Frequency Dispersion: Dielectrics, Conductors, and Plasmas

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Simple Model for $\epsilon(\omega)$
Simple Model for $\epsilon(\omega)$

Electron of charge $-e$ bound by a harmonic force to an atom, and acted on by an electric field

Equation of motion:

$$m \left[ \dddot{x} + \gamma \ddot{x} + \omega_0^2 x \right] = -e \vec{E}(\vec{x}, t)$$

(1)
If the field varies harmonically in time with frequency $\omega$.

\[
\vec{E} \sim \vec{E} e^{-i\omega t} \tag{2}
\]
\[
\vec{x} \sim \vec{x}_0 e^{-i\omega t} \tag{3}
\]

Dipole moment contributed by one electron:

\[
\vec{p} = - e \vec{x} = \frac{e^2}{m(\omega_0^2 - \omega^2 - i\omega\gamma)} \vec{E} = \frac{e^2}{m} \frac{(\omega_0^2 - \omega^2) + i\omega\gamma}{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2} \vec{E} \tag{4}
\]

Difference of phase between the electric field and the dipole moment:

\[
\phi = \tan^{-1} \left( \frac{\omega\gamma}{\omega_0^2 - \omega^2} \right) \tag{5}
\]
Atomic contribution to the dielectric constant for $N$ molecules per unit volume, $f_j$ electrons per molecule with binding frequency $\omega_j$ and damping constant $\gamma_j$:

\[
\frac{\varepsilon(\omega)}{\varepsilon_0} = 1 + \chi_e = 1 + N \frac{e^2}{\varepsilon_0 m} \sum_j \frac{f_j}{(\omega_j^2 - \omega^2 - i\omega \gamma_j)}
\]  (6)
Anomalous Dispersion and Resonant Absorption
Anomalous Dispersion and Resonant Absorption

In a dispersive medium, the wave equation for the electric field reads

\[ \nabla^2 \vec{E} = \epsilon \mu_0 \frac{\partial^2 \vec{E}}{\partial t^2} \]  

(7)

it admits plane wave solutions

\[ \vec{E}(z, t) = \vec{E}_0 e^{i(kz - \omega t)} \]  

(8)

with the complex wave number

\[ k = \sqrt{\mu_0 \epsilon \omega} \]  

(9)
We can write the wave number in terms of this real and imaginary parts

\[ k = \beta + i \frac{\alpha}{2} \] (10)

and the electric field becomes

\[ \vec{E}(z, t) = \vec{E}_0 e^{-\frac{\alpha}{2}z} e^{i(\beta z - \omega t)} \] (11)

- Intensity proportional to \( E^2 \) (to \( e^{-\alpha z} \)), \( \alpha \) is called the absorption coefficient.
- Index of refraction: \( n = \frac{c \beta}{\omega} \)
\[
\frac{\epsilon(\omega)}{\epsilon_0} = 1 + N \frac{e^2}{\epsilon_0 m} \sum_j \frac{f_j}{(\omega_j^2 - \omega^2 - i\omega\gamma_j)}
\]

- The damping constants \(\gamma_j\) are generally small compared with the resonant frequencies \(\omega_j\).
- Normal dispersion region is associated with the increase of \(\text{Re}(\epsilon)\) with \(\omega\).
- Anomalous dispersion (resonant absorption) is where the imaginary part of \(\epsilon\) is appreciable, and that represents dissipation of energy.
Low-Frequency Behavior, Electric Conductivity
Low-Frequency Behavior, Electric Conductivity

\[
\frac{\varepsilon(\omega)}{\varepsilon_0} = 1 + N \frac{e^2}{\varepsilon_0 m} \sum_j \frac{f_j}{(\omega_j^2 - \omega^2 - i\omega\gamma_j)}
\]

Fraction \(f_0\) of electrons per molecule free \((\omega_0 = 0)\)

\[
\varepsilon(\omega) = \varepsilon_b(\omega) + \frac{Ne^2}{m} \frac{f_0}{-\omega^2 - i\omega\gamma_0} = \varepsilon_b(\omega) + i \frac{Ne^2}{m} \frac{f_0}{\omega(\gamma_0 - i\omega)}
\] (12)
Maxwell-Ampere equation: \( \vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \)

- Assuming medium obeys Ohm’s law: \( \vec{J} = \sigma \vec{E}, \quad \vec{D} = \epsilon_b \vec{E} \)

\[
\vec{\nabla} \times \vec{H} = -i \omega \left( \epsilon_b + i \frac{\sigma}{\omega} \right) \vec{E}
\]  

(13)

- Assuming all the properties due the medium: \( \vec{D} = \epsilon(\omega) \vec{E} \)

\[
\vec{\nabla} \times \vec{H} = -i \omega \epsilon(\omega) \vec{E}
\]  

(14)
By comparison we get the electric conductivity (model of Drude)

$$\sigma = \frac{f_0 Ne^2}{m(\gamma_0 - i\omega)}$$  \hspace{1cm} (15)$$

The dispersive properties of the medium can be attributed as well to a complex dielectric constant, as to a frequency-dependent conductivity and a dielectric constant.
High-Frequency Limit, Plasma Frequency
At frequencies far above the highest resonant frequency

$$\frac{\epsilon(\omega)}{\epsilon_0} \approx 1 - \frac{\omega_p^2}{\omega^2} \quad (16)$$

where $\omega_p = \frac{NZe^2}{\epsilon_0 m}$ is called the plasma frequency of the medium.
The wave number is given by

\[ ck = \sqrt{\omega^2 - \omega_p^2} \]  \hspace{1cm} (17)

- In dielectric media, the approximation holds only for \( \omega^2 >> \omega_p^2 \).
- In plasmas, the electrons are free and the damping force is negligible, so the approximation holds over a wide range of frequencies, including \( \omega << \omega_p \).
- In conductors, the approximation holds for frequencies \( \omega >> \gamma_0 \) and the behavior of incident waves for \( \omega << \omega_p \) is similar to the behavior for plasma.
Example: Liquid Water
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Index of refraction $n$

Visible
(4000–7000 Å)
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Absorption coefficient \( a \) (cm\(^{-1}\))

Frequency (Hz)

1 km  
1 m  
1 cm  
1 \( \mu \)m  

1 km  
1 m  
1 cm  
1 \( \mu \)m  

\( K \) edge in oxygen  

Sea water
Bibliography

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Thank You