Geometrical interpretation of the free Klein-Gordon equation

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We study the question whether the laws of quantum mechanics can be derived from a purely classical setting with one additional dimension. It is shown that the coupling of the classical higher dimensional Einstein-tensor to a conserved energy-momentum tensor can be chosen in such a way that one finds the quantum Klein-Gordon equation.

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Early attempts to find an alternative explanation for quantum phenomena have been discussed within the so called Bohmian mechanics [1, 2]. Similarities between a higher dimensional wave equation and the non-relativistic quantum theory have already been pointed out in [3]. In a recent paper such similarities have also been found without the assumption of extra dimensions, by comparing the solutions of a Friedmann-Lemaítre-Robertson-Walker universe and the solutions of the Schrödinger equation [4]. In a field theoretical approach, the so called stochastic quantization [5], the classical Langevin equation for an auxiliary time coordinate $\tilde{t}$ is used in order to obtain euclidian quantum field theory in the limit $\tilde{t} \to \infty$. Other papers, relate quantum field theories on the horizon of a black hole with a corresponding classical theory in the higher dimensional space-time [6, 7]. In the context of supergravity and string theory similar ideas involving a holographic principle and extra time dimensions obtained a lot of attention [8–10]. But also for theories without the string background the general question was discussed whether quantum field theory might emerge from a chaotic classical theory with friction [11, 12]. Finally, the possibility of an additional time dimension without holographic effects has been considered in the context of high energy phenomenology [13] and in the context of a conceptual alternative to quantization [14].

All those ideas can be partially seen as motivation for the approach of this paper where the quantum Klein-Gordon equation is derived from classical differential geometry with one additional time dimension. As continuation of the successful interpretation of non-relativistic quantum wave functions in terms of classical particles moving with respect to an additional time dimension [15] we will now derive the relativistic Klein Gordon.

I. THE FREE KLEIN-GORDON EQUATION

Before starting with the alternative interpretation of quantum mechanics, we will rewrite the relativistic Klein-Gordon equation

$$\partial^\mu \partial_\mu \Phi(x,t) = -\frac{m^2}{\hbar^2}\Phi(x,t) \quad . \tag{1}$$

This equation for the complex wave function Φ can be expressed in terms of the real function $S(x,t)$ and the positive function $\rho(x,t)$ by defining $\Phi(x,t) = \sqrt{\rho}\exp(iS/\hbar)$. This gives two coupled differential equations

$$\partial^\mu \left(\rho(\partial_\mu S)\right) \;=\; 0 \quad , \tag{2}$$

$$\sqrt{\rho}\partial^\mu\partial_\mu\sqrt{\rho} \;=\; \frac{\rho}{\hbar^2}\left((\partial^\mu S)(\partial_\mu S) - m^2\right) \quad . \tag{3}$$

This redefinition does not change the meaning or the interpretation of Eq. (1).

II. GENERAL RELATIVITY WITH AN ADDITIONAL DIMENSION

In this section we will consider the geometric structure of Einstein equations with one additional coordinate $\bar{t}$. Those equations contain a classical energy-momentum tensor T_{AB} and a constant term Λ. Instead of taking the standard gravitational coupling we use a coupling constant for k_2 the energy-momentum tensor T_{AB}. A simple ansatz for the metric in $2+3$ dimensions will be made. We then show that it is possible to choose the coupling k_2 and the constant Λ in such a way that the classical equations of motion in the higher dimensional theory correspond to the relativistic quantum equation for a spinless particle. We use the coordinate notation $x_A = (\bar{t}, x_\mu) = (\bar{t}, t, x, y, z)$, where capital latin indices A run from 0 to 4 and greek indices run from 1 to 4. As starting point we take the general covariant equation

$$G_{AB} = R_{AB} - \frac{1}{2}g_{AB}R = k_2 T_{AB} - \frac{1}{2}g_{AB}\Lambda \quad , \tag{4}$$

where R_{AB} is the higher dimensional Ricci tensor and R is it's contraction. Usually the constant term Λ is called the cosmological constant. By taking the trace of Eq. (4) one obtains a single scalar equation

$$R = -\frac{1}{3}(T^A_A - 5\Lambda) \quad , \tag{5}$$

which will be sufficient for our purposes. Following the reasoning in [15] we make an ansatz for for the metric

$$g_{\mu\nu} = \begin{pmatrix} k_1\alpha(\bar{t})\rho(t,x) & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix} \quad . \qquad (6)$$

Note that this metric corresponds to the Kaluza-Klein metric for a vanishing electromagnetic field $A_\mu = 0$ [3, 16, 17]. However, in contrast to the Kaluza-Klein approach the energy momentum Tensor T_{AB} and the constant Λ are not assumed to be identically zero. An other difference is that (due to the function $\alpha(\bar{t})$) a $\bar{t}$ dependence is allowed. After using the metric (6) and after a multiplication with $-\rho/2$ the scalar equation (5) reads

$$\sqrt{\rho}\partial^\mu\partial_\mu\sqrt{\rho} = \frac{\rho}{3}\left(k_2(T^0_{\ 0} + T^\mu_{\ \mu}) - \frac{5}{2}\Lambda\right) \quad . \qquad (7)$$

The left hand side of this equation contains only the four dimensional Laplace operator since the dependence on $\alpha(\bar{t})$ and k_1 dropped out. For the right hand side a definition of the energy momentum tensor is needed. The Hamilton-Jakobi definition of the energy momentum tensor of a free particle in curved space-time is

$$\begin{aligned} T^A_{\ B} &= \frac{1}{VM}(\partial^A S)(\partial_B S), \quad \text{with} \qquad (8) \\ T^A_{\ A} &= \frac{1}{VM}\left((\partial^0 S)(\partial_0 S) + (\partial^\mu S)(\partial_\mu S)\right) \quad . \end{aligned}$$

Here, S is Hamilton's principal function [18, 19], V is the normalizing volume, and M is the normalizing mass which might be, but does not necessarily have to be, the same as m. Using Eq. (8) one can rewrite Eq. (7) as

$$\begin{aligned} \sqrt{\rho}\partial^\mu\partial_\mu\sqrt{\rho} = \tfrac{\rho}{\hbar^2} \Big[& \tfrac{\hbar^2 k_2}{3VM}((\partial^0 S)(\partial_0 S) + \\ & (\partial^\mu S)(\partial_\mu S)) - \tfrac{5\hbar^2}{6}\Lambda\Big] \quad . \end{aligned} \qquad (9)$$

A priory the derivative of S with respect to the additional coordinate $\bar{t}$ is not known. However, we will see that $\partial_0 S = k_3\sqrt{\rho}$ (where k_3 is some constant) is the only form that leaves this theory self-consistent. After choosing the constants $k_2 = 3MV/\hbar^2$ and $k_3^2 = k_1(5\hbar^2\Lambda/6 - m^2)$ Eq. (9) reads

$$\sqrt{\rho}\partial^\mu\partial_\mu\sqrt{\rho} = \frac{\rho}{\hbar^2}\left((\partial^\mu S)(\partial_\mu S) - m^2\right) \quad . \qquad (10)$$

This is the real part of the Klein-Gordon equation (3). One sees that "cosmological" constant Λ is not necessary for this theory to work. But one also sees that k_1 has to be negative as soon as $\Lambda < 6m^2/(5\hbar^2)$ and a negative k_1 corresponds to a spatial signature of extra dimension $\bar{t}$. Note that the coupling k_2 has an additional factor 3/2 in comparison to the previous derivation of the Schrödinger equation, where only one spatial dimension was considered.

The imaginary part of the Klein-Gordon equation (2) can found from the covariant conservation law for the energy-momentum tensor T_{AB}.

$$\begin{aligned} 0 &= \nabla^B(T_{AB}) = \nabla^B\left(g_{AC}T^C_{\ B}\right) \qquad (11) \\ &= \frac{1}{Vm}\nabla^B\left(g_{AC}(\partial^C S)(\partial_B S)\right) \quad . \end{aligned}$$

Now we take the 0 component of this equation and integrate over the unobservable coordinate $\bar{t}$. This yields

$$\begin{aligned} 0 &= \int d\bar{t}\ \nabla^B\left(g_{0C}(\partial^C S)(\partial_B S)\right) \qquad (12) \\ &= k_1\int d\bar{t}\ \partial^\mu\left(\sqrt{\rho}(\partial_0 S)(\partial_\mu S)\right) \quad , \end{aligned}$$

where a $\bar{t}$ boundary term was dropped. Again, just like for equation (9) one sees that $\partial_0 S = k_3\sqrt{\rho}$ is the only consistent choice and Eq. (12) simplifies to

$$0 = k_1 k_3\int d\bar{t}\ \partial^\mu\left(\rho(\partial_\mu S)\right) \quad . \qquad (13)$$

After dropping the constants k_1 and k_3, this is the desired imaginary part of the Klein-Gordon equation (2).

III. SUMMARY AND DISCUSSION

We studied the classical movement of a free particle in a higher dimensional space-time which is governed by the geometric equations (5) and by the continuity equation for its energy-momentum tensor (11). In those equations, we made an ansatz for the additional component of the metric g_{00} and for the additional component of the energy momentum tensor T_{00} by taking $\partial_0 S = k_3\sqrt{\rho}$. Since all the equations that were used are classical, the only way the "quantumness" in terms of $\hbar$ could enter into our equations was through the constants k_2, k_3, and Λ. After choosing the constants $k_2 = 3MV/\hbar^2$ and $k_3^2 = k_1(5\hbar^2\Lambda/6 - m^2)$ we found the two differential equations (10, 13) which are equivalent to the quantum-Klein-Gordon equation (1).

The free Schrödinger equation can be found from equation (1) in the limit of $m \gg \partial_0 S$. For bound state problems a non-relativistic potential has to be introduced on the side of the source term in Eq. (4). Therefore, the apparent uncertainty of quantum physics might be understood from a classical theory with one hidden dimension. Further generalizations of this results to interacting fields, spinors and so on will have to be studied.

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