PONTIFICIA UNIVERSIDAD CATÓLICA DE CHILE
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FIM 8440 $\star$ Mecánica Cuántica Avanzada

Tarea N º 4 (a)

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Resolución de Problemas

1. Relaciones de anti-conmutación

Recordemos que:

$$\psi(x) = \int \frac{d^3k}{(2\pi)^3}\frac{m}{k_0}\sum_\alpha \left(b_\alpha(k)u^{(\alpha)}(k)e^{-ikx} + {d_\alpha}^\dagger(k)v^{(\alpha)}e^{ikx}\right) \tag{1}$$

De esta forma, y utilizando la notación de spinores adjuntos $\bar{A} \equiv A^\dagger\gamma^0$,

$$\psi^\dagger(x) = \int \frac{d^3k}{(2\pi)^3}\frac{m}{k_0}\sum_\alpha \left({b_\alpha}^\dagger(k)\bar{u}^{(\alpha)}(k)\gamma^0 e^{ikx} + d_\alpha(k)\bar{v}^{(\alpha)}(k)\gamma^0 e^{-ikx}\right) \tag{2}$$

donde $(\gamma^0)^2 = \mathbb{1}$. Note que, en notación matricial, se cumple que:

$$(u^\dagger)_i = (\bar{u}\gamma^0)_i = (\bar{u})_k(\gamma^0)_{ki}$$

donde los índices repetidos están sumados.

1.1. Primera relación

$$\begin{aligned}
\{\psi_i(\mathbf{x},t), {\psi^\dagger}_j(\mathbf{x}',t)\} &= \sum_{\alpha,\alpha'}\iint \frac{d^3k}{(2\pi)^3}\frac{d^3k'}{(2\pi)^3}\frac{m^2}{k_0{k_0}'}\Bigg[\left(b_\alpha(k){u_i}^{(\alpha)}(k)e^{-ikx} + {d_\alpha}^\dagger(k){v_i}^{(\alpha)}e^{ikx}\right) \\
&\left({b_{\alpha'}}^\dagger(k'){\bar{u}_l}^{(\alpha')}(k'){\gamma^0}_{lj}\, e^{ik'x'} + d_{\alpha'}(k'){\bar{v}_l}^{(\alpha')}(k'){\gamma^0}_{lj}\, e^{-ik'x'}\right) + \\
&\left({b_{\alpha'}}^\dagger(k'){\bar{u}_l}^{(\alpha')}(k'){\gamma^0}_{lj}\, e^{ik'x'} + d_{\alpha'}(k'){\bar{v}_l}^{(\alpha')}(k'){\gamma^0}_{lj}\, e^{-ik'x'}\right) \\
&\left(b_\alpha(k){u_i}^{(\alpha)}(k)e^{-ikx} + {d_\alpha}^\dagger(k){v_i}^{(\alpha)}e^{ikx}\right)\Bigg] \\
&= \sum_{\alpha,\alpha'}\iint \frac{d^3k}{(2\pi)^3}\frac{d^3k'}{(2\pi)^3}\frac{m^2}{k_0{k_0}'}\Bigg[{u_i}^{(\alpha)}(k){\bar{u}_l}^{(\alpha')}(k'){\gamma^0}_{lj}\, e^{-i(kx-k'x')}\{b_\alpha(k), {b_{\alpha'}}^\dagger(k')\} \\
&+ {v_i}^{(\alpha)}(k){\bar{v}_l}^{(\alpha')}(k'){\gamma^0}_{lj}\, e^{-i(k'x'-kx)}\{{d_\alpha}^\dagger(k), d_{\alpha'}(k')\} \\
&+ {u_i}^{(\alpha)}(k){\bar{v}_l}^{(\alpha')}(k'){\gamma^0}_{lj}\, e^{-i(k'x'+kx)}\{b_\alpha(k), d_{\alpha'}(k')\} \\
&+ {v_i}^{(\alpha)}(k){\bar{u}_l}^{(\alpha')}(k'){\gamma^0}_{lj}\, e^{i(k'x'+kx)}\{{d_\alpha}^\dagger(k), {b_{\alpha'}}^\dagger(k')\}\Bigg]
\end{aligned}$$

Reemplazando las relaciones de anti-conmutación de los operadores,

$$
\begin{aligned}
= \; & \sum_\alpha \iint \frac{d^3k}{(2\pi)^3}\frac{m}{k_0}\Bigg[{u_i}^{(\alpha)}(k){\bar{u}_l}^{(\alpha)}(k){\gamma^0}_{lj}\, e^{-ik(x-x')} \\
& + {v_i}^{(\alpha)}(k){\bar{v}_l}^{(\alpha)}(k){\gamma^0}_{lj}\, e^{-ik(x'-x)} \Bigg]
\end{aligned}
$$

Utilizando la completitud de los spinores,

$$
\begin{aligned}
&= \iint \frac{d^3k}{(2\pi)^3}\frac{m}{k_0}\left[(\gamma\cdot k+m)_{il}\, {\gamma^0}_{lj}\, e^{-ik(x-x')} + (\gamma\cdot k-m)_{il}\, {\gamma^0}_{lj}\, e^{-ik(x'-x)} \right] \\
&= \iint \frac{d^3k}{(2\pi)^3}\frac{m}{k_0}\left[\Big((\gamma\cdot k+m)\gamma^0\Big)_{ij} e^{-ik(x-x')} + \Big((\gamma\cdot k-m)\gamma^0\Big)_{ij} e^{-ik(x'-x)} \right]
\end{aligned}
$$

Pero $k(x-x') = k_0(t-t) - \mathbf{k}\cdot(\mathbf{x}-\mathbf{x}') = \mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')$. Así,

$$
= \iint \frac{d^3k}{(2\pi)^3}\frac{m}{k_0}\left[\Big((\gamma\cdot k+m)\gamma^0\Big)_{ij} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} + \Big((\gamma\cdot k-m)\gamma^0\Big)_{ij} e^{-i\mathbf{k}(\mathbf{x}-\mathbf{x}')} \right]
$$

Para la segunda integral, cambiamos $\mathbf{k} \mapsto -\mathbf{k}$. La integral mantiene su valor, pero analicemos lo que ocurre con los términos que acompañan a la exponencial:

$$
(\gamma^0 k_0 - \gamma^i k_i + m)\gamma^0 + (\gamma^0 k_0 + \gamma^i k_i - m)\gamma^0 = 2k_0
$$

Así,

$$
\begin{aligned}
\{\psi_i(\mathbf{x},t), {\psi^\dagger}_j(\mathbf{x}',t)\} &= \int \frac{d^3k}{(2\pi)^3}\, e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}\,\delta_{ij} \\
&= \delta^{(3)}(\mathbf{x}-\mathbf{x}')\,\delta_{ij}
\end{aligned}
$$

Finalmente,

$$
\boxed{\{\psi_i(\mathbf{x},t), {\psi^\dagger}_j(\mathbf{x}',t)\} = \delta^{(3)}(\mathbf{x}-\mathbf{x}')\,\delta_{ij}} \tag{3}
$$

1.2. Segunda relación

Procedemos de manera análoga,

$$
\begin{aligned}
\{\psi_i(\mathbf{x},t),\psi_j(\mathbf{x}',t)\} &= \sum_{\alpha,\alpha'}\iint \frac{d^3k}{(2\pi)^3}\frac{d^3k'}{(2\pi)^3}\frac{m^2}{k_0k_0{}'}\Bigg[u_i{}^{(\alpha)}(k)u_j{}^{(\alpha')}(k')\,e^{-i(kx+k'x')}\{b_\alpha(k),b_{\alpha'}(k')\} \\
&+ v_i{}^{(\alpha)}(k)v_j{}^{\alpha'}(k')\,e^{i(k'x'+kx)}\{d_\alpha{}^\dagger(k),d_{\alpha'}{}^\dagger(k')\} \\
&+ u_i{}^{(\alpha)}(k)v_j{}^{(\alpha')}(k')\,e^{-i(kx-k'x')}\{b_\alpha(k),d_{\alpha'}{}^\dagger(k')\} \\
&+ v_i{}^{(\alpha)}(k)u_j{}^{(\alpha')}(k')\,e^{i(kx-k'x')}\{d_\alpha{}^\dagger(k),b_{\alpha'}(k')\}\Bigg]
\end{aligned}
$$

Reemplazando las relaciones de anti-conmutación de los operadores,

$$
= \quad 0
$$

Así,

$$
\boxed{\{\psi_i(\mathbf{x},t),\psi_j(\mathbf{x}',t)\} = 0} \tag{4}
$$

1.3. Tercera relación

Nuevamente procedemos de la misma forma que en la primera relación,

$$
\begin{aligned}
\{\psi^\dagger{}_i(\mathbf{x},t),\psi^\dagger{}_j(\mathbf{x}',t)\} &= \sum_{\alpha,\alpha'}\iint \frac{d^3k}{(2\pi)^3}\frac{d^3k'}{(2\pi)^3}\frac{m^2}{k_0k_0{}'}\Bigg[\bar{u}_l{}^{(\alpha)}\gamma^0{}_{li}(k)\bar{u}_m^{(\alpha')}(k')\gamma^0{}_{mj}\,e^{i(kx+k'x')}\{b_\alpha{}^\dagger(k),b_{\alpha'}{}^\dagger(k')\} \\
&+ \bar{v}_l{}^{(\alpha)}\gamma^0{}_{li}(k)\bar{v}_m^{(\alpha')}(k')\gamma^0{}_{mj}\,e^{-i(k'x'+kx)}\{d_\alpha(k),d_{\alpha'}(k')\} \\
&+ \bar{u}_l{}^{(\alpha)}\gamma^0{}_{li}(k)\bar{v}_m^{(\alpha')}(k')\gamma^0{}_{mj}\,e^{i(kx-k'x')}\{b_\alpha{}^\dagger(k),d_{\alpha'}(k')\} \\
&+ \bar{v}_l{}^{(\alpha)}\gamma^0{}_{li}(k)\bar{u}_m^{(\alpha')}(k')\gamma^0{}_{mj}\,e^{i(k'x'-kx)}\{d_\alpha(k),b_{\alpha'}{}^\dagger(k')\}\Bigg]
\end{aligned}
$$

Reemplazando las relaciones de anti-conmutación de los operadores,

$$
= \quad 0
$$

Finalmente,

$$
\boxed{\{\psi^\dagger{}_i(\mathbf{x},t),\psi^\dagger{}_j(\mathbf{x}',t)\} = 0} \tag{5}
$$