g_k = k^2 G_k \qquad \lambda_k = \frac{\Lambda_k}{k^2} \tag{4}$$

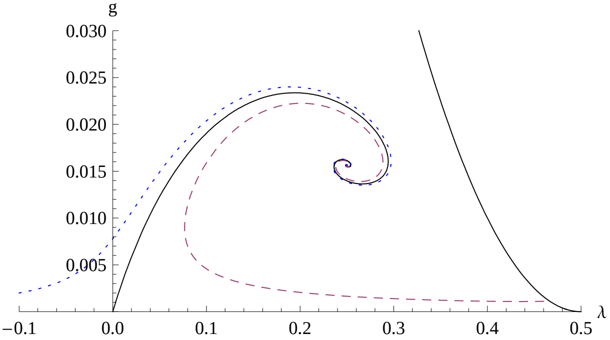
G_0: Newtons constant, Λ_0: Cosmological constant

With Wetterichs equation one can get running gravitational couplings [*]

$$\begin{aligned} \beta_\lambda &= \partial_t \lambda_k = \frac{P_1}{P_2 + 4(d + 2g_k)} \\ \beta_g &= \partial_t g_k = \frac{2 g_k P_2}{P_2 + 4(4 + 2g_k)} \end{aligned} \tag{5}$$

[*]Reuter ..., but here use: D. F. Litim, Phys. Rev. Lett. 92, 201301 (2004)

Solve numerically:



Numerical solution of (5), [R1]

Has UV fixed points → "Asymptotic Safety"
Would be nice to have analytical expression to work with ...

Expand beta functions for small couplings $g, \lambda \ll 1$:

$$\beta_g = g(k)(2 - 24g(k)) \tag{6}$$

$$\beta_\lambda = 12g(k) - 2\lambda(k) \tag{7}$$

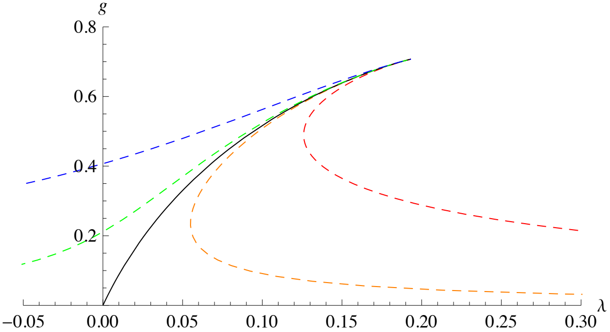
Solve

$$g_{FRG}(k) = \frac{k^2 G_0 g_U^*}{g_U^* + G_0 k^2} \tag{8}$$

$$\lambda(k)_{FRG} = \lambda_U^* + \frac{1}{k^2}\Lambda_0 - \frac{g_U^* \lambda_U^*}{G_0 k^2} \text{Log}\left[\left(1 + G_0 \frac{k^2}{g_U^*}\right)\right] \tag{9}$$

with fixed points g_U^* and λ_U^* used as free parameters.

Analytically approximated flow



Analytical approximation (8, 9)

II From FRG to black holes

B. K. and F. Saueressig, arXiv:1306.1546 [hep-th].

Work pioneered by Bonanno and Reuter [*],
but now do not neglect cosmological term Λ_k

Take classical Schwarzschild (A)dS-solution

$$ds^2 = f(r)dt^2 - f^{-1}(r)dr^2 - r^2 d\Omega_{d+2} \tag{10}$$

$$\text{with} \quad f(r) = 1 - \frac{2G_k M}{r^1} - \frac{1}{3}\Lambda_k r^2 \ .$$

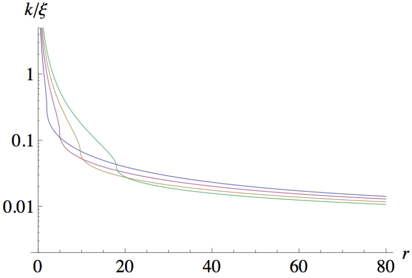
But: Take couplings scale dependent $G = G_k$ and $\Lambda = \Lambda_k$

∗ Phys. Rev. D **62**, 043008 (2000);

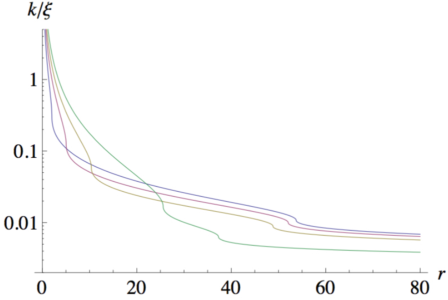
Sensible scale setting $k \leftrightarrow r$

$$k(P(r)) = \frac{\xi}{d(P(r))} \quad \text{with} \quad d(r) = \int_{\mathcal{C}_r} \sqrt{|ds^2|} \tag{11}$$

along purely radial curve $\mathcal{C}_r$ with parameter ξ



$k(r)$ for AdS

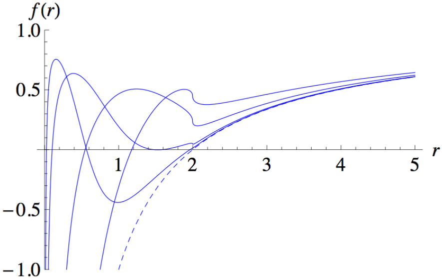


$k(r)$ for dS

- In the UV close to the NGFP one finds

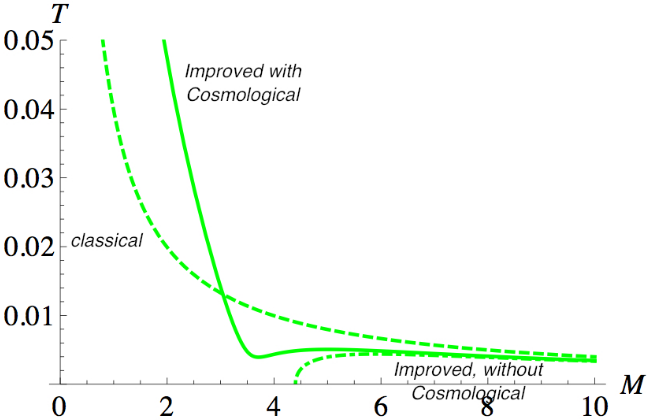
$$f_*(r) = 1 - \frac{2\,G_0\,M}{r}\left(\frac{3}{4}\lambda_*\,\xi^2\right) - \frac{1}{3}\left(\frac{4\,g_*}{3\,G_0\,\xi^2}\right)\,r^2\,. \qquad (12)$$

- For general scales one studies $f(r)$ numerically



$f(r)$ for $\xi = \{\xi_{sc}, 1{,}5,\ 0{,}6,\ 0{,}3\}$

Thermodynamics: Temperature as function of mass parameter M



Summary: II From FRG to black holes

- Cosmological term makes huge difference
- Universal short distance behavior independent of G_0 and Λ_0
- UV: Singularity persists
- UV limit same form as classical solution, but g and λ change role
- Permits analytic micro states counting
- No completely stable remnant
- UV: Improved solution also solution of improved equations of motion, (but not for all scales k)

III
From black holes to FRG?

C. Contreras, B. K. and P. Rioseco, arXiv:1303.3892

New strategy

Equations of motion for scale dependent couplings

$$
\begin{aligned}
G_{\mu\nu} &= -g_{\mu\nu}\Lambda_r + 8\pi G_r T_{\mu\nu} - \Delta t_{\mu\nu} \quad \text{with,} \qquad (13)\\
\Delta t_{\mu\nu} &= G_r \left(g_{\mu\nu}\Box - \nabla_\mu \nabla_\nu\right)\frac{1}{G_r}
\end{aligned}
$$

Find solution to those improved equations of motion

A priory nothing to do with FRG

But: Can compare this $g(r)$, $\lambda(r)$ to FRG g_k, λ_k

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Equations of motion

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Ansatz

$$ds^2 = -f(r)dt^2 + 1/f(r)dr^2 + r^2 d\theta^2 + r^2 \sin(\theta) d\phi^2 \tag{15}$$

where

$$f(r) = (1 - 2\frac{MG(r)}{r} - \frac{l(r)}{3} r^2). \tag{16}$$

Solution

$$\begin{aligned}
G(r) &= -\frac{16\pi c_2}{r-2c_1} \\
\Lambda(r) &= \frac{-1}{12r(r-2c_1)^2 c_1^3}\left\{\left(c_1^2\left(12c_1^2+384\Sigma\pi c_2+c_3\right)+24r^3c_1^3c_4+\dots\right\}\right. \\
I(r) &= c_4+\frac{1}{48c_1^4}\left[\frac{576\Sigma\pi c_1 c_2}{r-2c_1}+\frac{8c_1^3\left(12c_1^2+96\Sigma\pi c_2+c_3\right)}{r^3}+\dots\right]
\end{aligned} \tag{17}$$

Four constants of integration c_1, c_2, c_3, c_4 & time rescaling $t->qt$ to "play" with

III From black holes to FRG?

Singularity at $r = 0$

Calculate invariant quantities

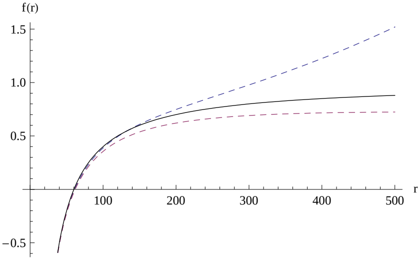
$$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = \frac{144c_1^4 + 9216\Sigma\pi c_1^2 c_2 + 147456\Sigma^2\pi^2 c_2^2 + 24c_1^2 c_3 + 768\Sigma\pi c_2 c_3 + c_3^2}{27c_1^2 r^6} + \mathcal{O}(r^{-5}) \quad (18)$$

and

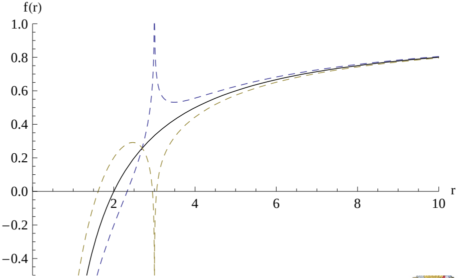
$$C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} \sim \frac{\cdots}{r^6} \quad (19)$$

Singularity persists like in classical Schwarzschild

Reproduce Newtons law in certain regimes



g_{00} reproducing Newton for small r



g_{00} reproducing Newton for large r

This is for g_{00} rescaled $t- > B \cdot t$, but note that g_{11} is tricky

Have dimensionfull couplings G, Λ and integration constants c_i
Want dimensionless expressions

III From black holes to FRG?

Induced coupling flow

Define 4 dimensionless integration constants

$$
\left.\begin{array}{rcl}
c_1 &=& -\frac{g_I}{2 g_U^* \Sigma} \\
c_2 &=& -\frac{g_I}{16 \Sigma^3 \pi} \\
c_3 &=& \frac{3 g_I (8 g_U^{*3} - g_I g_U^* + 2 g_I^2 \lambda_U^*)}{g_U^{*3} \Sigma^2} \\
c_4 &=& -\frac{\Sigma^2 l_I}{2}
\end{array}\right\} \leftrightarrow \left\{\begin{array}{rcl}
\lambda_U^* &=& -\frac{12 c_1^2 + c_3 + 384 c_2 \Sigma \pi}{48 c_1^3 \Sigma} \\
l_I &=& -\frac{2 c_4}{\Sigma^2} \\
g_U^* &=& \frac{8 c_2 \Sigma^2 \pi}{c_1} \\
g_I &=& -16 c_2 \Sigma^3 \pi
\end{array}\right.
$$

Define 4 dimensionless coupling constants

$$
\begin{aligned}
g(r) &= G(r)\Sigma^2 \\
\lambda(r) &= \Lambda(r)\frac{(\Sigma r)}{\Sigma^2}
\end{aligned} \tag{20}
$$

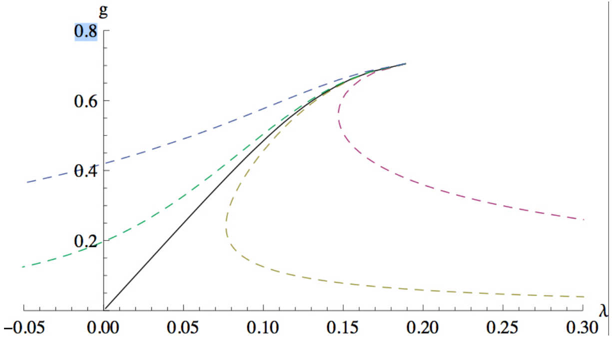
Note: Now only r and Σ have scale dimension

One finds a non-trivial UV fixed point only if $a = 0\, c = 1$:

$$
\begin{aligned}
g_U(r \to 0) &= g_U^* \\
\lambda_U(r \to 0) &= \lambda_U^*
\end{aligned}
$$

Justifies notation of the previously chosen dimensionless parameters.

Obtain flow for an UV fixed point



BH induced flow

Wow, that looks familiar ⇒ compare to FRG

III From black holes to FRG?

Induced coupling flow

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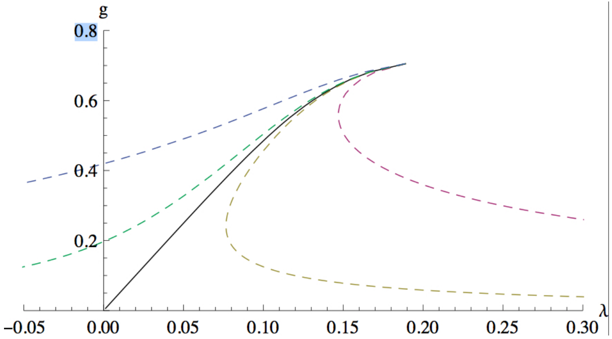

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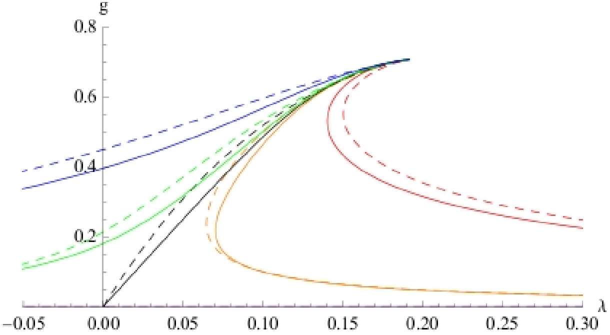
III From black holes to FRG?

FRG and BH induced comparison

Compare flow from BH solution and from FRG approach



BH induced: solid line, FRG: dashed line

Looks so good, compare analytically

FRG:

$$g_{FRG}(k) = \frac{k^2 G_0 g_U^*}{g_U^* + G_0 k^2} \quad (21)$$

$$\lambda(k)_{FRG} = \lambda_U^* + \ldots \ln(\ldots) \quad (22)$$

BH induced:

$$g_U(r) = \frac{g_I}{\left(\frac{g_I}{g_U^*} + \Sigma r\right)} \quad (23)$$

$$\lambda(k)_U \approx \lambda_U^* + \ldots \ln(\ldots) \quad (24)$$

Perfect match for **scale setting**

$$r = \frac{g_I}{k^2 G_0 \Sigma} \quad (25)$$

and for $g_I = 3g_U^*/(5\lambda_U^*)$ and $l_I = \Lambda_0 G_0 / g_I$

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Perfect match for **scale setting**

$$r \equiv \frac{g_I}{k^2 G_0 \Sigma} \tag{25}$$

and for $g_I = 3g_U^*/(5\lambda_U^*)$ and $l_I = \Lambda_0 G_0/g_I$

Summary: III From black hole to FRG?

- New solution of improved equations of motion
- $\Lambda(r)$ essential for finding new solution
- Singularity at origin persists
- No completely stable remnant (up to now)
- Similarity to FRG flow

Summary: Common findings of II and III

- $\Lambda(r)$ essential in context of scale dependent couplings
- Singularity at origin persists
- No completely stable remnant (up to now)
- Similarity of FRG flow and new solution

Thank you

Backups

Looks good, but still many issues and open questions

- Experimental restrictions on parameters? (D.& O.)
- Possible dark matter interpretation? (D.& O.)
- Thermodynamics and horizon structure? (...?)
- Calculate energy for arbitrary parameters (...?)
- Is this a coincidence or something deeper?

Black holes are wise guys, so maybe some truth in this result

Scale setting

intuition → something like 1/ distance

$$k(r) = \frac{\xi}{d(r)} \tag{26}$$

Something with r, M, G_0, usually [*]

$$d_{(2)}(r) = \int_{\mathcal{C}_r} \sqrt{|ds^2|} \approx |_{UV} \frac{1}{R_H^{\frac{1}{2}}} \frac{2}{3} r^{\frac{3}{2}} \tag{27}$$

Put this into $f(r)$

[*] A. Bonanno and M. Reuter, Phys. Rev. D **62**, 043008 (2000) [arXiv:hep-th/0002196];

No solution

Plug improved solution $f(r)$ into Einstein equations

$$G_{\mu\nu} \neq -g_{\mu\nu}\Lambda_k + 8\pi G_k T_{\mu\nu} \tag{28}$$

Why?
Because $G \to G_k \to G(r)$
Need take into account variable $G(r)$ when deriving the eoms,

$$\begin{aligned} G_{\mu\nu} &\neq -g_{\mu\nu}\Lambda_k + 8\pi G_k T_{\mu\nu} - \Delta t_{\mu\nu} \quad \text{with,} \\ \Delta t_{\mu\nu} &= G_k \left(g_{\mu\nu}\Box - \nabla_\mu \nabla_\nu \right) \frac{1}{G_k} \end{aligned} \tag{29}$$

Still no solution

Induced coupling flow

FRG black holes

Dimensionless couplings after redefinition:

$$g_U(r) = \frac{g_I}{\left(\frac{g_I}{g_U^*} + \Sigma r\right)} \tag{30}$$

$$\begin{aligned} \lambda_U(r) \quad &= \quad \frac{1}{2g_I^2(g_I + g_U^*\Sigma r)^2} \left\{ g_I \left(g_I^3(\Sigma r l_I + 2\lambda_U^*) - 12 g_U^{*3}\Sigma^2 r^2 + 3g_I^2 g_U^* \Sigma r(\Sigma r l_I + 8\lambda_U^*) \right) \right. \\ &+ \quad \left. g_I^2 g_U^{*2} \Sigma r \left(2\Sigma^2 r^2 l_I - 11 + 24\Sigma r \lambda_U^* \right) + 6 g_U^* \Sigma r \left(g_I^2 + 3 g_I g_U^* \Sigma r + 2 g_U^{*2}\Sigma^2 r^2 \right) (g_U^* - 2 g_I \lambda_U^*) \ln\left[\frac{g_I}{g_U^*\Sigma r} + 1 \right] \right\} \end{aligned} \tag{31}$$

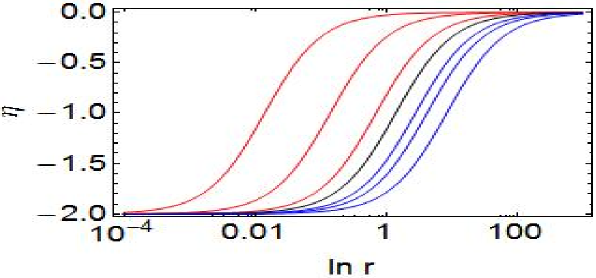
Parametric plot of these functions

Important "observable" in quantum theories of gravity:
Anomalous Dimension η

$$\partial_t g(k) = \beta_g(\lambda_k, g_k) = [d - 2 + \eta(k)] g(k) \tag{32}$$

here

$$\eta(r) = -2 + 2 \frac{r/g_I}{\frac{1}{g_U^* \Sigma} + r/g_I}. \tag{33}$$



Induced coupling flow

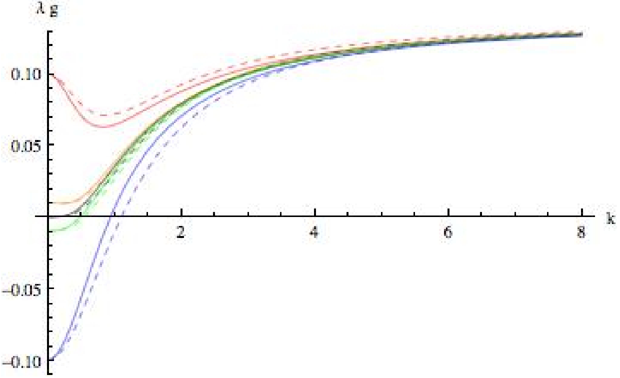
Product of adimensional couplings

In "each" FRG calculation, different values of the fixed points g_U^*, λ_U^*.

Product $g_U^* \cdot \lambda_U^*$ much more stable

$\Rightarrow$

Expect $g(k) \cdot \lambda(k)$ to be more stable



choose

$$\hat{c}_2 = -\frac{12c_1^2 + c_3}{384\Sigma\pi} \tag{34}$$

Singularity improves Kretschmann

$$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}|_{\hat{c}_2} = \frac{2}{c_1^2 r^2} + \mathcal{O}(r^{-1}) \tag{35}$$

Singularity en $r = 0$ vanishes in Weyl

$$C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}|_{\hat{c}_2} = \frac{1}{(3r - 2c_1)^4} \tag{36}$$

(Conformal nature of remaining singularity) further choose

$$\hat{c}_1 = c_1 \to \infty \tag{37}$$

Finite tensor density

$$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}|_{\hat{c}_2,\hat{c}_1} = \frac{8}{3}c_4^2 \tag{38}$$

but simple metric

$$f(r)|_{\hat{c}_2,\hat{c}_1} = 1 - \frac{c_4}{3}r^2 \tag{39}$$

$\Rightarrow$ Either boring or singularity persists

At least learned that c_4 something to do with Λ_0

III From black holes to FRG?

Newton regime

g_{11} problem in simple r expansion:
Ignore Λ should get Brans Dicke metric *

$$ds^2 = (1-2\frac{MG}{r}+\frac{3M^2G^2}{2r^2}+\dots)dt^2-(1+\frac{MG}{r}+\dots)dr^2-r^2d\theta^2-r^2\sin^2\theta d\phi^2 \quad (40)$$

Ignore in our solution c_4 and expand in $1/r$

$$ds^2 = (1-2\frac{MG}{r}+\frac{3M^2G^2}{2r^2}+\dots)dt^2-(2+2\frac{MG}{r}+\dots)dr^2-r^2d\theta^2-r^2\sin^2\theta d\phi^2 \quad (41)$$

Bad for geodesics and gravitational lensing of relativistic trajectories

* Weinberg, Gravitation and Cosmology

Try to find constants c_i such that the new solution is in agreement with the confirmed classical solution

$$f_s(r) = 1 - 2\frac{G_0 M_0}{r} - r^2 \frac{\Lambda_0}{3} \tag{42}$$

for the observational range

$$\{f_s(r) \approx f(r) | \quad r_{min} < r < r_{max}\} \tag{43}$$

up to now invented four conditions, which fixed four constants

$$c_{4,s} = \frac{12c_1^2 + \frac{4\sqrt{3}\tilde{c}_3 c_1}{\sqrt{\tilde{c}_3+12c_1^2}} + \frac{16\sqrt{3}c_1^3}{\sqrt{\tilde{c}_3+12c_1^2}} - \tilde{c}_3 \ln[3] - \tilde{c}_3 \ln\left[\tilde{c}_3 + 12c_1^2\right] + 2\tilde{c}_3 \ln\left[-6c_1 + \sqrt{3}\sqrt{\tilde{c}_3 + 12c_1^2}\right]}{32c_1^4} \quad (44)$$

where $\tilde{c}_3 = c_3 + 382\pi\Sigma c_2$.

$$c_{1,s} = \frac{3^{2/3}}{4(2G_0 M_0 \Lambda_0^2)^{1/3}} \quad (45)$$

$$c_{3,s} = \frac{12 \cdot 6^{2/3} G_0 (G_0 M_0)^{2/3} (-4\Sigma + 3M_0) \Lambda_0^{4/3} - 9 \cdot 6^{1/3} \left(G_0 M_0 \Lambda_0^2\right)^{1/3}}{8 G_0 M_0 \Lambda_0^2} \quad (46)$$

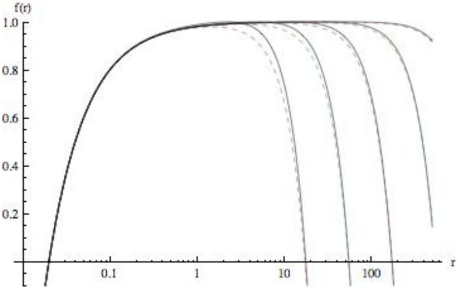
$$c_{2,s} = \frac{G_0}{32\pi(2G_0 M_0 \Lambda_0^2/9)^{1/3}} \quad (47)$$

This gives

$$f(r) = \quad ugly\ldots \qquad (48)$$

but approximately

$$f(r) = 1 - 2\frac{G_0 M_0}{r} + \mathcal{O}\left(\Lambda_0^{2/3}\right) \qquad (49)$$



OK, works but very restricted ...

Now found better way to approximate Newton ...

(Paola Rioseco, Carlos Contreras, Davi Rodrigues, Ilya Shapiro, Oliver Piatella)

Better way to approximate Newton:
First solve approx. eom for Ansatz

$$f(r) = f_{SdS}(r) + \epsilon \cdot \phi(r) + \ldots \tag{50}$$

use this solution to interpret integration constants c_i
Constraints etc. work in progress ...
$\Rightarrow$
Confidence to continue with studying couplings

Induced coupling flow

FRG black holes

Have dimensionfull couplings G, Λ and integration constants c_i
Want dimensionless expressions

Redefine 4 dimensionless integration constants

$$
\left.\begin{array}{lcl}
c_1 & = & -\frac{g_I}{2g_U^*\Sigma} \\
c_2 & = & -\frac{g_I}{16\Sigma^3\pi} \\
c_3 & = & \frac{3g_I(8g_U^{*3}-g_I g_U^*+2g_I^2\lambda_U^*)}{g_U^{*3}\Sigma^2} \\
c_4 & = & -\frac{\Sigma^2 l_I}{2}
\end{array}\right\} \leftrightarrow \left\{\begin{array}{lcl}
\lambda_U^* & = & -\frac{12c_1^2+c_3+384c_2\Sigma\pi}{48c_1^3\Sigma} \\
l_I & = & -\frac{2c_4}{\Sigma^2} \\
g_U^* & = & \frac{8c_2\Sigma^2\pi}{c_1} \\
g_I & = & -16c_2\Sigma^3\pi
\end{array}\right.
$$

Metric reads

$$
\begin{aligned}
f(r) \quad = \quad & \frac{1}{6g_I^2 g_U^{*2}\Sigma r}\left\{g_I\left(-6g_U^{*3}\Sigma^2 r^2+4g_I^3\lambda_U^*-6g_I^2 g_U^*\Sigma r\lambda_U^*+g_I g_U^{*2}\Sigma r\left(6+\Sigma^2 r^2 l_I+12\Sigma r\lambda_U^*\right)\right)\right. \\
& \left.+6g_U^{*3}\Sigma^3 r^3(g_U^*-2g_I\lambda_U^*)\mathrm{Log}\left[\frac{g_I}{g_U^*\Sigma r}+1\right]\right\}
\end{aligned} \qquad (51)
$$

Note: Now only r and Σ have scale dimension