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FIM 8440 ⋆ Mecánica Cuántica Avanzada

Tarea N º 3

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<u>*Resolución de Problemas*</u>

1. Operadores de creación y destrucción

En clases se probó que:

$$\langle f_k{}^*, f_{k'} \rangle \equiv \int d^3x \; f_k{}^*(x)\, i\overleftrightarrow{\partial_0}\, f_{k'}(x) = \delta^{(3)}(\vec{k} - \vec{k'}) \tag{1}$$

Ahora, realicemos el siguiente cálculo:

$$\begin{aligned}
\langle f_k{}^*, f_{k'}{}^* \rangle &= \int d^3x \; f_k{}^*(x)\, i\overleftrightarrow{\partial_0}\, f_{k'}{}^*(x) && (2)\\
&= \frac{i}{2(2\pi)^3\sqrt{\omega_k\omega_{k'}}} \int d^3x \; e^{ikx}e^{ik'x}(ik_0 - ik'_0) && (3)\\
&= \frac{(k'_0 - k_0)}{2(2\pi)^3\sqrt{\omega_k\omega_{k'}}} \int d^3x \; e^{ikx}e^{ik'x} && (4)\\
&= \frac{(k'_0 - k_0)\, e^{i(k_0+k'_0)t}}{2(2\pi)^3\sqrt{\omega_k\omega_{k'}}} \int d^3x \; e^{-i(\vec{k}+\vec{k'})\cdot\vec{x}} && (5)
\end{aligned}$$

Haciendo el cambio de variable $\vec{x} \to -\vec{x}$,

$$\begin{aligned}
&= \frac{(k'_0 - k_0)\, e^{i(k_0+k'_0)t}}{2(2\pi)^3\sqrt{\omega_k\omega_{k'}}} \int d^3x \; e^{i(\vec{k}+\vec{k'})\cdot\vec{x}} && (6)\\
&= \frac{(k'_0 - k_0)\, e^{i(k_0+k'_0)t}}{2(2\pi)^3\sqrt{\omega_k\omega_{k'}}} \delta^{(3)}(\vec{k} + \vec{k'}) && (7)
\end{aligned}$$

Como $\omega_k = \sqrt{\vec{k}^2 + m^2} = k_0$, entonces la distribución delta de Dirac obliga a que:

$$\vec{k} = -\vec{k'} \quad \longrightarrow \quad k_0 = \omega_k = \omega_{k'} = k'_0 \tag{8}$$

Por tanto,

$$\langle f_k{}^*, f_{k'}{}^* \rangle = 0 \tag{9}$$

El resultado es análogo para

$$\langle f_k, f_{k'} \rangle = 0 \tag{10}$$

Notemos que:

$$\int d^3x \; f_{k'}{}^*(x)\, i\overleftrightarrow{\partial_0}\, \phi(x) \;=\; \int \frac{d^3k}{\sqrt{(2\pi)^3\, 2\omega_k}} \Big(\mathbf{a}(k)\langle f_{k'}{}^*, f_k \rangle + \mathbf{a}^\dagger(k)\langle f_{k'}{}^*, f_k{}^* \rangle \Big)$$

y, usando las relaciones (1) y (9),

$$= \int \frac{d^3k}{\sqrt{(2\pi)^3\, 2\omega_k}}\; \mathbf{a}(k)\, \delta^{(3)}(\vec{k'} - \vec{k})$$
$$= \frac{\mathbf{a}(k')}{\sqrt{(2\pi)^3\, 2\omega_{k'}}}$$

Por tanto,

$$\boxed{\mathbf{a}(k) = \int d^3x\ \sqrt{(2\pi)^3\, 2\omega_k}\, {f_k}^*(x)\, i\overleftrightarrow{\partial_0}\, \phi(x)} \tag{11}$$

De manera análoga,

$$\int d^3x\ \phi(x)\, i\overleftrightarrow{\partial_0}\, f_{k'} \ = \ \int \frac{d^3k}{\sqrt{(2\pi)^3\, 2\omega_k}}\ \Big(\mathbf{a}(k)\langle f_k, f_{k'}\rangle + \mathbf{a}^\dagger(k)\langle {f_k}^*, f_{k'}\rangle\Big)$$

y, usando las relaciones (1) y (10),

$$= \int \frac{d^3k}{\sqrt{(2\pi)^3\, 2\omega_k}}\; \mathbf{a}^\dagger(k)\, \delta^{(3)}(\vec{k} - \vec{k'})$$
$$= \frac{\mathbf{a}^\dagger(k')}{\sqrt{(2\pi)^3\, 2\omega_{k'}}}$$

Por tanto,

$$\boxed{\mathbf{a}^\dagger(k) = \int d^3x\ \sqrt{(2\pi)^3\, 2\omega_k}\, \phi(x)\, i\overleftrightarrow{\partial_0}\, f_k(x)} \tag{12}$$

2. Relaciones de conmutación

En la presente sección, emplearemos reiteradamente el siguiente lema:

Lema 1.

$$[\phi(x)\overleftrightarrow{\partial_0}A(x), B(y)\overleftrightarrow{\partial_0}\phi(y)] = B(y)\, i\overleftrightarrow{\partial_0}\, A(x)\, \delta^{(3)}(\vec{x} - \vec{y}) \tag{13}$$

cuya demostración puede revisarse en el **Anexo I**. De las expresiones (11) y (12),

$$[\mathbf{a}(k_1), \mathbf{a}^\dagger(k_2)] \ = \ \iint d^3x_1\, d^3x_2\ (2\pi)^3(4\omega_{k_1}\omega_{k_2})^{1/2}[{f_{k_1}}^*(x_1)\, i\overleftrightarrow{\partial_0}\, \phi(x_1), \phi(x_2)\, i\overleftrightarrow{\partial_0}\, f_{k_2}(x_2)] \tag{14}$$

$$= \iint d^3x_1\, d^3x_2\ (2\pi)^3(4\omega_{k_1}\omega_{k_2})^{1/2}[\phi(x_2)\, \overleftrightarrow{\partial_0}\, f_{k_2}(x_2), {f_{k_1}}^*(x_1)\, \overleftrightarrow{\partial_0}\, \phi(x_1)] \tag{15}$$

usando el lema 1,

$$= \iint d^3x_1\, d^3x_2\ (2\pi)^3 (4\omega_{k_1}\omega_{k_2})^{1/2}\, f_{k_1}{}^*(x_1)\, i\overleftrightarrow{\partial_0}\, f_{k_2}(x_2)\, \delta^{(3)}(\vec{x}_1 - \vec{x}_2) \tag{16}$$

$$= (2\pi)^3\, (4\omega_{k_1}\omega_{k_2})^{1/2} \int d^3x_1\ f_{k_1}{}^*(x_1)\, i\overleftrightarrow{\partial_0}\, f_{k_2}(x_1) \tag{17}$$

ahora, usando la relación de ortonormalidad (1),

$$= (2\pi)^3\, 2\omega_{k_1}\, \delta^{(3)}(\vec{k}_1 - \vec{k}_2) \tag{18}$$

ya que la delta de Dirac forzará que $\omega_{k_1} = \omega_{k_2}$, por lo que se puede reemplazar por cualquiera de dichas expresiones.

Por otra parte,

$$\begin{aligned}
[\mathbf{a}(k_1), \mathbf{a}(k_2)] &= \iint d^3x_1\, d^3x_2\ (2\pi)^3 (4\omega_{k_1}\omega_{k_2})^{1/2} [f_{k_1}{}^*(x_1)\, i\overleftrightarrow{\partial_0}\, \phi(x_1), f_{k_2}{}^*(x_2)\, i\overleftrightarrow{\partial_0}\, \phi(x_2)] \\
&= \iint d^3x_1\, d^3x_2\ (2\pi)^3 (4\omega_{k_1}\omega_{k_2})^{1/2} [f_{k_2}{}^*(x_2)\, \overleftrightarrow{\partial_0}\, \phi(x_2), f_{k_1}{}^*(x_1)\, \overleftrightarrow{\partial_0}\, \phi(x_1)] \\
&= -\iint d^3x_1\, d^3x_2\ (2\pi)^3 (4\omega_{k_1}\omega_{k_2})^{1/2} [\phi(x_2)\, \overleftrightarrow{\partial_0}\, f_{k_2}{}^*(x_2), f_{k_1}{}^*(x_1)\, \overleftrightarrow{\partial_0}\, \phi(x_1)]
\end{aligned}$$

pero, por el lema 1,

$$\begin{aligned}
&= -\iint d^3x_1\, d^3x_2\ (2\pi)^3 (4\omega_{k_1}\omega_{k_2})^{1/2}\, f_{k_1}{}^*(x_1)\, i\overleftrightarrow{\partial_0}\, f_{k_2}{}^*(x_2)\, \delta^{(3)}(\vec{x_1} - \vec{x}_2) \\
&= -(2\pi)^3 (4\omega_{k_1}\omega_{k_2})^{1/2} \int d^3x_1\ f_{k_1}{}^*(x_1)\, i\overleftrightarrow{\partial_0}\, f_{k_2}{}^*(x_1) \\
&= -(2\pi)^3 (4\omega_{k_1}\omega_{k_2})^{1/2}\, \langle f_{k_1}{}^*, f_{k_2}{}^* \rangle
\end{aligned}$$

usando la propiedad (9),

$$= 0$$

De manera análoga,

$$\begin{aligned}
[\mathbf{a}^\dagger(k_1), \mathbf{a}^\dagger(k_2)] &= \iint d^3x_1\, d^3x_2\ (2\pi)^3 (4\omega_{k_1}\omega_{k_2})^{1/2} [\phi(x_1)\, i\overleftrightarrow{\partial_0}\, f_{k_1}(x_1), \phi(x_2)\, i\overleftrightarrow{\partial_0}\, f_{k_2}(x_2)] \\
&= \iint d^3x_1\, d^3x_2\ (2\pi)^3 (4\omega_{k_1}\omega_{k_2})^{1/2} [\phi(x_2)\, \overleftrightarrow{\partial_0}\, f_{k_2}(x_2), \phi(x_1)\, \overleftrightarrow{\partial_0}\, f_{k_1}(x_1)] \\
&= -\iint d^3x_1\, d^3x_2\ (2\pi)^3 (4\omega_{k_1}\omega_{k_2})^{1/2} [\phi(x_2)\, \overleftrightarrow{\partial_0}\, f_{k_2}(x_2), f_{k_1}(x_1)\, \overleftrightarrow{\partial_0}\, \phi(x_1)]
\end{aligned}$$

pero, por el lema 1,

$$
\begin{aligned}
&= -\iint d^3x_1\, d^3x_2\, (2\pi)^3 (4\omega_{k_1}\omega_{k_2})^{1/2}\, f_{k_1}(x_1)\, i\overleftrightarrow{\partial_0}\, f_{k_2}(x_2)\, \delta^{(3)}(\vec{x_1} - \vec{x}_2) \\
&= -(2\pi)^3 (4\omega_{k_1}\omega_{k_2})^{1/2} \int d^3x_1\, f_{k_1}(x_1)\, i\overleftrightarrow{\partial_0}\, f_{k_2}(x_1) \\
&= -(2\pi)^3 (4\omega_{k_1}\omega_{k_2})^{1/2} \langle f_{k_1}, f_{k_2} \rangle
\end{aligned}
$$

pero ahora empleando la propiedad (10)

$$
= 0
$$

Finalmente,

$$
\boxed{[\mathbf{a}(k_1), \mathbf{a}(k_2)] = [\mathbf{a}^\dagger(k_1), \mathbf{a}^\dagger(k_2)] = 0} \tag{19}
$$

$$
\boxed{[\mathbf{a}(k_1), \mathbf{a}^\dagger(k_2)] = (2\pi)^3\, 2\omega_{k_1}\, \delta^{(3)}(\vec{k}_1 - \vec{k}_2)} \tag{20}
$$

Anexo I: Demostración del lema 1

Usamos la siguiente notación:

$$A\partial_0 B \equiv A(\partial_0 B) \qquad \wedge \qquad A\overleftrightarrow{\partial_0} B \equiv A(\partial_0 B) - B(\partial_0 A) = -B\overleftrightarrow{\partial_0} A$$

$$[\phi(x)\overleftrightarrow{\partial_0} A(x), B(y)\overleftrightarrow{\partial_0}\phi(y)] = [\phi(x)\partial_0 A(x) - \pi(x)A(x), B(y)\pi(y) - \phi(y)\partial_0 B(y)] \tag{21}$$

$$= [\phi(x)\partial_0 A(x), B(y)\pi(y)] - [\pi(x)A(x), B(y)\pi(y)] - [\phi(x)\partial_0 A(x), \phi(y)\partial_0 B(y)] + [\pi(x)A(x), \phi(y)\partial_0 B(y)] \tag{22}$$

$$= B(y)\partial_0 A(x)[\phi(x), \pi(y)] - A(x)B(y)[\pi(x), \pi(y)] - \partial_0 B(y)\partial_0 A(x)[\phi(x), \phi(y)] + A(x)\partial_0 B(y)[\pi(x), \phi(y)] \tag{23}$$

$$= iB(y)\partial_0 A(x)\, \delta^{(3)}(\vec{x} - \vec{y}) - iA(x)\partial_0 B(y)\, \delta^{(3)}(\vec{x} - \vec{y}) \tag{24}$$

$$= B(y)\, i\overleftrightarrow{\partial_0}\, A(x)\, \delta^{(3)}(\vec{x} - \vec{y}) \tag{25}$$