Applying exact renormalization group with optimal scale to cosmology

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Outline

- Scale dependent couplings
- Consistent scale in Einstein-Hilbert action
- Exact renormalization group
- Cosmology
- Conclusions

Scale dependent Couplings

Lesson from QFT

couplings run

$$\alpha_1 \to \alpha_1(k),\ \alpha_2 \to \alpha_2(k),\ \alpha_s \to \alpha_s(k), \ldots$$

Expect the same from quantum gravity

$$G \to G(k),\ \Lambda \to \Lambda(k),\ \ldots$$

Action:

Einstein-Hilbert action coupled to matter with scale dependent couplings

$$S[g] = \int d^4x\sqrt{-g}\left(\frac{R-2\Lambda_k}{16\pi G_k}+\mathcal{L}_m\right) \quad . \tag{1}$$

In practical applications one has to choose the scale-dependence

$$k = k(x_\mu)$$

this affects equations of motion

Equations of motion:

$$G_{\mu\nu} = -g_{\mu\nu}\Lambda_k + 8\pi G_k T_{\mu\nu} - \Delta t_{\mu\nu} \quad , \tag{2}$$

with G_k → additional stress energy tensor

$$\Delta t_{\mu\nu} = G_k \left(g_{\mu\nu}\Box - \nabla_\mu \nabla_\nu \right) \frac{1}{G_k} \quad . \tag{3}$$

Diffeomorphism invariance dictates:

$$G_{\mu\nu}{}^{;\nu} = 0 \tag{4}$$

Would like to conserve also matter $T_{\mu\nu}$

Problem with choice of $k(x_\mu)$

Consistent Scale in Einstein-Hilbert action

Choice of $k(x_\mu)$:

Usual choice: $k \sim 1/(\int ds)$ (De Broglie) has high price:

- either loose conservation of $T_{\mu\nu}$
- or the obtained is $g_{\mu\nu}$ not solution of eom any more "improved solution"

We suggest, choose k in a cleverer way such that

- $T_{\mu\nu}{}^{;\nu} = 0$
- solution stays solution

Choice of $k(x_\mu)$:

This choice is dictated by a consistency condition:

$$\left(8\pi G'_{\bar{k}} T_{\mu\nu} - g_{\mu\nu}\Lambda'_{\bar{k}}\right)\partial^\nu k - \nabla^\nu \Delta t_{\mu\nu} \quad = \quad 0 \tag{5}$$

rewrite using eom (2)

$$R\nabla_\mu \left(\frac{1}{G_k}\right) - 2\nabla_\mu \left(\frac{\Lambda_k}{G_k}\right) = 0 \tag{6}$$

For given G_k, Λ_k this is “just” an algebraic relation for $k(x_\mu)$

“optimal scale”

Apply to candidate of quantum gravity that predicts G_k and Λ_k

Exact renomalization group:

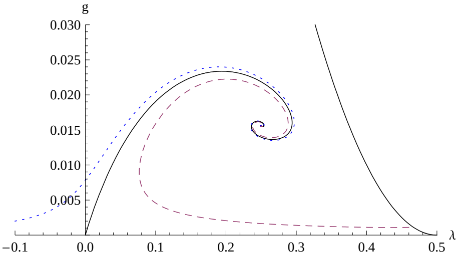
The exact renormalization group (ERGE) approach predicts running couplings independent of the background:

$$\begin{aligned}\beta_\lambda &= \partial_t \lambda_k = \frac{P_1}{P_2 + 4(d + 2g_k)} \\ \beta_g &= \partial_t g_k = \frac{2 g_k P_2}{P_2 + 4(4 + 2g_k)}\end{aligned} \quad (7)$$

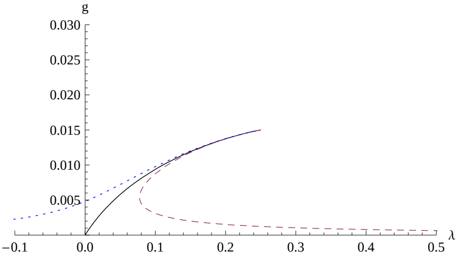
with the dimensionless couplings defined as

$$g_k = k^2 G_k \quad , \quad \lambda_k = \frac{\Lambda_k}{k^2} \quad (8)$$

ERGE solutions:



Numerical solution of (7)



Analytical approximation of (7) using $\lambda \ll 1$

We use analytical approximation

$$\lambda(g) = \frac{g^*\lambda^*}{g}\left((5+e)\left[1-g/g^*\right]^{3/2} - 5 + 3g/(2g^*)(5-g/g^*)\right)$$

$$g(k) = \frac{k^2}{1+k^2/g^*} \quad ,$$

With the UV fixed points λ^* and g^*

The metric:

Assuming homogenous background

$$ds^2 = -dt^2 + a(t)^2 d\vec{x}^2 \tag{9}$$

$T_{\mu\nu}$ is the standard fluid
plus the contribution induced by $G = G_k$

$$\begin{aligned} \Delta t_{00} &= -\frac{3\dot{G}_k \dot{a}}{G_k a} \\ \Delta t_{ii} &= \frac{a}{G_k^2}(a\ddot{G}_k G_k - 2\dot{G}_k^2 a + 2\dot{G}_k \dot{a} G_k) \\ \Delta t_{\mu\neq\nu} &= 0 \end{aligned} \tag{10}$$

Equations of motion:

Generalized Friedmann equations

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G_k}{3}\left(\frac{a_0^4\rho_r}{a^4}+\frac{a_0^3\rho_m}{a^3}\right)+\frac{\Lambda_k}{3}-\frac{\kappa}{a^2}+\frac{\dot{G}_k\dot{a}}{G_k a} \quad (11)$$

$$\frac{\ddot{a}}{a} = -\frac{8\pi G_k}{3}\left(\frac{a_0^4\rho_r}{a^4}+\frac{a_0^3\rho_m}{a^3}\right)+\frac{\Lambda_k}{3}+\frac{\dot{G}_k\dot{a}}{2G_k a}+\frac{\dot{G}_k\ddot{G}_k-2\dot{G}_k}{G_k^2} \quad (12)$$

Usually equivalent, here not generally equivalent!

What is wrong?

Consistency condition:

If we impose consistency condition (5), the problem is solved and the generalized Friedmann equations become equivalent again.
For cosmology the consistency condition reads

$$8\pi\dot{G}_k\left(\frac{a_0^4\rho_r}{a^4}+\frac{a_0^3\rho_m}{a^3}\right)+\dot{\Lambda}_k+3\dot{G}_k\frac{\dot{a}\dot{G}_k+G_k\ddot{a}}{aG_k^2}=0 \tag{13}$$

Rewrite using equations of one of the Friedmann equations

$$3\alpha(t)-\Lambda_k+G_k\frac{\dot{\Lambda}_k}{\dot{G}_k}=0 \tag{14}$$

where $\alpha(t)=(\dot{a}/a)^2+\ddot{a}/a+\kappa/a^2$.

Solving consistency condition:

find ugly large analytical expression

$$k^2(\alpha) = \ldots \tag{15}$$

Insert this back into generalized Friedmann equation gives even more ugly differential equation

$$\ddot{a} = \ldots \tag{16}$$

Impossible (for us) to solve

$\Rightarrow$ Study asymptotics

Infra-red consistency:

Energy scale is way below the Planck scale $k^2 \ll 1/G_0$.
Expand consistency condition in a Taylor series around $k^2 = 0$

$$3\alpha + \frac{k^2\lambda^*}{4}(-3+e) + \frac{e\lambda^* g^*}{2G_0} = 0 + \mathcal{O}(k^4 G_0^2) \tag{17}$$

Gives simple solution

$$k_{IR}^2 = \begin{cases} \frac{2e\lambda^* g^* - 12 G_0 \alpha}{(-3+e) G_0 2\lambda^*} & \text{for } \alpha \geq e\lambda^* g^*/(6G_0) \\ 0 & \text{for } \alpha < e\lambda^* g^*/(6G_0) \end{cases} \tag{18}$$

Insert into Friedmann equation

Infra-red Friedmann equation:

Infrared generalized Friedmann equation

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G_0}{3}\left(\frac{a_0^4\rho_r^0}{a^4} + \frac{a_0^3\rho_m^0}{a^3}\right) + \frac{eg^*\lambda^*}{3G_0} - \frac{\kappa}{a^2} \quad . \tag{19}$$

Match to constant cosmological constant:

$$e = \Lambda_{observed} \cdot \frac{G_0}{g^*\lambda^*} \quad . \tag{20}$$

Interesting:
Observed value of Λ determines RG trajectory

Ultra-violet consistency:

For Planckian and pre-Planckian epoch approximate $(1/(k^2 G_0) \ll 1$

$$3\alpha - k_{UV}^2 2\lambda^* + \frac{10\lambda^* g^*}{4G_0} = 0 + \mathcal{O}(1/(k^4 G_0^2)) \quad . \tag{21}$$

gives

$$k_{UV}^2 \approx \frac{3\alpha}{2\lambda^*} \quad . \tag{22}$$

Insert into Friedmann equation

Ultra-violet Friedmann equation:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G_{k_{UV}}}{3}\left(\frac{a_0^4\rho_r}{a^4}+\frac{a_0^3\rho_m}{a^3}\right)+\frac{\Lambda_{k_{UV}}}{3}-\frac{\kappa}{a^2}+\frac{\dot{G}_{k_{UV}}\dot{a}}{G_{k_{UV}}a} \quad (23)$$

Ansatz for small t

$$a = C\cdot t \quad (24)$$

Solves the equation for

$$C=\frac{1}{3}\sqrt{-3\kappa+2\sqrt{-24a_0^4\pi\rho_r\lambda^*g^*+9\kappa^2}} \quad (25)$$

Nice: linear expansion solves Horizon problem
Not nice: C can be complex

Ultra-violet nice:

Causal horizon scales as

$$h_c = \int_{t_i}^{t_f} dt \frac{c}{a(t)} = \frac{c}{C}\left[\ln\left(\frac{t_f}{t_i}\right)\right] \quad . \tag{26}$$

In contrast to this, the Hubble Horizon in this epoch scales as

$$h_H = \frac{1}{t_f - t_i}\int_{t_f}^{t^i} \frac{c}{\dot{a}} = \frac{c}{C} \quad . \tag{27}$$

Thus, the early epoch of linear expansion can create arbitrarily high homogeneities for $t_i \to 0$.

Ultra-violet not nice:

Complex values of C can only be avoided if

$$\frac{3}{4} G_0 H_0^2 \frac{|\Omega_k|}{\Omega_r} > \lambda^* g^* \quad , \tag{28}$$

This determines the values of the fixed points to be veeeery small

$$g^* \lambda^* \approx 10^{-120}.$$

In contrast the numerical ERGE solution suggests

$$g_N^* \lambda_N^* \approx 10^{-2}$$

One can not get around this problem by simple tricks

Conclusions:

- Formulated general consistency condition that allows to determine scale k^2
- Found analytic parametrization for λ_k and g_k
- Applied the framework to cosmology
- IR cosmology determines ERGE tracetory
- UV cosmology allows for nice solution of horizon problem but predicts terribly wrong values of fixed points λ^* and g^*
- Further studies on the way

Literature

- This work: Benjamin Koch, Israel Ramirez, e-Print: arXiv:1010.2799 , accepted by CQG
- Related work: M. Reuter and F. Saueressig, Phys. Rev. D **65**, 065016 (2002); D. F. Litim, Phys. Rev. Lett. **92**, 201301 (2004); K. Groh and F. Saueressig, J. Phys. A **43**, 365403 (2010); A. Bonanno and M. Reuter, Phys. Lett. B **527**, 9 (2002) ...