Quantizing Geometry or Geometrizing the Quantum?

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The unsatisfactory status of the search for a consistent and predictive quantization of gravity is taken as motivation for discussing an alternative approach. We study the question whether quantum mechanical laws could emerge from a geometrical theory. A toy model that incorporates the idea is presented and its necessary formulation in configuration space is emphasized.

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I. QUANTIZING GEOMETRY

The dream of finding a unified description of all physical phenomena is facing a profound problem: "*The deep incompatibility between the indefinite nature of quantum mechanics and the rigid geometrical formulation of general relativity.*" A common assumption is that quantum mechanics, as it is usually formulated, is a fact of "nature" and thus it is more fundamental than general relativity. Consequently most approaches to solve the problem try to apply one of the well defined quantization procedures to the physical degrees of freedom of space-time (or to some deeper theory that gives rise to space-time). Some of the most popular approaches along this line are:

String theory is for many the most promising candidate for a unified theory of nature [1]. It lead to interesting conjectures about the relation between certain tree level string theories and quantum field theory [2]. But until today it could not live up to its promises concerning the uniqueness of what this theory actually predicts and explains.

Loop quantum gravity is a canonical approach for the quantization of space-time. In earlier stages of its development it lead to the development of geometrodynamics [3] and [4] supergravity that has very nice features at the Planck scale [5]. However, up to now it was not possible to show that it really contains general relativity in some classical limit [6].

Causal dynamical triangulation and causal sets are disciplines that earn more and more attention [7]. They show the emergence of four dimensional space-time by starting from a discrete causal structure. Until now those approaches are limited to asking very basic questions on such as the dimensionality of space-time and don't say anything about an effective gravitational action or the possible unification of all forces.

Induced gravity theories try to show the emergence of curved space-time in a mean field approximation of some underlying microscopic degrees of freedom [8]. It is assumed that this mechanism is similar to the mechanism that allows to get fluid dynamics from Bose-Einstein condensation. Up to now those models manage to mimic some possible features of (quantized) general relativity but a complete picture is still missing.

Renormalization group approaches are working in the imaginary time formalism. Given an ultra violet (UV) completion and the existence of a non-trivial fixed point in the running couplings of the completed gravitational action this approach might present a renormalizable version of gravity [9, 10]. Until now the strict applicability of the imaginary time formalism and the form of the UV completion are open issues.

Anisotropic models postulate a different scaling behavior of space and time in the UV regime, which allows to construct a power counting renormalizable theory [11] in the UV. However, recent studies claim that the infrared limit of the theory is not identical to massless gravity [12].

Further research has been done on asymptotic quantization [13], twistors [14], non-commutative [15] and discretized [16] geometry.

Despite of impressive progress in some directions, the original task remains unsolved in all those approaches.

II. GEOMETRIZING THE QUANTUM

Given the problems in applying the laws of quantum mechanics to the geometry of space-time we want to ask the following question:

"*Could it be that (classical) geometry is more fundamental than the rules of quantization?*"

A. Conceptual problems

Necessarily, answering this question with "yes" would mean that the undeniable observable effects of quantization have to emerge from the deeper theory (in this case a classical geometrical theory). Such an approach faces immediately two mayor problems

- *Determinism*
  is, in contrast to quantum mechanics, part of most geometric theories (such as general relativity). This means for example that in causal geometrical theories uncertainties are just a result of unknown ini-

tial conditions, whereas in standard quantum mechanics they are an irrenunciable concept.

- *Non-locality:*
  In principle it is possible to construct deterministic (hidden variable) theories that are in agreement with the predictions of quantum mechanics. However, those theories have to pay a price in order to evade "no go" theorems such as the Bell inequalities [17]. They have to contain non-local interactions.

B. A conceptual bridge

There exists a self consistent deterministic formulation of quantum mechanics, which also reproduces all typical experimental results [40]. It was first suggested by de Broglie, then shown to be consistent by Bohm [18, 19] and later further developed by several authors [17, 20, 21]. It will be referred to as dBB (de Broglie-Bohm) theory. In this proposal, the dBB theory will be an essential piece when building the bridge from classical geometry to a quantum theory. We will now shortly present its formulation for the case of a relativistic system of n-bosonic particles as given by [21]: Let $|0\rangle$ be state vector of the vacuum and $|n\rangle$ be an arbitrary n-particle state. The corresponding n-particle wave function is [21]

$$\psi(x_1;\,\dots\,;x_n) = \frac{\mathcal{P}_s}{\sqrt{n!}}\langle 0|\hat{\Phi}(x_1)\dots\hat{\Phi}(x_n)|n\rangle \quad , \qquad (1)$$

where the $\hat{\Phi}(x_j)$ are scalar Klein-Gordon field operators. The symbol $\mathcal{P}_s$ denotes symmetrization over all positions x_j which we will keep in mind but not write explicitly any more. For free fields, the wave function (1) satisfies the equation

$$\left(\sum_j^n \partial_j^m\partial_{jm} + n\frac{M^2}{\hbar^2}\right)\psi(x_1;\,\dots\,;x_n) = 0 \quad . \qquad (2)$$

The mass of a single particle is given by M. The index j indicates on which one of the n particle coordinates the differential operator has to act and the index m is the typical space-time index in four flat dimensions. The key step to the dBB interpretation comes from splitting the wavefunction up $\psi = P\exp(iS/\hbar)$ and postulating the four-momentum of the particle j to be $-\partial_{jm}S$. Including this definition one has three coupled real differential equations. For further convenience the four dimensional coordinates for the n particles can be labeled as

$$x^L = (x_1^0, x_1^1, x_1^2, x_1^3;\,\dots\,;x_n^0, x_n^1, x_n^2, x_n^3) \quad , \qquad (3)$$

which also implies the $4\times n$ dimensional co- and contravariant derivatives $\partial_{jm} \leftrightarrow \partial_L$ and $\partial_j^m \leftrightarrow \partial^L$. Now the three real equations of the dBB theory read

$$\begin{aligned} 2MQ &\equiv (\partial^L S)(\partial_L S) - nM^2 \quad \text{with} && (4)\\ Q &\equiv \frac{\hbar^2}{2M}\frac{\partial^L\partial_L P}{P} \quad , &&\\ 0 &\equiv \partial_L\left(P^2(\partial^L S)\right) \quad , && (5)\\ p^L &\equiv M\frac{dx^L}{ds} \equiv -\partial^L S \quad . && (6)\end{aligned}$$

Applying the total derivative $d/ds \equiv dx^L/ds\partial_L$ to eq. (6) gives a Newtonian type of equation of motion

$$\frac{d^2x^L}{ds^2} = \frac{(\partial^N S)(\partial^L\partial_N S)}{M^2} \quad . \qquad (7)$$

It is crucial to note that this theory addresses the two previously mentioned conceptual issues.
Fist, it is deterministic in the sense that given initial positions and given initial field configurations for S and P determine the final state of the system.
Second, it is deeply non-local, because the functions S and P simultaneously depend on the positions of all the n-particles. A further remark: the dBB theory is not affected by the Kochen-Specker theorem [22] since it is a contextual. Thus, the dBB theory could be a useful intermediate step for the program of geometrizing the quantum.

III. EMERGENT QUANTUM MECHANICS

The idea that quantum mechanics might not be fundamental but rather emerge from an underlying classical system has been proposed in various ways.

A. Various appearances of the idea

Although the focus of this paper is on the possible geometric origin of quantum mechanics it is instructive to give a list of proposals that point into a similar direction.

Statistical emergence of quantum mechanics:
In [23, 24] it was shown that quantum mechanical correlations arise when considering finite subsystems of classical statistical systems with originally infinite degrees of freedom. An application of this observation to quantum gravity is perceivable but was not attempted yet.

Gauge emergence of quantum mechanics:
Based on a new kind of local gauge transformation a non-linear field theory has been proposed that contains quantum field theory as an infrared limit [25]. Also a special classical supersymmetric model was suggested to give rise to a quantum mechanical system [26]. A possible unification with general relativity was not explored yet.

Dissipative emergence of both, quantum and gravity:
Dissipative deterministic systems can give rise to quantum operators and symmetries that are not present in the original theory at the microscopic scale [27, 28]. Further

conjecturing that those symmetries are the ones of diffeomorphism invariance (general relativity) might give an identikit picture of a future theory of quantum gravity.

Geometrical emergence of quantum mechanics:
The similarity between Weyl geometry and the structure of quantum mechanical equations was first noticed in [29]. Other studies in this direction focused on the Ricci flow [30, 31] or on a geometric reduction of the dimensionality of space-time [32, 33]. Using local conformal transformations (Weyl geometry) it was even possible to formulate a geometrical theory that contains in certain limits both general relativity and the equations of Bohmian mechanics [34–36]. The impressive success of those (Weyl geometry) models is limited to the single particle case because the dBB theory is only consistent if it also contains the non-local interactions due to multi particle dynamics.

B. Geometry of configuration space

It was shown that existing models for the geometrical emergence of quantum mechanics are incomplete, since they can't explain the non-local interactions in the multi particle dBB theory. Continuing previous work in this direction [37, 38] a possible way to fill this gap will be presented.

The $4\times n$ dimensional configuration space of n-particles will be considered. Following the notation in eq. (3) the coordinates in this (possibly curved) space-time will be denoted as

$$\hat{x}^\Lambda = (\hat{x}_1^0, \hat{x}_1^1, \hat{x}_1^2, \hat{x}_1^3;\ \dots\ ; \hat{x}_n^0, \hat{x}_n^1, \hat{x}_n^2, \hat{x}_n^3) \quad . \tag{8}$$

The toy model for the curvature of this space will be a single scalar equation which is a $4\times n$ dimensional analog to the Nordstrom theory [39]

$$\hat{R}\Big|_{\mathcal{S}} = \kappa\, \hat{\mathcal{T}}_M\Big|_{\mathcal{S}} \quad . \tag{9}$$

The left hand side contains the Ricci scalar (corresponding to a metric $\hat{g}_{\Lambda\Sigma}$). The right hand side contains some coupling constant κ and the trace of the energy momentum tensor $\hat{T}$. The symbol $|_{\mathcal{S}}$ indicates complete symmetrization of the terms with respect to the interchange of two configuration coordinates $\hat{x}_i^\mu \leftrightarrow \hat{x}_j^\mu$. Just like in the case of the bosonic Klein-Gordon equation we will keep this in mind without explicitly writing it into the following equations. The symmetrization further fixes the coordinate system for the four dimensional subspaces and forces all block diagonal submetrics to be identical $\hat{g}_i^{\mu\nu} \leftrightarrow \hat{g}_j^{\mu\nu}$. In order to describe the local conformal part of this theory separately and for simplification one assumes the metric $\hat{g}$ to split up into a conformal function $\phi(x)$ and a flat part η

$$\hat{g}_{\Lambda\Gamma} = \phi^{\frac{2}{2n-1}} \eta_{LG} \quad . \tag{10}$$

The inverse of the metric (10) is

$$\hat{g}^{\Lambda\Gamma} = \phi^{-\frac{2}{2n-1}} \eta^{LG} \quad . \tag{11}$$

Indices with a lower Greek and a lower Roman index can be identified $\hat{\partial}_\Lambda \equiv \partial_L$. From this follows for example that the adjoint derivatives are not identical, in both notations

$$\hat{\partial}^\Lambda = \hat{g}^{\Lambda\Sigma}\hat{\partial}_\Sigma = \phi^{-\frac{2}{2n-1}}\eta^{LS}\partial_S = \phi^{-\frac{2}{2n-1}}\partial^L \quad . \tag{12}$$

The geometrical dual to the first dBB equation:
An Extension of the Hamilton Jacobi stress energy tensor $\hat{\mathcal{T}}_M$ can be defined by subtracting a mass term $\hat{M}^2$ for every particle

$$\begin{aligned} \hat{\mathcal{T}}_M &= \hat{p}^\Lambda \hat{p}_\Lambda - n\hat{M}_G^2 \\ &= (\hat{\partial}^\Lambda S_H)(\hat{\partial}_\Lambda S_H) - n\hat{M}_G^2 \\ &= \phi^{\frac{-2}{2n-1}} \left((\partial^L S_H)(\partial_L S_H) - nM_G^2 \right) \\ &= \phi^{\frac{-2}{2n-1}} \mathcal{T}_M \quad . \end{aligned} \tag{13}$$

The Hamilton principle function S_H defines the local momentum $\hat{p}^\Lambda = \hat{M}_G\, d\hat{x}^\Lambda/d\hat{s} = -\hat{\partial}^\Lambda S_H$. Combining (13), (12), (10), and (9) gives

$$\frac{2(4n-1)}{\kappa(1-2n)} \frac{\partial^L\partial_L\phi}{\phi} = (\partial^L S_H)(\partial_L S_H) - nM_G^2 \quad . \tag{14}$$

This is exactly the first dBB equation (4) if one identifies

$$\begin{aligned} \phi(x) &= P(x) \quad , \\ S_H(x) &= S(x) \ , \\ \kappa &= \frac{2(4n-1)}{1-2n}/\hbar^2 \ , \\ M^2 &= M_G^2 \quad . \end{aligned} \tag{15}$$

Note that the matching conditions demand a negative coupling κ.

The geometrical dual to the second dBB equation:
In order to find the dual to the second Bohmian equation one can exploit that the stress-energy tensor (13) is covariantly conserved

$$\hat{\nabla}_\Lambda \hat{T}^{\Lambda\Delta} = 0 \quad . \tag{16}$$

This is true if the following relations are fulfilled

$$\hat{\nabla}_\Lambda(\hat{\partial}^\Lambda S_H) = 0 \quad , \tag{17}$$

$$(\hat{\partial}^\Lambda S_H)\hat{\nabla}^\Delta(\hat{\partial}_\Lambda S_H) = 0 \quad , \tag{18}$$

$$(\hat{\partial}^\Lambda S_H)\hat{\nabla}_\Lambda(\hat{\partial}^\Delta S_H) = 0 \quad . \tag{19}$$

In addition to the covariant conservation of momentum (17) and the conservation of squared momentum (18) the tensor nature of (13) also demands (19). In order to calculate the covariant derivatives in (17-19), one needs to know the Levi Civita connection

$$\begin{aligned} \Gamma^\Sigma_{\Lambda\Delta} &= \frac{1}{2} g^{\Sigma\Xi}\left(\partial_\Lambda g_{\Delta\Xi} + \partial_\Delta g_{\Xi\Lambda} - \partial_\Xi g_{\Lambda\Delta}\right) \\ &= \frac{1}{2}\phi^{-\frac{2}{2n-1}} \Big[(\partial_L \phi^{\frac{2}{2n-1}})\delta^S_D + (\partial_D \phi^{\frac{2}{2n-1}})\delta^S_L \\ &\qquad - (\partial^S \phi^{\frac{2}{2n-1}})\eta_{LD} \Big] \quad . \end{aligned} \tag{20}$$

It is this form of the connection that gives rise to the non-metricity in Weyl geometry. Using eq. (20), the condition (17) reads

$$\hat{\nabla}_\Lambda(\hat{\partial}^\Lambda S_H) = \phi^{-\frac{4n}{2n-1}}\partial_L\left[\phi^2(\partial^L S_H)\right] = 0 \quad . \tag{21}$$

With the matching conditions (15), the above equation is identical to the second Bohmian equation (5).

The geometrical dual to the third dBB equation: According to the Hamilton-Jacobi formalism the derivatives of the Hamilton principle function (S_H) define the momenta

$$\hat{p}_\Lambda \equiv -(\hat{\partial}_\Lambda S_H) \quad . \tag{22}$$

Therefore, with the prescription (12) and the matching condition (15) one sees that the third Bohmian equation (6) is fulfilled.

The geometrical dual to the dBB equation of motion: From differential geometry it is known that the validity of the geodesics equation of motion results in the conservation of the stress energy tensor. Nevertheless, it is a good consistency check [38] to explicitly calculate the geodesic equation

$$\frac{d^2\hat{x}^\Lambda}{d\hat{s}^2} + \hat{\Gamma}^\Lambda_{\Delta\Sigma}\frac{d\hat{x}^\Delta}{d\hat{s}}\frac{d\hat{x}^\Sigma}{d\hat{s}} = \hat{p}^\Lambda \cdot f(\hat{x}) \quad . \tag{23}$$

Inserting eq. (20) into eq. (23) and using the matching conditions eq. (15) the dBB equation of motion (7) is obtained.

IV. SUMMARY

It is advocated that "geometrizing the quantum" might be a viable alternative to the standard approaches to quantum gravity. The main conceptual problems of the new approach are discussed. Using the example of a scalar geometrical toy model (incorporating gravity is beyond the scope of this proposal) and mapping this model to the dBB interpretation of the multi particle Klein-Gordon equation, it is shown how those problems can be evaded. It is argued that such a mechanism only can work consistently if the geometrical theory is formulated in the ($4 \times n$ dimensional) configuration space of the system.

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