

- $\vec{\nabla} \cdot \vec{B}(\vec{x}) = 0$ No hay monopolos magnéticos.
- $\vec{\nabla} \times \vec{B}(\vec{x}) = \mu_0 \vec{J}(\vec{x})$ Ley de Ampère

Si $\vec{J}(\vec{x}) = 0$, se tiene: $\vec{B}(\vec{x}) = -\nabla \Phi_m(\vec{x})$, $\Phi_m(\vec{x})$ satisface la ecuación de Laplace. Podemos usar los métodos desarrollados en Electrostática para encontrar en campo magnético.

En general: $\vec{\nabla} \cdot \vec{B}(\vec{x}) = 0$ implica: $\vec{B}(\vec{x}) = \vec{\nabla} \times \vec{A}(\vec{x})$, $\vec{A}(\vec{x})$ es el potencial vector.

Transformación de gauge: $\vec{A}'(\vec{x}) = \vec{A}(\vec{x}) + \nabla \psi$, no cambia $\vec{B}(\vec{x})$.

$$\vec{\nabla} \times \vec{B}(\vec{x}) = \vec{\nabla} \times (\vec{\nabla} \times \vec{A}(\vec{x})) = -\nabla^2 A + \nabla(\nabla A) = \mu_0 \vec{J}(\vec{x})$$

Utilizando la libertad de gauge podemos fijar el gauge de Coulomb $\nabla A = 0$. En efecto:

$$0 = \nabla A' = \nabla A + \nabla^2 \psi$$

$$\nabla^2 \psi = -\nabla A \text{ tiene siempre solución para } \psi$$

$$\nabla^2 A = -\mu_0 \vec{J}(\vec{x})$$

En el espacio abierto:

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \int d^3x' \frac{\vec{J}(x')}{|\vec{x} - \vec{x}'|}$$

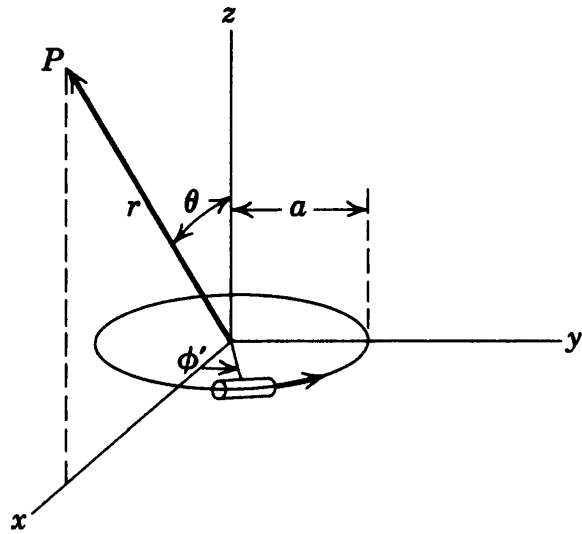


Figura 1.

$$J_{\phi} = I \sin \theta' \delta(\cos \theta') \frac{\delta(r' - a)}{a}$$

$$J = -J_{\phi} \sin \phi' \hat{i} + J_{\phi} \cos \phi' \hat{j}$$

Por simetría, podemos escoger el punto de observación con $\phi = 0$.

$$\begin{aligned}
 A_\phi(r, \theta) &= \frac{\mu_0}{4\pi} \int d\phi' \int d\theta' \sin \theta' \int dr' r'^2 I \sin \theta' \delta(\cos \theta') \frac{\delta(r' - a)}{a} \frac{\cos \phi'}{|x - x'|} = \\
 &= \frac{\mu_0 I a}{4\pi} \int d\phi' \frac{\cos \phi'}{(r^2 + a^2 - 2ra \sin \theta \cos \phi')^{1/2}} = \\
 &= \frac{\mu_0 I a}{4\pi} \frac{4}{(r^2 + a^2 + 2ra \sin \theta)^{1/2}} \left[\frac{(2 - k^2)K(k) - 2E(k)}{k^2} \right], \\
 & \quad k^2 = \frac{4ar \sin \theta}{r^2 + a^2 + 2ra \sin \theta}
 \end{aligned} \tag{1}$$

Ejercicio: Expandir el potencial vector (1) en armónicos esféricos.

$$\frac{1}{|\vec{x} - \vec{x}'|} = \frac{1}{|\vec{x}|} + \frac{\vec{x} \cdot \vec{x}'}{|\vec{x}|^3}$$

$$\vec{A}_i(\vec{x}) = \frac{1}{|\vec{x}|} \int d^3x' J_i(x') + \frac{\vec{x}_j}{|\vec{x}|^3} \int d^3x' x'_j J_i(x')$$

$$\int d^3x' J_i(x') = \int d^3x' \{(x'_j J_i(x'))_{,i} - x'_j J_{i,i}(x')\} = \oint dS_i x'_j J_i(x') = 0$$

$$\int d^3x' \{x'_j x'_k J_i(x')\}_{,i} = \oint dS_i x'_j x'_k J_i(x') = 0$$

$$\int d^3x' \{\delta_{ji} x'_k J_i(x') + x'_j \delta_{ki} J_i(x')\} = \int d^3x' \{x'_k J_j(x') + x'_j J_k(x')\}$$

$$\int d^3x' x'_j J_i(x') = \frac{1}{2} \int d^3x' [x'_j J_i(x') - x'_i J_j(x')] = \varepsilon_{jik} \int d^3x' (\vec{x}' \times \vec{J}(\vec{x}'))_k$$

$$\mathcal{M}(x') = \frac{1}{2} \vec{x}' \times \vec{J}(\vec{x}')$$

es la magnetización.

$$\vec{m}_k = \frac{1}{2} \int d^3x' (\vec{x}' \times \vec{J}(\vec{x}'))_k$$

es el momento magnético.

$$\vec{A}(\vec{x}) = \frac{\mu_0 m \times x}{4\pi |x|^3}$$

es el potencial vector del dipolo magnético.

$$B_k(x) = B_k(0) + x \cdot \nabla B_k(0) + \dots$$

$$F = \int d^3x J(x) \times B(x)$$

$$F_i = \varepsilon_{ijk} \int d^3x J_j(x) B_k(x) \sim$$

$$\varepsilon_{ijk} \int d^3x J_j(x) [B_k(0) + x \cdot \nabla B_k(0)] =$$

$$\varepsilon_{ijk} \int d^3x J_j(x) x_l B_{k,l}(0) = -\varepsilon_{ijk} \varepsilon_{jln} m_n B_{k,l}(0) =$$

$$(\delta_{il} \delta_{kn} - \delta_{in} \delta_{kl}) m_n B_{k,l}(0) =$$

$$m_k B_{k,i} - m_i B_{k,k} = \nabla_i (m \cdot B)$$

$$N = \int d^3x x \times (J(x) \times B(x)) \sim$$

$$\int d^3x x \times (J(x) \times B(0))$$

$$N_i = \varepsilon_{ijk} \int d^3x x_j \varepsilon_{klm} J_l B_m(0) = (\delta_{il} \delta_{jn} - \delta_{in} \delta_{jl}) \int d^3x x_j J_l B_n(0) =$$

$$\int d^3x x_j J_i B_j(0) - \int d^3x x_j J_j B_i(0) =$$

$$\varepsilon_{jik} m_k B_j(0) - 0 = (m \times B(0))_i$$

$$\int d^3x (x_k x_k J_j)_{,j} = \int d^3x (2x_j J_j + x_k x_k J_{j,j}) = 2 \int d^3x x_j J_j = 0$$

$$\vec{N} = \vec{m} \times \vec{B}(0)$$

Energía Potencial: $\vec{F} = -\vec{\nabla}U$, $U = -\vec{m} \cdot \vec{B}$

$$\nabla B = 0, B = \nabla \times A$$

$$\vec{M}(\vec{x}) = \sum_i N_i \langle \vec{m}_i \rangle$$

$M(x)$: Magnetización.

$$\begin{aligned} \vec{A}(\vec{x}) &= \frac{\mu_0}{4\pi} \int d^3x' \left[\frac{\vec{J}(x')}{|\vec{x} - \vec{x}'|} + \frac{\vec{M}(\vec{x}') \times (\vec{x} - \vec{x}')}{|x - x'|^3} \right] \\ \int d^3x' \left[\frac{\vec{M}(\vec{x}') \times (\vec{x} - \vec{x}')}{|x - x'|^3} \right] &= \int d^3x' \left[\vec{M}(\vec{x}') \times \vec{\nabla}' \frac{1}{|x - x'|} \right] = \\ &= \int d^3x' \left[\vec{\nabla}' \times \vec{M}(x') \frac{1}{|x - x'|} \right] \end{aligned}$$

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \int d^3x' \left[\frac{\vec{J}(x') + \vec{\nabla}' \times \vec{M}(x')}{|\vec{x} - \vec{x}'|} \right]$$

Corriente de Magnetización efectiva: $\vec{J}_M(\vec{x}') = \vec{\nabla}' \times \vec{M}(x')$.

$$\begin{aligned} \vec{\nabla} \times \vec{B}(\vec{x}) &= \mu_0 \vec{J}(\vec{x}) + \mu_0 \vec{\nabla} \times \vec{M}(x) \\ \vec{H} &= \frac{1}{\mu_0} \vec{B} - \vec{M} \\ \vec{\nabla} \times \vec{H}(\vec{x}) &= \vec{J}(\vec{x}) \end{aligned}$$

\vec{H} : Campo magnético

Para materiales diamagnéticos y paramagnéticos isotrópicos :

$$B = \mu H$$

μ : permeabilidad magnética.

Paramagnético: $\mu > \mu_0$

Diamagnético: $\mu < \mu_0$

Ferromagnético: $B = F(H)$

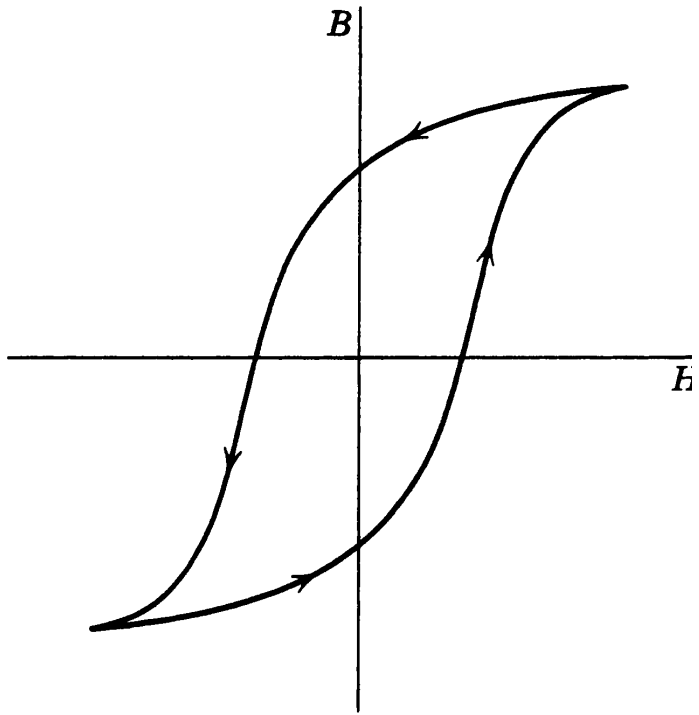


Figura 2. Histéresis

En una superficie S que separa las regiones 1 y 2:

$$\begin{aligned}(\vec{B}_2 - \vec{B}_1) \cdot \hat{n} &= 0 \\ \hat{n} \times (\vec{H}_2 - \vec{H}_1) &= \vec{K}\end{aligned}$$

\hat{n} : normal a la superficie que apunta de 1 a 2.

\vec{K} : densidad de corriente superficial (no incluye corriente de magnetización).

1. Potencial vector, medios lineales:

$$\begin{aligned}\vec{B}(\vec{x}) &= \vec{\nabla} \times \vec{A}(\vec{x}) \\ \nabla \times \frac{1}{\mu}(\vec{\nabla} \times \vec{A}(\vec{x})) &= J \\ -\nabla^2 A + \nabla(\nabla A) &= \mu J \quad \nabla \cdot A = 0 \text{ gauge de Coulomb} \\ \nabla^2 A &= -\mu J\end{aligned}$$

2. $\vec{J} = \vec{0}, \vec{H} = -\vec{\nabla}\Phi_M, \Phi_M$ es el potencial escalar magnético.

$$\vec{\nabla} \cdot \vec{B} = 0, \nabla^2 \Phi_M = 0$$

3. Ferromagnetos duros: $J = 0, M$ dado.

a) Potencial escalar:

$$\begin{aligned}\vec{H} &= -\vec{\nabla}\Phi_M, \quad \vec{\nabla} \cdot \vec{B} = 0 = \mu_0 \vec{\nabla} \cdot (\vec{H} + \vec{M}) \\ \nabla^2 \Phi_M &= -\rho_M, \quad \rho_M = -\vec{\nabla} \cdot \vec{M}\end{aligned}$$

En el espacio abierto:

$$\begin{aligned}\Phi_M(\vec{x}) &= -\frac{1}{4\pi} \int d^3x' \frac{\vec{\nabla}' \cdot \vec{M}(x')}{|\vec{x} - \vec{x}'|} = \frac{1}{4\pi} \int d^3x' \vec{M}(x') \cdot \vec{\nabla}' \frac{1}{|\vec{x} - \vec{x}'|} = \\ &= -\frac{1}{4\pi} \vec{\nabla} \int d^3x' \frac{\vec{M}(x')}{|\vec{x} - \vec{x}'|}\end{aligned}$$

Si hay una discontinuidad en la magnetización se induce una densidad superficial de magnetización: $\sigma_M = \hat{n} \cdot \vec{M}$ con lo cual:

$$\Phi_M(\vec{x}) = -\frac{1}{4\pi} \int d^3x' \frac{\vec{\nabla}' \cdot \vec{M}(x')}{|\vec{x} - \vec{x}'|} + \frac{1}{4\pi} \oint dS' \frac{\hat{n} \cdot \vec{M}(x')}{|\vec{x} - \vec{x}'|} =$$

$$\Phi_M \sim_{r \rightarrow \infty} \frac{\vec{m} \cdot \vec{x}}{4\pi r^3}, \quad \vec{m} = \int d^3x' \vec{M}(x')$$

b) Potencial vector.

$$\begin{aligned} \vec{B}(\vec{x}) &= \vec{\nabla} \times \vec{A}(\vec{x}) \\ \vec{\nabla} \times \vec{H}(\vec{x}) &= 0 = \vec{\nabla} \times \left(\frac{\vec{B}}{\mu_0} - \vec{M} \right) \\ \nabla^2 A &= -\mu_0 \vec{\nabla} \times \vec{M}(\vec{x}) \\ \vec{A}(\vec{x}) &= \frac{\mu_0}{4\pi} \int d^3x' \frac{\vec{\nabla}' \times \vec{M}(x')}{|\vec{x} - \vec{x}'|} \end{aligned}$$

Si hay discontinuidades en la magnetización:

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \int d^3x' \frac{\vec{\nabla}' \times \vec{M}(x')}{|\vec{x} - \vec{x}'|} + \frac{\mu_0}{4\pi} \oint_S dS' \frac{M(x') \times \vec{n}'}{|x - x'|}$$

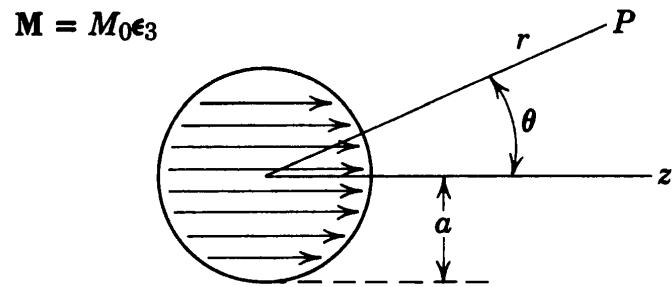


Figura 3.

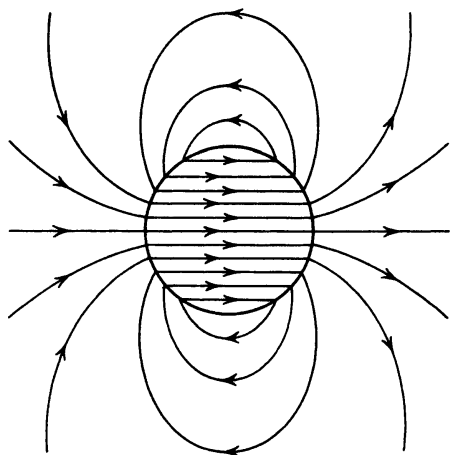
$$\vec{M} = M_0 \hat{z} \theta(a - r), \sigma_M = \hat{n} \cdot \vec{M} = M_0 \cos \theta$$

$$\Phi_M(\vec{x}) = \frac{1}{4\pi} \oint dS' \frac{\hat{n} \cdot \vec{M}(x')}{|\vec{x} - \vec{x}'|} =$$

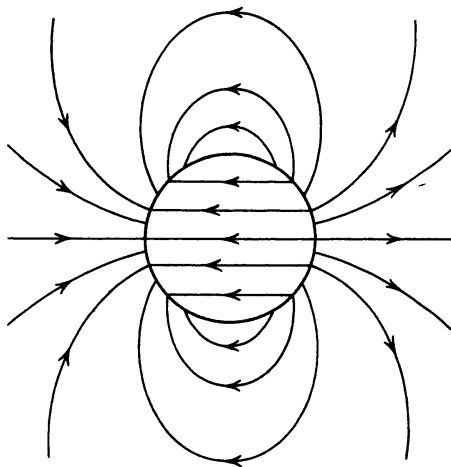
$$\frac{M_0 a^2}{4\pi} \int d\Omega' \frac{\cos \theta'}{|\vec{x} - \vec{x}'|}$$

$$\frac{1}{|\mathbf{x} - \mathbf{x}'|} = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^l \frac{1}{2l+1} \frac{r_{<}^l}{r_{>}^{l+1}} Y_{lm}^*(\theta', \phi') Y_{lm}(\theta, \phi)$$

$$\frac{M_0 a^2}{4\pi} \int d\Omega' \frac{\cos \theta'}{|\vec{x} - \vec{x}'|} = \frac{M_0 a^2}{3} \frac{r_{<}}{r_{>}^2} \cos \theta, \quad r_{<} = \min \{r, a\}$$



B



H

Figura 4.

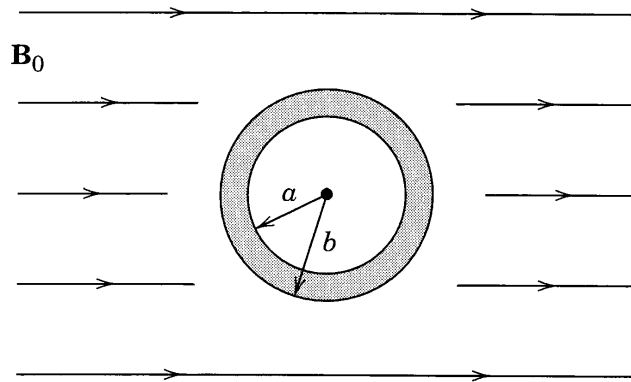


Figura 5.

$$H = -\nabla\Phi_M \quad \nabla B = \mu\nabla H = 0, \quad \nabla^2\Phi_M = 0$$

El problema tiene simetría azimutal con $\vec{B}_0 = B_0\hat{z}$

$$1) r > b, \Phi_1(r, \theta) = \sum_{\ell=0}^{\infty} [A_{\ell}r^{\ell} + \alpha_{\ell}r^{-(\ell+1)}] P_{\ell}(\cos \theta), \quad A_1 = -H_0, \quad A_{\ell} = 0, \ell \neq 1$$

$$2) a < r < b, \Phi_2(r, \theta) = \sum_{\ell=0}^{\infty} [\beta_{\ell}r^{\ell} + \gamma_{\ell}r^{-(\ell+1)}] P_{\ell}(\cos \theta)$$

$$3) r < a, \Phi(r, \theta) = \sum_{\ell=0}^{\infty} [\delta_{\ell}r^{\ell}] P_{\ell}(\cos \theta)$$

Condiciones de borde en $r = a, b$.

$$\begin{aligned} \frac{\partial}{\partial \theta} \Phi_1|_{r=b} &= \frac{\partial}{\partial \theta} \Phi_2|_{r=b}, & \frac{\partial}{\partial \theta} \Phi_3|_{r=a} &= \frac{\partial}{\partial \theta} \Phi_2|_{r=a} \\ \mu_0 \frac{\partial}{\partial r} \Phi_1|_{r=b} &= \mu \frac{\partial}{\partial r} \Phi_2|_{r=b}, & \mu_0 \frac{\partial}{\partial r} \Phi_3|_{r=a} &= \mu \frac{\partial}{\partial r} \Phi_2|_{r=a} \end{aligned}$$

Todos los coeficientes con $l \neq 1$ se anulan. Para $l = 1$ se tiene:

$$\begin{aligned} \alpha_1 - b^3 \beta_1 - \gamma_1 &= b^3 H_0 \\ 2\alpha_1 + \mu' b^3 \beta_1 - 2\mu' \gamma_1 &= -b^3 H_0 \\ a^3 \beta_1 + \gamma_1 - a^3 \delta_1 &= 0 \\ \mu' a^3 \beta_1 - 2\mu' \gamma_1 - a^3 \delta_1 &= 0 \end{aligned}$$

$$\mu' = \frac{\mu}{\mu_0}.$$

$$\alpha_1 = \left[\frac{(2\mu' + 1)(\mu' - 1)}{(2\mu' + 1)(\mu' + 2) - 2\frac{a^3}{b^3}(\mu' - 1)^2} \right] (b^3 - a^3) H_0$$

$$\delta_1 = - \left[\frac{9\mu'}{(2\mu' + 1)(\mu' + 2) - 2\frac{a^3}{b^3}(\mu' - 1)^2} \right] H_0$$

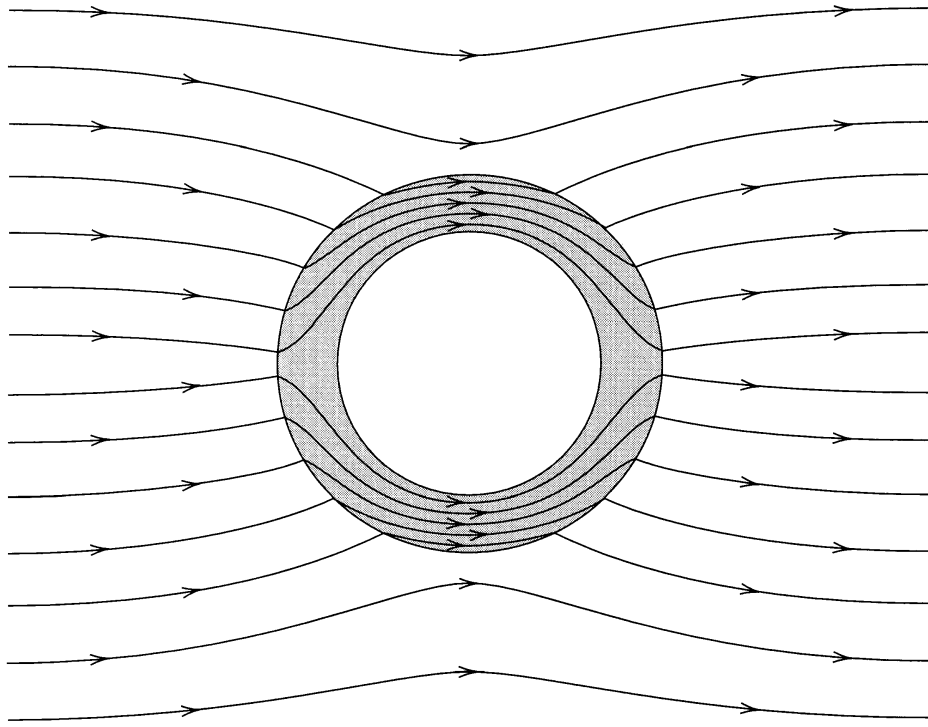


Figura 6.

Para $\mu \gg \mu_0$

$$\alpha_1 = b^3 H_0$$

$$\delta_1 = -\frac{9}{2\mu'} \frac{H_0}{1 - \frac{a^3}{b^3}} \rightarrow_{\mu' \rightarrow \infty} 0$$

El campo magnético puede ser muy débil para $r < a$.

