A Tractable Lagrangian for Arbitrary Spin

Benjamin Koch, Nicolás Rojas
Pontificia Universidad Católica de Chile
Campus San Joaquín. Vicuña Mackenna 4860. Macul. Santiago

Abstract

We propose simple Lagrangian that by the choice of the representation of SU(2), gives rise to field equations for arbitrary spin. In explicit examples it is shown, how the Klein-Gordon, the Dirac, and the Proca equation can be obtained from this Lagrangian. On the same footing, field equations for arbitrary spin are given. Finally, symmetries are discussed, the fields are quantized, their statistics is deduced, and Feynman rules are derived. This poster is based on [5].

Introduction

There are many attemps to unify the description of arbitrary spin fields, starting from Dirac, Pauli, and Fierz [1,2]. We propose a specifically simple Lagrangian whose equations of motion may be written as the well known Klein Gordon, Dirac, and Proca equations. Furthermore, the Lagrangian gives a known equation of motion compatible with the arbitrary spin case. Finally the quantization of an arbitrary spin field is studied.

The Lagrangian

The following Lorentz and Gauge invariant Lagrangian is proposed:

$$L=(D^{\mu}\bar{\Psi})(D_{\mu}\Psi)-ieg_s\bar{\Psi}H^{\mu\nu}_{(S)}F_{\mu\nu}{}^{a}T_a\Psi-m^2\bar{\Psi}\Psi$$

The field Ψ is composed by two spinors with 2s+1 components each one, corresponding to the dimension of the representation of Lorentz group that governs the fields. One also may define the adjoint spinor of Ψ:

$$\bar{\Psi}=\Psi\,\gamma_0.\quad \gamma_0=\begin{pmatrix}\mathbf{0}&\mathbf{1}\\ \mathbf{1}&\mathbf{0}\end{pmatrix}.$$

Let Σ_k be the generators of SU(2) in a 2s+1 dimensional representation, then the symbol $H^{\mu\nu}$ is defined via

$$H^{\mu\nu}_{(s)}=\begin{pmatrix}h^{\mu\nu}&0\\0&h'^{\mu\nu}\end{pmatrix},\quad h^{0i}=-h^{i0}=\Sigma^i,\quad h^{ij}=-\frac{1}{2}\left[\Sigma^i,\Sigma^j\right].$$

$$h'^{0i}=-h'^{i0}=-\Sigma^i,\quad h'^{ij}=-\frac{1}{2}\left[\Sigma^i,\Sigma^j\right]$$

The g factor depends on the spin of the field:

$$g_s=\begin{cases}1 & \text{for } s=0\\ \dfrac{1}{4s} & \text{for } s\neq 0\end{cases}$$

Equations of Motion

The classical equations of motion for the cases zero, half and spin one are obtained from the Lagrangian by simply picking the respective representations of the Lorentz group.

Spin Zero

The representation of SU(2) for this spin is identically zero, so the Lagrangian acquires the form:

$$L_{(s=0)}=(D_{\mu}\bar{\Psi})(D^{\mu}\Psi)-m^2\bar{\Psi}\Psi$$

Note that Ψ field is now a two component vector. Both components fulfill the Klein Gordon equation

$$(D_{\mu}D^{\mu}+m^2)\Psi=0$$

Spin One Half

For spin half one picks the Pauli matrices as two dimensional representation of the Lorentz group. The resulting Lagrangian and its equations of motion were first studied by Laurie Brown [3]. It is based on two bispinors Ψ in the chiral representation of the Lorentz group.

$$\Psi=\begin{pmatrix}\psi\\ \Omega\end{pmatrix}$$

The spinor ψ is a Left handed bispinor, and Ω is a Right handed bispinor. For this case the Lagrangian for arbitrary spin acquires the form

$$L_{(s=1/2)}=(D_{\mu}\Omega)\,(D^{\mu}\psi)-ie\Omega\,h^{\mu\nu}F_{\mu\nu}{}^{a}T_a\psi-m^2\Omega\,\psi+\text{h.c}$$
$$=(D^{+}\Omega)\,(D^{-}\psi)-m^2\Omega\,\psi+\text{h.c}$$

Where the following definitions have been made:

$$D^{+}=\bar{\sigma}^{\mu}D_{\mu}.\quad D^{-}=\sigma^{\mu}D_{\mu}$$
$$\sigma^{\mu}=(1,\sigma^k).\quad \bar{\sigma}^{\mu}=(1,-\sigma^k).$$

Thus one writes:

$$\left.\begin{aligned}D^{-}\psi&=-im\Omega\\ D^{+}\Omega&=-im\psi\end{aligned}\right\}\Rightarrow\left(D_0-\begin{pmatrix}\sigma_i&0\\0&-\sigma_i\end{pmatrix}D_i-m\begin{pmatrix}0&1\\-1&0\end{pmatrix}\right)\begin{pmatrix}\psi\\ \Omega\end{pmatrix}=0$$

$$\left(\gamma^{\mu}D_{\mu}-m\right)\begin{pmatrix}\psi\\ \Omega\end{pmatrix}=0$$

That means that the Ψ field satisfies the Dirac equation

Spin One

For spin one, the representation of SU(2) is the adjoint representation. Again, the components ψ and Ω of the former spinor is taken into account. Note that the field components are complex. The eom's are:

$$(D^2+m^2)\psi_m-ieF^{oj}\varepsilon_{jm}{}^{n}\psi_n+ieF_m{}^{n}\psi_n=0$$
$$(D^2+m^2)\Omega_m+ieF^{oj}\varepsilon_{jm}{}^{n}\Omega_n+ieF_m{}^{n}\Omega_n=0$$

Giving the transformation laws of ψ and Ω spimors under the Lorentz group, one can see that a certain combination of them tranforms as electric and magnetic fields. So one defines:

$$\vec{E}_m=\frac{\psi_m-\Omega_m}{2i}\Rightarrow G_{0m}=\vec{E}_m$$
$$\vec{B}_m=\frac{\psi_m+\Omega_m}{2}\Rightarrow G_{mn}=\varepsilon_{mn}{}^{p}\vec{B}_p$$

Where the tensor G may be identifyied with the field strength tensor of a Proca field. And the eom's may be rewritten in a more familiar form, the interactive Proca equation for a massive spin one particle;

$$(D^2+m^2)G_{\mu\nu}+ieF_{\alpha\mu}G^{\alpha}{}_{\nu}-ieF_{\alpha\nu}G^{\alpha}{}_{\mu}=0$$

Arbitrary Spin

Leaving the representation of SU(2) open, the equations of motion of this lagrangian turn out to be identical to the equations found by Hurley from a first oder theory defined with (12s+2) components

$$D_{\mu}D^{\mu}\psi+ig_seH^{s}_{\mu\nu}F^{\mu\nu}\psi+m^2\psi = 0$$
$$D_{\mu}D^{\mu}\Omega+ig_seH'^{s}_{\mu\nu}F^{\mu\nu}\Omega+m^2\Omega = 0$$

Quantization

The equation of motion for the free field in the arbitrary spin case, is the Klein Gordon equation. Thus, one may perform plane wave expansion including creation-destruction modes. An arbitrary spin field may be written as:

$$\Psi=\int\frac{d^3k}{\sqrt{(2\pi)^3 2\omega_k}}\sum_{\sigma}\left(u_{\sigma}(k)a_{\sigma}(k)e^{-ip\cdot x}+v_{\sigma}(k)b_{\sigma}(k)e^{ip\cdot x}\right)$$

Now one performes the usual canontical quantization procedure...

$$\left[a(p,\sigma),a^{\dagger}(p',\sigma')\right]_{\pm} = (2\pi)^3 2k_0\delta^3(\mathbf{p}-\mathbf{p}')\delta_{\sigma\sigma'}$$
$$\left[b(p,\sigma),b^{\dagger}(p',\sigma')\right]_{\pm} = (2\pi)^3 2k_0\delta^3(\mathbf{p}-\mathbf{p}')\delta_{\sigma\sigma'}$$

$$:H:=\int d^3k\,\frac{k_0^2+k_i^2+m^2}{4(2\pi)^3\omega_k^2}\left(a_{\sigma}(k)a_{\sigma}(k)+b_{\sigma}(k)b_{\sigma}\right)$$

Perform many consistency checks:
- + Feynman rules
- + Causality of the propagator
- + Positive Hamiltonian
- + Creation & annihilation condition
- + Charge conservation

} All fields have to obey anti-commutation relations

integer spin = ghosts

Conclusions

A novel and simple approach for an arbitrary spin Lagrangian was studied. On the level of equations of motion this Lagrangian turns out to reproduce the Klein Gordon (s=0) the Dirac equation (s=1/2), the Proca equation (s=1), and the Hurley equations (s=arbitrary).
By performing field quantization in this framework we proof that thishigh level of simplicity has the price that all fields (including integer spin) have to obey Fermi statistics.

References

[1] P. A. M. Dirac, Proc. Roy. Soc. Lond **155A**, 447-459 (1936).
[2] M. Fierz, W. Pauli, Proc. Roy. Soc. Lond. **A173**, 211-232 (1939).
[3] L. M. Brown, Phys. Rev. **111**, 957 (1958).
[4] C. R. Hagen, W. J. Hurley, Phys. Rev. Lett. **24**, 1381-1384 (1970).
[5] B. Koch, N. Rojas, arXiv:1107.3144.