Frequency Dispersion: Dielectrics, Conductors, and Plasmas

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Simple Model for $\epsilon(\omega)$

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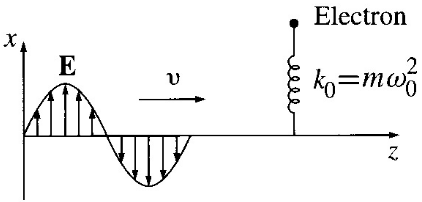
Electron of charge $-e$ bound by a harmonic force to an atom, and acted on by an electric field



Equation of motion:

$$m\left[\ddot{\vec{x}} + \gamma\dot{\vec{x}} + \omega_0^2\vec{x}\right] = -e\vec{E}(\vec{x}, t) \tag{1}$$

If the field varies harmonically in time with frequency ω.

$$\vec{E} \sim \vec{E}e^{-i\omega t} \tag{2}$$

$$\vec{x} \sim \vec{x}_0 e^{-i\omega t} \tag{3}$$

Dipole moment contributed by one electron:

$$\vec{p} = -e\vec{x} = \frac{e^2}{m(\omega_0^2 - \omega^2 - i\omega\gamma)}\vec{E} = \frac{e^2}{m}\frac{(\omega_0^2 - \omega^2) + i\omega\gamma}{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}\vec{E} \tag{4}$$

Difference of phase between the electric field and the dipole moment:

$$\phi = tan^{-1}\left(\frac{\omega\gamma}{\omega_0^2 - \omega^2}\right) \tag{5}$$

Atomic contribution to the dielectric constant for N molecules per unit volume, f_j electrons per molecule with binding frequency ω_j and damping constan γ_j:

$$\frac{\epsilon(\omega)}{\epsilon_0} = 1 + \chi_e = 1 + N\frac{e^2}{\epsilon_0 m}\sum_j \frac{f_j}{(\omega_j^2 - \omega^2 - i\omega\gamma_j)} \tag{6}$$

Anomalous Dispersion and Resonant Absorption

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In a dispersive medium, the wave equation for the electric field reads

$$\nabla^2 \vec{E} = \epsilon \mu_0 \frac{\partial^2 \vec{E}}{\partial t^2} \tag{7}$$

it admits plane wave solutions

$$\vec{E}(z,t) = \vec{E}_0 e^{i(kz - \omega t)} \tag{8}$$

with the complex wave number

$$k = \sqrt{\mu_0 \epsilon}\,\omega \tag{9}$$

We can write the wave number in terms of this real and imaginary parts

$$k = \beta + i\frac{\alpha}{2} \tag{10}$$

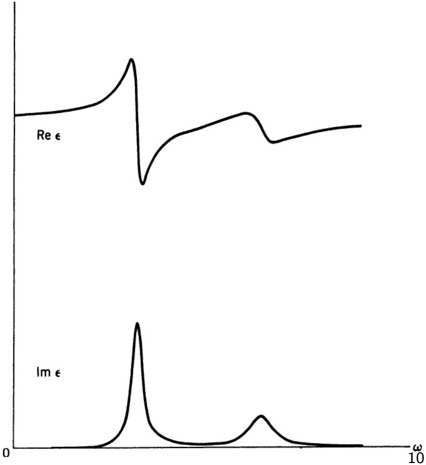
and the electric field becomes

$$\vec{E}(z,t) = \vec{E}_0 e^{-\frac{\alpha}{2}z} e^{i(\beta z - \omega t)} \tag{11}$$

- Intensity proportional to E^2 (to $e^{-\alpha z}$), α is called the absorption coefficient.
- Index of refraction: $n = \dfrac{c\beta}{\omega}$

$$\frac{\epsilon(\omega)}{\epsilon_0} = 1 + N\frac{e^2}{\epsilon_0 m}\sum_j \frac{f_j}{(\omega_j^2 - \omega^2 - i\omega\gamma_j)}$$

- The damping constants γ_j are generally small compared with the resonant frequencies ω_j.
- Normal dispersion region is associated with the increase of $Re(\epsilon)$ with ω.
- Anomalous dispersion (resonant absorption) is where the imaginary part of ϵ is appreciable, and that represents dissipation of energy.



Low-Frequency Behavior, Electric Conductivity

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$$\frac{\epsilon(\omega)}{\epsilon_0} = 1 + N\frac{e^2}{\epsilon_0 m}\sum_j \frac{f_j}{(\omega_j^2 - \omega^2 - i\omega\gamma_j)}$$

Fraction f_0 of electrons per molecule free $(\omega_0 = 0)$

$$\epsilon(\omega) = \epsilon_b(\omega) + \frac{Ne^2}{m}\frac{f_0}{-\omega^2 - i\omega\gamma_0} = \epsilon_b(\omega) + i\frac{Ne^2}{m}\frac{f_0}{\omega(\gamma_0 - i\omega)} \tag{12}$$

Maxwell-Ampere equation: $\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial D}{\partial t}$

- Assuming medium obbeys Ohm's law: $\vec{J} = \sigma \vec{E}$, $\vec{D} = \epsilon_b \vec{E}$

$$\vec{\nabla} \times \vec{H} = -i\omega \left(\epsilon_b + i\frac{\sigma}{\omega} \right) \vec{E} \tag{13}$$

- Assuming all the properties due the medium: $\vec{D} = \epsilon(\omega)\vec{E}$

$$\vec{\nabla} \times \vec{H} = -i\omega\epsilon(\omega)\vec{E} \tag{14}$$

By comparison we get the electric conductivity (model of Drude)

$$\sigma = \frac{f_0 N e^2}{m(\gamma_0 - i\omega)} \tag{15}$$

The dispersive properties of the medium can be attributed as well to a complez dielectric constant, as to a frequency-dependent conductivity and a dielectric constant.

High-Frequency Limit, Plasma Frequency

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$$\frac{\epsilon(\omega)}{\epsilon_0} = 1 + N\frac{e^2}{\epsilon_0 m}\sum_j \frac{f_j}{(\omega_j^2 - \omega^2 - i\omega\gamma_j)}$$

At frequencies far above the highest resonant frequency

$$\frac{\epsilon(\omega)}{\epsilon_0} \approx 1 - \frac{\omega_p^2}{\omega^2} \tag{16}$$

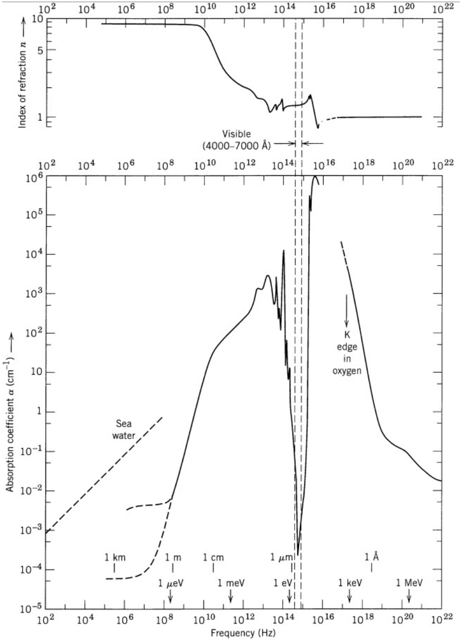
where $\omega_p = \frac{NZe^2}{\epsilon_0 m}$ is called the plasma frequency of the medium.

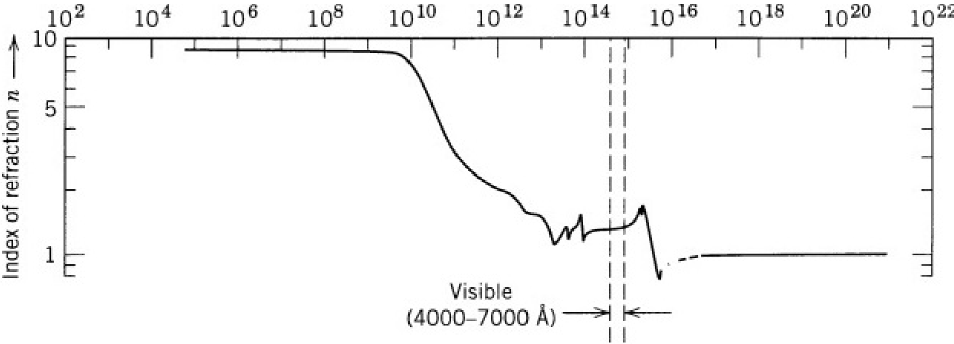
The wave number is given by

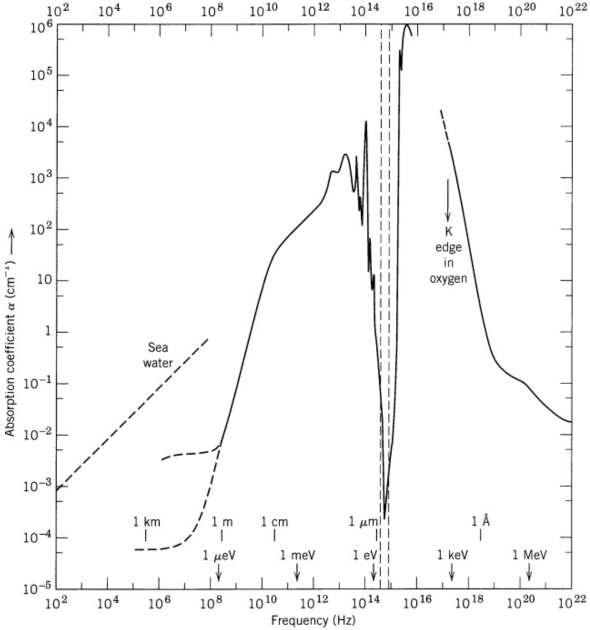
$$ck = \sqrt{\omega^2 - \omega_p^2} \tag{17}$$

- In dielectric media, the approximation holds only for $\omega^2 >> \omega_p^2$.
- In plasmas, the electrons are free and the damping force is negligible, so the approximation holds over a wide range of requencies, including $\omega < \omega_p$.
- In conductors, the approximation holds for frequencies $\omega >> \gamma_0$ and the behavior of incident waves for $\omega << \omega_p$ is similar to the behavior for plasma

Example: Liquid Water


Index of refraction n
Visible
(4000–7000 Å)
Absorption coefficient α (cm⁻¹)
Sea
water
K
edge
in
oxygen
1 km
1 m
1 cm
1 μm
1 Å
1 μeV
1 meV
1 eV
1 keV
1 MeV
Frequency (Hz)


10^2
10^4
10^6
10^8
10^{10}
10^{12}
10^{14}
10^{16}
10^{18}
10^{20}
10
5
1
Index of refraction n
Visible
(4000–7000 Å)


Absorption coefficient α (cm⁻¹)
Frequency (Hz)
Sea water
K edge in oxygen
1 km
1 m
1 cm
1 μm
1 Å
1 μeV
1 meV
1 eV
1 keV
1 MeV

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Thank You