Multipolos

Índice

1 Desarrollo en serie

$$\Phi(\vec{x}\,) = \frac{1}{4\pi\varepsilon_0}\int_V d^3x'\frac{\rho(\vec{x}\,')}{|\vec{x}-\vec{x}\,'|}$$

V es acotado. Calculemos el potencial lejos de las cargas $|\vec{x}\,| \gg |\vec{x}\,'|$:

$$\frac{1}{|\vec{x}-\vec{x}\,'|} = \frac{1}{|\vec{x}\,|}\left[1-2\frac{\vec{x}}{|\vec{x}\,|^2}.\vec{x}\,'+\frac{|\vec{x}\,'|^2}{|\vec{x}\,|^2}\right]^{-1/2} =$$

$$\frac{1}{|\vec{x}\,|}\left(1+\frac{\vec{x}}{|\vec{x}\,|^2}.\vec{x}\,'-\frac{1}{2}\frac{|\vec{x}\,'|^2}{|\vec{x}\,|^2}+\frac{3}{2}\left(\frac{\vec{x}}{|\vec{x}\,|^2}.\vec{x}\,'\right)^2+\ldots.\right)$$

$$\Phi(\vec{x}\,) = \frac{1}{4\pi\varepsilon_0}\left(\frac{Q}{|\vec{x}\,|}+\frac{\vec{p}\,.\vec{x}}{|\vec{x}\,|^3}+\frac{1}{2}Q_{ij}\frac{x_ix_j}{|\vec{x}\,|^5}+\ldots.\right.$$

$Q = \int_V d^3x' \rho(\vec{x}\,')$, carga, monopolo
$\vec{p} = \int_V d^3x' \rho(\vec{x}\,')\vec{x}\,'$, momento dipolar

$$Q_{ij} = \int_V d^3x' \rho(\vec{x}\,')(3x_i'x_j' - |\vec{x}\,'|^2\delta_{ij})$$

es el momento cuadripolar, Es un tensor simétrico, de traza nula.

2 Desarrollo en Armónicos Esféricos

$$\Phi(\vec{x}) = \frac{1}{\varepsilon_0}\int_V d^3x' \rho(\vec{x}\,') \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{2l+1} \frac{r'^{\ell}}{r^{\ell+1}} Y_{lm}^{\star}(\theta', \phi') \; Y_{lm}(\theta, \phi) =$$

$$\frac{1}{\varepsilon_0} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{2l+1} \frac{Y_{lm}(\theta, \phi)}{r^{\ell+1}} q_{lm}$$

$$q_{lm} = \int_V d^3x' \rho(\vec{x}\,') r'^{l} Y_{lm}^{\star}(\theta', \phi')$$

3 Campo eléctrico

Ejercicio: Mostrar que el campo eléctrico debido a un multipolo es:

$$E_r = \frac{l+1}{\varepsilon_0(2l+1)} q_{lm} \frac{Y_{lm}(\theta, \phi)}{r^{l+2}}$$

$$E_\theta = -\frac{1}{\varepsilon_0(2l+1)} q_{lm} \frac{1}{r^{l+2}} \frac{\partial}{\partial\theta} Y_{lm}(\theta,\phi)$$

$$E_\phi = -\frac{1}{\varepsilon_0(2l+1)} q_{lm} \frac{1}{r^{l+2}} \frac{im}{\operatorname{sen}\theta} Y_{lm}(\theta,\phi)$$

4 Energía de una distribución de carga en un campo externo

$$W = \int_V d^3x \rho(x)\Phi(x) \sim$$

$$\int_V d^3x \rho(x) \left[\Phi(0) + x\nabla\Phi(0) + \frac{1}{2} x_i x_j \Phi_{,ij}(0) + \ldots \right] =$$

$$q\Phi(0) - \vec{p}.\vec{E}(0) + \frac{1}{2}\Phi_{,ij}(0) \int_V d^3x \rho(x) x_i x_j$$

$$\int_V d^3x \rho(x) x_i x_j = \frac{1}{3}\left(Q_{ij} + \delta_{ij} \int_V d^3x \rho(x) \vec{x}^{\,2} \right)$$

No hay cargas en 0,$\Phi_{,ii}(0) = 0$

$$W = q\Phi(0) - \vec{p}.\vec{E}(0) - \frac{1}{6} Q_{ij} E_{i,j}(0) + \ldots.$$