Working paper on the geometrical interpretation of the Schrödinger equation

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We study the question whether the laws of quantum mechanics can be derived from a purely classical setting with one additional time dimension. The additional time dimension in this theory is not observable and has therefore to be integrated out. Can this integration lead to the fuzzy laws of quantum mechanics? For the simple example of a static potential and for weakly time dependent problems, it is explicitly shown that such an interpretation is indeed possible. This is done by deriving the Schrödinger equation from the equations of classical general relativity with one additional time dimension.

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Early attempts to find an alternative explanation for quantum phenomena have been discussed within the so called Bohmian mechanics [1, 2]. Similarities between a higher dimensional wave equation and the non-relativistic quantum theory have been pointed out in [3]. More recent papers, relate quantum field theories on the horizon of a black hole with a corresponding classical theory in the higher dimensional space-time [4, 5]. In the context of supergravity and string theory similar ideas involving a holographic principle and extra time dimensions obtained a lot of attention [6–8]. Later on, the general question was discussed whether quantum field theory might emerge from a chaotic classical theory with friction [9, 10]. The possibility of an additional time dimension without holographic effects has been studied in the context of high energy phenomenology [11] and from a conceptual point of view in [12]. All those ideas can be partially seen as motivation for the approach of this paper where the Schrödinger equation is derived from classical general relativity with one additional time dimension.

I. NON-RELATIVISTIC QUANTUM MECHANICS

First we will rephrase non-relativistic single particle quantum mechanics in terms of a probability density $\rho(x,t)$ and a phase $S(x,t)$. Usually this theory is expressed in terms of a wave function $\psi(x,t)$ which is a solution of the Schrödinger equation

$$i\hbar\partial_t\psi(x,t) = -\frac{\hbar^2}{2m}\partial_x^2\psi(x,t) + V(x)\psi(x,t) \quad . \qquad (1)$$

The probability for measuring a particle at the position x at the time t is $\rho(x,t) = \psi(x,t)^*\psi(x,t)$. Since $\rho(x,t)$ is the quantity which is actually related to experiment, one might want to rephrase Eq. (1) in terms of ρ. To do this one can split the wave function up into its amplitude $\sqrt{\rho(x,t)}$ and its phase $S(x,t)$: $\psi(x,t) = \sqrt{\rho(x,t)}\exp(iS(x,t)/\hbar)$ which gives

$$\partial_x\left(\rho(x,t)\frac{\partial_x S(x,t)}{m}\right) = -\partial_t\rho(x,t) \quad , \qquad (2)$$

$$\left(\partial_x^2\rho(x,t) - \frac{1}{2}\frac{(\partial_x\rho(x,t))^2}{\rho(x,t)}\right) = \tfrac{4m}{\hbar^2}\rho(x,t) \cdot \left(\partial_t S(x,t) + V(x) + \tfrac{(\partial_x S(x,t))^2}{2m}\right) \quad . \qquad (3)$$

Those two equations explain every quantum phenomena that can be described by the Schrödinger equation, since they are just a rephrasing of the latter.

The simplest application for the system of equations (2,3) is given for the static case. Here, the time derivative of the density vanishes $\partial_t\rho(x,t) = 0$ and the time derivative of the phase gives the negative of the energy $\partial_t S(x,t) = -E$. The spatial derivative of the phase is

$$\partial_x S(x,t) = p_x \quad , \qquad (4)$$

where p_x is the momentum of the wave package. This reduces Eqs. (2,3) to

$$\left(\rho'' - \frac{1}{2}\frac{\rho'^2}{\rho}\right) = \frac{4m}{\hbar^2}\rho\left(V(x) - E + \frac{p_x^2}{2m}\right) \quad . \qquad (5)$$

Note that also in the Bohmian interpretation of quantum mechanics [1, 2] the spatial derivative of the phase defines the particle momentum $\partial_x S(x,t) = p_x$ and Eq. (2) is the continuity equation.

II. CLASSICAL PHYSICS WITH TWO TIME VARIABLES - THE STATIC CASE

We will now derive the static Schrödinger equation (5) from general relativity with two time dimensions with a space dependent metric tensor. Therefore we will assume that particles move in a (2+1) dimensional manifold with the coordinates $x^A = (\bar{t}, x^\mu) = (\bar{t}, t, x)$, where capital latin indices run from 0 to 2, greek indices run from 1 to 2, $\bar{t}$ is the additional unobservable time coordinate, t is the usual time coordinate [18]. The fact that the additional time coordinate is not visible, calls for some concealing mechanism like compactification, but for the purpose of this notice it will suffice that we assume that $\bar{t}$ exists, but is not directly visible.

The idea of this new approach becomes most clear for a system that does not change in the observable time direction t and which we will therefore call a **"static"** system. In a deterministic system, the position x and velocity $v = dx/d\tau$ of the particle are known as soon the initial position x_0, the initial velocity v_0 and the propagation time τ are known. Not knowing the initial conditions one has to deal with a probability density $f(x_0, v_0, \tau)$. Without loss of generality we choose some fixed starting point (for instance the minimum of the potential x_0) which leaves a probability density $g(v_0, \tau)$. In a deterministic system the initial velocity can be calculated as a function of the actual velocity and the time $v_0 = v_0(v_x(\tau), \tau)$. Therefore, the whole system can also be described by a probability density $h(v_x(\tau))$. The probability $\rho(x)$ of finding the particle at a point x will than be obtained by an integration over the proper time variable

$$\rho(x) = \int d\tau h(v_x(\tau), \tau) \quad . \tag{6}$$

The faster a particle moves at a certain point x the smaller is the probability of finding the particle at this point $h(v_x(\tau), \tau) \sim 1/v_x(\tau)$ and hence

$$\rho(x) \sim \int d\tau \frac{1}{v_x(\tau)} = \int d\tau \frac{1}{\frac{dx}{d\tau}} \quad . \tag{7}$$

Now we will connect this intuitive argument with the metric in the higher dimensional space-time. Following the flat solution of Einsteins field equations, we take

$$\begin{aligned} d\tau^2 &= g_{AB} dx^A dx^B \\ &= A(x) d\bar{t}^2 + dt^2 - dx^2 \quad , \end{aligned} \tag{8}$$

as ansatz for the metric tensor $g_{AB}(x)$. Here, $A(x)$ encodes our ignorance of the metric concerning the additional time variable $\bar{t}$. A physical interpretation for $A(x)$ can be obtained by assuming that the particle is non-relativistic with respect to the additional time coordinate ($d\bar{t}/d\tau \gg dx/d\tau > dt/d\tau$). In this limit the differential of the proper time τ can be approximated by the differential of the additional time variable $d\tau \approx d\bar{t}\,\sqrt{A}$. Under the assumption that A is independent of $\bar{t}$ it can be pulled out of the integral in Eq. (7) and one finds

$$\rho(x) \sim A(x) \int d\bar{t} \frac{1}{\frac{dx}{d\bar{t}}} \equiv A(x) C(x) \quad . \tag{9}$$

For $\partial_x A(x) \gg \partial_x C(x)$ we can replace $A(x)$ by $k_1 \rho(x)$, where k_1 is a constant, which leads to the metric

$$g_{AB} = \begin{pmatrix} k_1\rho(x) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad . \tag{10}$$

This metric has to be a solution of the classical Einstein tensor G_{AB} coupled to matter

$$G_{AB} = R_{\mu\nu} - \frac{1}{2} g_{AB} R = k_2 T_{AB} \quad , \tag{11}$$

with the constant k_2 and the energy-momentum tensor T_{AB}. The constant k_2 is a new coupling in addition to the standard coupling of gravity G that should be dominant at small distance scales. Only one of these nine differential equations is non zero

$$\frac{\rho'^2 - 2\rho\rho''}{4\rho^2} = k_2 T_{tt} \quad . \tag{12}$$

The standard form of the tt component of the energy-momentum tensor is

$$T_{tt} = \frac{1}{\text{Vol}} \left(E_0 + \frac{p_x^2}{2m} + V(x) \right) \quad . \tag{13}$$

Here, $p_x = m v_x$ is the momentum at the point x, $V(x)$ is the potential at the point x, and Vol is the spatial volume that makes T_{tt} a density. With this energy-momentum tensor the differential equation for the density ρ is

$$\left(\rho'' - \frac{1}{2} \frac{\rho'^2}{\rho} \right) = -2k_2 \rho \left(V(x) + E_0 + \frac{p_x^2}{2m} \right) \quad . \tag{14}$$

After identifying $E = -E_0$ and $k_2 = -2m\text{Vol}/\hbar^2$, this is identical to the time independent Schrödinger equation (5) for the probability density $\rho(x)$.

III. CLASSICAL PHYSICS WITH TWO TIME VARIABLES - WEAK TIME DEPENDENCE

Now we will derive the time dependent Schrödinger equation from the two time setting. Therefore we assume a density distribution $\rho(x, t)$ which only depends weakly on the time t. In this case the arguments used in the previous discussion hold asymptotically. The straightforward generalization of the ansatz in Eq. (10) is

$$g_{\mu\nu} = \begin{pmatrix} k_1\rho(x,t) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad . \tag{15}$$

This leaves equation (12) unchanged, but the xx component now reads

$$\frac{\dot{\rho}^2 - 2\rho\ddot{\rho}}{4\rho^2} = k_2 T_{xx} \quad . \tag{16}$$

Since the time derivatives of ρ in Eq. (16) are all of second order (either $\sim \ddot{\rho}(x,t)|_{t=0}$, or $\sim \dot{\rho}^2(x,t)|_{t=0}$), they can be neglected in a linear approximation for the time evolution.

The time dependent energy-momentum tensor needs a more general treatment than in the static case Eq. (13). The Hamilton-Jakobi definition of the energy momentum tensor of a free particle in curved space-time is

$$T^A_{\ B} = \frac{1}{Vm}(\partial^A S)(\partial_B S), \quad \text{with} \quad T^A_{\ A} = \frac{m}{\mathrm{Vol}} \quad . \tag{17}$$

Here, S is Hamilton's principal function [13, 14], V is the normalizing volume, and m is the invariant mass of the particle. It has to be a covariant conserved quantity such that

$$\begin{aligned} 0 &= \nabla^B (T_{AB}) = \nabla^B \left(g_{AC} T^C_{\ B}\right) & (18)\\ &= \frac{1}{Vm}\nabla^B \left(g_{AC}(\partial^C S)(\partial_B S)\right) \quad . \end{aligned}$$

Now we take the 0 component of this equation

$$\begin{aligned} 0 &= \int \mathrm{d}\bar{\mathrm{t}}\ \nabla^{\mathrm{B}} \left(\mathrm{g_{0C}}(\partial^{\mathrm{C}}\mathrm{S})(\partial_{\mathrm{B}}\mathrm{S})\right) & (19)\\ &= k_1 \int \mathrm{d}\bar{\mathrm{t}}\ \partial^\mu \left(\sqrt{\rho}(\partial_0 \mathrm{S})(\partial_\mu \mathrm{S})\right) \quad , \end{aligned}$$

where the boundary term was dropped. By using the definition (17) the partial derivative $(\partial_0 S)$ which appears in Eq. (19) can be rewritten

$$(\partial_0 S) = \pm k_1 \sqrt{M^2 - (\partial^\mu S)(\partial_\mu S)}\sqrt{\rho} \quad . \tag{20}$$

We assume that the squared four-momentum $p^\mu p_\mu = (\partial^\mu S)(\partial_\mu S) = m^2$ does depend on space (x) and time (t). As long as $m^2 \neq M^2$ this assumption allows us to define the constant k_3 by $(\partial_0 S) = k_3\sqrt{\rho}$, which simplifies Eq. (19) to

$$0 = k_1 k_3 \int \mathrm{d}\bar{\mathrm{t}}\ \partial^\mu \left(\rho(\partial_\mu \mathrm{S})\right) \quad . \tag{21}$$

This is already one of the two time dependent Schrödinger equations (2). The tt component of the energy-momentum tensor consists of a free part and an interaction part

$$T_{I\ tt} = T^{\text{free}}_{I\ tt} + T^{\text{int}}_{I\ tt} \quad . \tag{22}$$

In the local rest frame of a free particle the energy-momentum tensor is

$$T^{\text{free}}_{I\ \mu\nu} = \mathcal{H}_{\prime} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad , \tag{23}$$

with $\mathcal{H}_{\prime}$ as the Hamiltonian density in the rest frame. A Lorentz boost to a moving frame with a particle velocity p_x/m (including the contraction of the density function $\mathcal{H}_{\prime} \to \mathcal{H} = H/\mathrm{Vol}$) gives

$$T^{\text{free}}_{I\ \mu\nu} = \frac{\mathcal{H}}{\sqrt{1-p^2/m^2}} \begin{pmatrix} 1 & -p_x/m \\ -p_x/m & p_x^2/m^2 \end{pmatrix} \quad . \tag{24}$$

In an expansion for small momenta the tt component of this tensor is

$$T^{\text{free}}_{I\ tt} \approx \frac{1}{\mathrm{Vol}}\left(H + \frac{p_x^2}{2m}\right) = \frac{1}{\mathrm{Vol}}\left(-\partial_t S + \frac{(\partial_x S)^2}{2m}\right) \quad . \tag{25}$$

where the Hamilton-Jacobi principal function $-\partial_t S = H$ and $\partial_x S = p_x$ was used. In the background field approximation all electromagnetic interactions are assumed to act in such a way that the tt component of the energy-momentum tensor can be described by a single potential $V(x,t)$

$$T^{\text{int}}_{I\ tt} = \frac{V(x,t)}{\mathrm{Vol}} \quad . \tag{26}$$

Combining Eqs. (22,25,26) in Eq. (12) gives

$$\left(\rho'' - \frac{\rho'^2}{2\rho}\right) = \frac{-2k_2}{\mathrm{Vol}}\rho\left(-\partial_t S + \frac{S'^2}{2m} + V(x,t)\right) \quad . \tag{27}$$

After identifying $S \to -S$ and $k_2 = -2m\mathrm{Vol}/\hbar^2$, this is the second time dependent Schrödinger equation (3). The justification of first identification will be discussed in the following section With the same replacement ($S \to -S$), the first time dependent Schrödinger equation (2) is identical to the equation for the conservation of probability Eq. (21). Therefore we conclude that the complete time dependent Schrödinger equation (5) can be derived from the higher dimensional ansatz (15).

IV. CRITICAL POINTS

We will now try to point out some possible criticism on the idea and the derivation presented here.

Why is the second time not visible?
This question is not addressed in this paper, but the easiest way to explain this would be a compactification such that $\bar{t} = \bar{t} + \bar{T}$. As long the "radius" is just short enough this would only lead to violations of Lorentz invariance on the small scale $\bar{T}$.

What about spin?
The discrete spin of elementary particles might be interpreted as rotations with respect to $\bar{t}$. Since the particle has to be in its original state after a time laps $\bar{t} = \bar{T}$, only discrete angular momenta would be possible. However, it has to be checked whether this intuitive interpretation is really consistent with the concept of a statistical average over the classical extra time. Up to now there is no explanation for the half integer spin of the fermionic particles.

Superposition of states?
The superposition of time independent states with different energies results in a total state which is time dependent and should therefore correctly be described by Eqs. (21, 27). Still, it has to checked that this works for the superposition of any operator definable in standard quantum mechanics.

Which parameters had to be engineered?
In the static case it was assumed that $\partial_x A(x) \gg \partial_x C(x)$ without further justification. In the time dependent case, $g_{\overline{tt}} = k_1 \rho(x,t)$ was used as an ansatz. The coupling to the energy-momentum tensor was adjusted to be $k_2 = -2m\mathrm{Vol}/\hbar^2$, but the constant parameter k_1 canceled out of the final equation. Therefore, even a negative choice of k_1 would lead to the same result. For the time dependent case it was assumed $S \to -S$ (or $E = -E_0$ for the static case). This replacement might be justified by using a so called extended canonical transformation, of the standard Hamilton-Jacobi formalism. An other way to get around the matching of $S \to -S$ would be to define the Schrödinger wave function as $\psi(x,t) = \sqrt{\rho(x,t)} \exp(-iS(x,t)/\hbar)$. Although, the replacements can be justified, the matter of negative energy states is a key ingredient of standard quantum mechanics, and has to be checked with highest caution.

Is the energy-momentum tensor unique?
No, although the actual choice is well motivated other choices are possible. For an originally relativistic energy-momentum tensor, the ground state energy E_0 should also contain the rest mass, which is not the case in the standard quantum mechanics. The standard Planck coupling ($G = 1/m_{Pl}^2$) of the energy-momentum tensor was neglected here due to the smallness of $1/m_{Pl}^2$ at microscopic ($\sim$1/eV) distance scales.

Is this also valid for gravitational potentials?
No, a gravitational potential has to appear on the curvature side of Einstein's equation. However, it turns out that one can still obtain an equation of the form of Eq. (14) but it involves T_{xx} as well. Studying quantum states in a gravitational potential could therefore help to distinguish this theory from the standard quantum mechanics.

Is causality and probability violated?
Although the scheme is by construction Lorentz invariant in the higher dimensional space-time, it can appear to violate causality in the integrated lower dimensional version. This is inevitable since also wave function in the Schrödinger formulation, which is derived here, is non causal. The probability is not violated, since the whole derivation is based on probability conserving arguments. Here particle propagates with respect to both time dimensions and hence arguments that were made for brane world scenarios [15] do not apply here.

Multi particle states, Pauli principle, QFT, gauge symmetries ...?
Working on that ☺. It is very tempting to combine this ansatz with the idea of Kaluza and Klein [3, 16, 17].

V. SUMMARY AND DISCUSSION

Now a short summary of the obtained results will be given. The classical motion of a particle with respect to an additional unobservable time dimension was studied. We showed that for this setting the Einstein equations lead under simple assumptions in the limit of non-relativistic velocities to a differential equation (14) for the probability distribution. This differential equation can be identified with the static Schrödinger equation (5).
For the non-static case we expressed the energy-momentum tensor Eq. (22) in terms of the Hamilton's classical principal function. In the limit of a weakly time dependent density ρ and small velocities S'/m, it is found that the continuity equation (21) and equation (27) can be identified with the time dependent Schrödinger equation (1).
However, there are many open questions which we tried to raise and discuss in the last section. In order to learn whether this result is a pure coincidence or has deeper meaning one has to answer those questions and study generalizations of this simple setting. For instance the relativistic equations of quantum field theory should be derivable in a similar way.

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[1] David Bohm. A suggested interpretation of the quantum theory in terms of "hidden" variables i. *Phys. Rev.*, 85:166–179, 1951.

[2] David Bohm. A suggested interpretation of the quantum theory in terms of "hidden" variables ii. *Phys. Rev.*, 85:180–193, 1951.

[3] Oskar Klein. *Z. f. Physik*, 37:895, 1926.

[4] Gerard 't Hooft. Dimensional reduction in quantum gravity. 1993.

[5] Leonard Susskind. The world as a hologram. *J. Math. Phys.*, 36:6377–6396, 1995.

[6] Juan M. Maldacena. The large n limit of superconformal field theories and supergravity. *Adv. Theor. Math. Phys.*, 2:231–252, 1998.

[7] Edward Witten. Anti-de sitter space and holography. *Adv. Theor. Math. Phys.*, 2:253–291, 1998.

[8] C. M. Hull. Duality and strings, space and time. 1999.

[9] Gerard 't Hooft. Quantum gravity as a dissipative deterministic system. *Class. Quant. Grav.*, 16:3263–3279, 1999.

[10] T. S. Biro, Sergei G. Matinyan, and B. Muller. Chaotic quantization of classical gauge fields. *Found. Phys. Lett.*,

14:471–485, 2001.
[11] Itzhak Bars. The standard model of particles and forces in the framework of 2t-physics. *Phys. Rev.*, D74:085019, 2006.
[12] S. Weinstein. Philosophy pulls strings. *Physicsworld*, September:18–19, 2007.
[13] W. Hamilton. On a general method of expressing the paths of light, and of the planets, by the coefficients of a characteristic function. *Dublin University Review pp.*, pages 795–826, 1833.
[14] L.D. Landau and L.M Lifshitz. Mechanics. *Elsevier, Amsterdam*, 1975.
[15] G. R. Dvali, Gregory Gabadadze, and Goran Senjanovic. Constraints on extra time dimensions. 1999.
[16] Theodor Kaluza. *Sitzungsber. Preuss. Akad. Wiss. Phys. math.*, Klasse:996, 1921.
[17] Oskar Klein. *Nature*, 118:516, 1926.
[18] The generalization to three spatial dimensions is trivial.