2 Theory

We observe a rotary disk D of the radius *r*, around which a thin thread has been wound according to Fig. 1. The thread is connected to a mass *m* via a pulley R. The disk is held at rest by the pin T of the magnet B. After closing the switch S, a current flows from the power supply *U* through the coil of the magnet. The holding pin T is pulled back by the resulting magnetic field, thereby unlocking the disc. The falling mass *m* then causes an accelerated rotation of the disk about the rotary axis H.

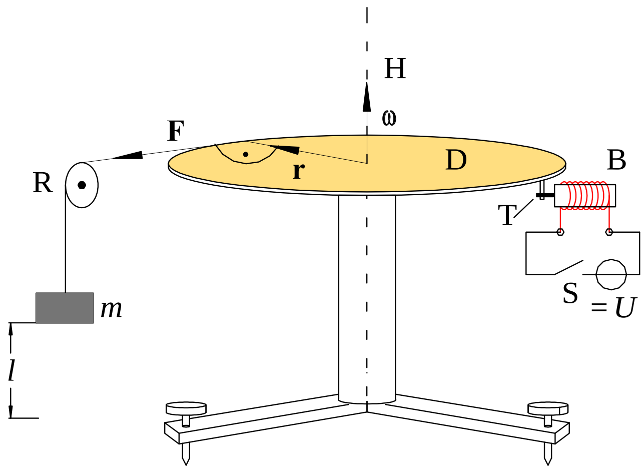

Fig. 1: Rotary disk for measuring moments of inertia. Refer to the text for labels.

Now we require an equation by means of which we can calculate the moment of inertia I_D of the rotary disk from known or measurable quantities.

For this purpose we first set up the equation of motion for the rotation of the rotary disk. It is very simple in this case: the rotary disk has the angular acceleration d**ω**/d*t* due to the rotational moment **r** × **F**. In analogy to NEWTON's law **F** = *m* **a** we thus obtain:

(1) $$\mathbf{r} \times \mathbf{F} = I_D \frac{\mathrm{d}\,\boldsymbol{\omega}}{\mathrm{d}t}$$

Then it follows from the chosen geometry (**r** ⊥ **F**) for the absolute values:

(2) $$F = \frac{I_D}{r} \frac{\mathrm{d}\omega}{\mathrm{d}\,t}$$

In this equation we have to replace *F* and *dω/dt* by known or measurable quantities. In order to find an expression for *dω/dt*, we first observe the motion of the mass *m*. If time *t* is needed for falling a distance *l*, we obtain for its acceleration *a*:

(3) $$a = \frac{2l}{t^2}$$

Because *m* and the rotary disk are connected via the thread, this must also be the tangential acceleration of a mass point on the edge of the rotary disk. Based on the well-known relationship between tangential and angular acceleration with Eg (3), we thus obtain for such a point:

(4) $$\frac{d\omega}{dt} = \frac{a}{r} = \frac{1}{r}\frac{2l}{t^2}$$

Inserting Eq. (4) into Eq. (2) yields:

(5) $$F = I_D \frac{2l}{r^2 t^2}$$

Since F cannot be measured directly, we need a relationship between F and measurable quantities. For this we look at the net force acting on the set-up. The accelerating force of gravity $G = mg$ (g: gravitational acceleration) must accelerate the mass m, overcome friction forces at pulley R and rotary disk D, and set the pulley and rotary disk into an accelerated rotation. For this the following forces are necessary:

F_m : Accelerating force for m
F_{RR} : Frictional force at the pulley
F_R : Accelerating force for the pulley
F_{RD}: Frictional force at the rotary disk
F: Accelerating force for the rotary disk

Thus we obtain:

(6) $$mg = F_m + F_{RR} + F_R + F_{RD} + F$$

The force which accelerates m, $F_m = ma$, is therefore considerably smaller than the force of gravity $G = mg$.

To simplify matters we now assume that the force of friction and the accelerating force are replaced by *one* force acting on the pulley, which is necessary for the translational acceleration of an equivalent mass m_e (here: $m_e \approx 2.2$ g):

(7) $$F_R + F_{RR} := m_e a$$

From Eq. (6) we therefore obtain for the required force F:

(8) $$F = mg - (m + m_e)a - F_{RD}$$

Inserting this equation into Eq. (5) we obtain:

(9) $$mg - (m + m_e)a = I_D \frac{2l}{r^2 t^2} + F_{RD}$$

For better readability we introduce a force

(10) $$F_1 := mg - (m + m_e)a$$

with the measurable quantities m and a and the known quantities m_e and g such that Eq. (9) becomes:

(11) $$F_1 = I_D \frac{2l}{r^2 t^2} + F_{RD}$$

The unknown quantity F_{RD} which cannot be measured directly is still bothering us in this equation for determining I_D. If we assume, however, that the friction at the rotary disc is a rolling and sliding friction independent of the velocity (the so-called COULOMB *friction*) which only depends on the mass of the rotary table inclusive of bodies spread on it, then F_{RD} can be considered a time-independent constant. In this case Eq. (11) represents a simple linear equation of the form

(12) $$y = cx + b$$

with

(13) $$y = F_1, \qquad x = \frac{2l}{r^2 t^2}, \qquad c = I_D, \qquad b = F_{RD}$$

Plotting the related quantity F_1 (to be calculated according to Eq. (10)) against $2l/(r^2t^2)$ for constant r and different accelerating masses m (Eq. (11)), we obtain a line with the slope I_D. Thus we have found a way to measure the moment of inertia without knowing the quantity F_{RD}.

We now observe the case in which an additional body is placed on the rotary disk. Suppose I_K is the moment of inertia of this body (mass m_K) when it rotates about one of its gravity axes (*principal axis*); if this gravity axis C corresponds with the rotary axis H of the table, then the overall moment of inertia of the system rotary disk/body is:

(14) $$I = I_D + I_K$$

If the axes H and C run parallel at a distance s we obtain according to STEINER*'s theorem*[3]:

(15) $$I = I_D + I_K + m_K s^2$$

Eq. (11) then reads:

(16) $$F_1 = I\,\frac{2l}{r^2 t^2} + F_{RD}$$

Using Eq. (3) and Eq. (10), it follows:

(17) $$I = \left(F_1 - F_{RD}\right)\frac{r^2}{2l}\,t^2 = \left(mg - F_{RD}\right)\frac{r^2}{2l}\,t^2 - \left(m + m_e\right)r^2$$

We want to use this relationship in order to determine the position of a gravity axis running parallel to the rotary axis of the table of a body of any possible shape lying on the rotary disc. We take the following steps: according to Eq. (15) I becomes has a minimum when $s = 0$, i.e., for the case that the gravity axis of the body is identical to that of the rotary axis of the disc. A minimum of I is equivalent to a minimum of the fall time t and t^2, respectively for constant quantities m, l, r, and F_{RD} according to Eq. (17). Shifting the body on the rotary disc (varying s), the fall time t must therefore show a minimum at a certain position. The related function $t = f(s)$ describing this behaviour will now be determined. For this we insert Eq. (15) into Eq. (16), solve for t^2 and obtain for t as a function of s:

3 JAKOB STEINER (1796 - 1863)

(18) $$t^2 = \underbrace{\frac{(I_D + I_K)2l}{(F_1 - F_{RD})r^2}}_{K_1} + \underbrace{\frac{2l\,m_k}{(F_1 - F_{RD})r^2}}_{K_2} s^2$$

or in a clear way with the auxiliary quantities K_1 and K_2:

(19) $$t^2 = K_1 + K_2\ s^2$$

Question 1:
- Which function (curve) does Eq. (19) represent? (Hint: Conic sections)

In order to determine the position of the required gravity axis C by means of Eq. (19) we proceed as follows: predetermine a coordinate system *XY* on the rotary disc, the origin of which coincides with the axis of rotation H (cf. Fig. 2). A line of holes is created along the *y*-axis of the rotary disk. A pin is fixed at an optional point P on the body, for which we find the position of the gravity axis. The pin and line of holes are placed such that the body can be shifted in Y direction on the rotary disc without changing its orientation with regard to the coordinate system XY (cf. remarks at the end of chapter 3.2).

Let point P (the pin) have the coordinates $(0, y_P)$ after placing the body on the rotary disc.
For the distance s of the gravity axis C from the rotary axis H we then obtain:

(20) $$s = \sqrt{\Delta x^2 + (y_P - \Delta y)^2}$$

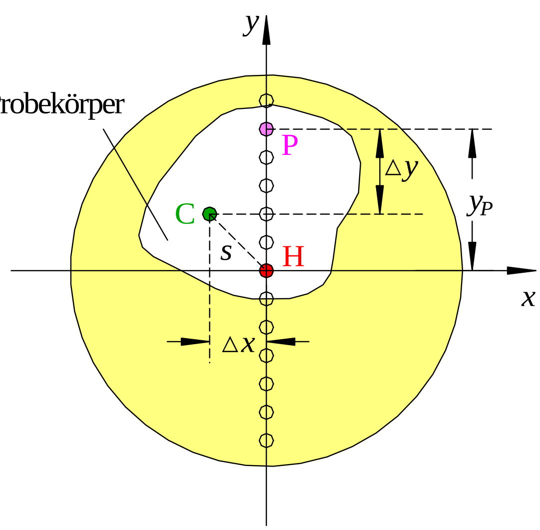

Fig. 2: Rotary disc (yellow) with sample body (white, top view). H is the rotary axis of the table, C the gravity axis of the sample body[4] and P is the sample body's point of fixation along the vertical line of holes on the disc. s is the distance between C and H.

According to Eq. (19) the fall time t for the accelerating mass m is a minimum when s is a minimum, which is the case with fixed Δx for the condition $y_P = \Delta y$ according to Eq. (20).

If we shift the body in y direction on the table and plot the fall time t over the shift y_P, we can determine the quantity Δy by finding the minimum in the produced curve. In an analogous way, the quantity Δx can

4 Note that the white area represents the *top view* of the sample body. For this reason, C does not need to be located at the centre of gravity of the white area.

be determined and proceeding from the optional point P, we can state the position of the desired gravity axis.

3 Experimental procedure

Equipment:

Rotary disc on tripod, acceleration masses ($m \approx (2, 3, 4, 5, 6)$ g), brass disk with locking pins, irregularly shaped sample body with locking pins, power supply (Phywe (0 - 15 / 0 - 30) V), magnetic holder, stand material for magnetic holder, switch, light barrier, electronic universal counter, digital oscilloscope, precision spirit level (accuracy 0.1 mm on 1 m), balance, metal measuring tape, sliding calliper, deceleration rod, thread.

Attention:

The rotary discs have very sensitive precision bearings which are easy to damage through improper handling. Only move the rotary discs with careful fingers! Take care that the thread does not get entangled in the bearing by timely deceleration! Only decelerate the discs using the small rod available!

Hint:

Usually the rotary discs are levelled exactly by the technical assistant using a precision water level prior to the lab course. Please make sure that this has actually been done!

3.1 Moment of inertia of a disc

The moment of inertia I_K of a brass disk (radius r_K, mass m) rotating about its symmetry axis C (Fig. 3) is to be determined by means of the set-up in Fig. 1. It is then calculated according to Eq. (14) as follows:

(21) $$I_K = I - I_D$$

In order to obtain I_K, first the moment of inertia of the rotary disc (I_D) has to be determined by means of Eq. (11) and then the moment of inertia of table and brass disk together (I) by means of Eq. (16). For this purpose

a) for the rotary disc
b) for rotary disk with brass disc

the fall time t (mean value from at least four single measurements each) is measured for five different acceleration masses (weigh masses!) and for a predetermined distance l (to be measured!). The fall time is measured by means of an electronic universal counter. The counter is started by the impulse, which causes the release of the holding pin of the magnetic holder, which is responsible for keeping the table in the starting position. The stopping impulse for the universal counter is given by a light barrier, which the accelerated masses pass at the end of the specified distance l.

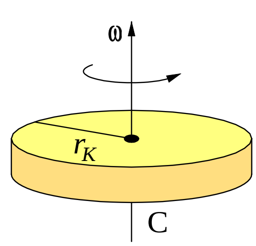

Fig. 3: Rotation of a disk of radius r_K about its symmetry axis C.

Subsequently F_1 is plotted against $2l/(r^2t^2)$ for a) and b) according to Eq. (11) and Eq. (16) in *one* diagram and the regression lines are calculated (measure *r carefully* using a metal measuring tape). An error analysis for the individual values of F_1 and $2l/(r^2t^2)$ is not required. The friction forces F_{RD} on the rotary disc as well as the moments of inertia I_D and I are calculated from the parameters of the regression line (including error) and from that I_K according to Eq. (21) (also including error).

Question 2:

- How can the moment of inertia I of a disk with the mass m and the radius r_K rotating about its symmetry axis C (cf. Fig. 3) be calculated from the relationship $I = \int R^2 \, \mathrm{d}m$ (cf. Chapter 1)? How large is the theoretically expected moment of inertia for the brass disk used (measure r_K and m!)? What are the possible sources of deviations between theory and experiment?

3.2 Determining the position of a gravity axis of an irregularly shaped body

According to the explanations given for Eqs. (18) - (20) the position of a gravity axis C running parallel to the rotary axis H of an irregularly shaped sample body shall be determined. For this purpose the pin mounted on the body is put into ten different holes of the hole row along the *y*-axis of the rotary disk and the coordinate y_P is determined[5]. At each position, the fall time t (mean of 4 single measurements) for a predetermined distance l is measured(cf. 3.1) for *one* mass m each. Afterwards, t is plotted against y_P (including error bars) and the value Δy is graphically determined, where t has a minimum.

Alternatively, the position of the minimum of t may be determined by a non-linear fit. The target function is, according to Eq. (19), given by:

(22) $$t = \sqrt{a + b\left(y_1 - \Delta y\right)^2}$$

with the fit parameters a, b and Δy. With the knowledge of these parameters, the value $y_P = \Delta y$, for which the fall time t is minimal can be determined from Eq. (22).

Analogously, it would be possible to determine Δx and to state the position of the centre of gravity C in the *xy*-plane relative to point P. In order to save time, however, we will confine ourselves to measuring only the distance Δy between P and C.

Remarks:

In order to make sure that the orientation of the sample body does not change when shifting along the *y*-axis, *two* pins are mounted on the body. Therefore, it has to be determined first, which of the two pins marks the position of point P.

[5] The distance between two holes on the disc is 10 mm (error free).