A Tractable Lagrangian for Arbitrary Spin
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Abstract

We propose simple Lagrangian that by the choice of the representation of SU(2), gives rise to field equations for arbitrary spin. In explicit examples it is shown, how the Klein-Gordon, the Dirac, and the Proca equation can be obtained from this Lagrangian. On the same footing, field equations for arbitrary spin are given. Finally, symmetries are discussed, the fields are quantized, their statistics is deduced, and Feynman rules are derived. This poster is based on [5].

Introduction

There are many attempts to unify the description of arbitrary spin fields, starting from Dirac, Pauli, and Fierz [1,2]. We propose a specifically simple Lagrangian whose equations of motion may be written as the well known Klein Gordon, Dirac, and Proca equations. Furthermore, the Lagrangian gives a known equation of motion compatible with the arbitrary spin case. Finally the quantization of an arbitrary spin field is studied.

The Lagrangian

The following Lorentz and Gauge invariant Lagrangian is proposed:

\[ L = (D^\mu \bar{\Psi}(D_\mu \Psi) - ie_\mu \bar{\Psi} H^{\mu \nu} F_{\nu} \cdot \Psi - m^2 \bar{\Psi} \Psi) \]

The field \( \Psi \) is composed by two spinors with 2s+1 components each one, corresponding to the dimension of the representation of Lorentz group that governs the fields. One also may define the adjoint spinor of \( \Psi \):

\[ \bar{\Psi} = \Psi^\dagger \gamma_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \]

Let \( \Sigma_k \) be the generators of SU(2) in a 2s+1 dimensional representation, then the symbol \( H^{\mu \nu} \) is defined via

\[ H^{\mu \nu} = \begin{pmatrix} h^{\mu \nu} & 0 \\ 0 & h^{\nu \mu} \end{pmatrix} \]

The factor depends on the spin of the field:

\[ s = \begin{cases} 1 & \text{for } s = 0 \\ \frac{1}{4s} & \text{for } s \neq 0 \end{cases} \]

Spin One

For spin one, the representation of SU(2) is the adjoint representation. Again, the components \( \psi \) and \( \Omega \) of the former spinor is taken into account. Note that the field components are complex. The equations are:

\[ (D^2 + m^2) \psi + ice_\mu F_{\mu} \psi = 0 \]

\[ (D^2 + m^2) \Omega + ice_\mu F_{\mu} \psi + ic_\psi \Omega = 0 \]

Giving the transformation laws of \( \psi \) and \( \Omega \) spinors under the Lorentz group, one can see that a certain combination of them transforms as electric and magnetic fields. So one defines:

\[ \psi = \psi + i \Omega / 2 \Rightarrow \psi = \psi \]

\[ \Omega = -i \Omega / 2 \Rightarrow \Omega = \Omega \]

Where the tensor \( G \) may be identified with the field strength tensor of a Proca field. And the equations may be rewritten in a more familiar form, the interactive Proca equation for a massive spin one particle;

\[ (D^2 + m^2) G_{\mu \nu} + ice_\mu G_{\nu}^\mu - ice_\nu G_{\mu}^\nu = 0 \]

Arbitrary Spin

Leaving the representation of SU(2) open, the equations of motion of this lagrangian turn out to be identical to the equations found by Hurley from a first order theory defined with \((12s+2)\) components.

The equation of motion for the free field in the arbitrary spin case, is the Klein Gordon equation. Thus, one may perform plane wave expansion including creation-destruction modes. An arbitrary spin field may be written as:

\[ \psi = \sum_{k} \frac{d^k}{\sqrt{(2\pi)^2} 2m} \left[ \psi_0(k) \omega_0(k) e^{-ik \cdot x} + \psi_1(k) \omega_1(k) e^{ik \cdot x} \right] \]

Quantization

The equation of motion for the free field is the arbitrary spin case, is the Klein Gordon equation. Thus, one may perform plane wave expansion including creation-destruction modes. An arbitrary spin field may be written as:

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Conclusions

A novel and simple approach for an arbitrary spin Lagrangian was studied. On the level of equations of motion this Lagrangian turns out to reproduce the Klein Gordon equation (s=0), the Dirac equation (s=1/2), the Proca equation (s=1), and the Hurley equations (s=arbitrary). By performing field quantization in this framework we proof that this high level of simplicity has the price that all fields (including integer spin) have to obey Fermi statistics.

References