- $\vec{\nabla}\vec{B}(\vec{x})=0$ No hay monopolos magnéticos.
- $\vec{\nabla}\times\vec{B}(\vec{x})=\mu_0\vec{J}(\vec{x})$ Ley de Ampére

Si $\vec{J}(\vec{x})=0$, se tiene: $\vec{B}(\vec{x})=-\nabla\Phi_m(\vec{x})$, $\Phi_m(\vec{x})$ satisface la ecuación de Laplace. Podemos usar los métodos desarrollados en Electrostática para encontrar en campo magnético.

En general:$\vec{\nabla}\vec{B}(\vec{x})=0$ implica:$\vec{B}(\vec{x})=\vec{\nabla}\times\vec{A}(\vec{x})$, $\vec{A}(\vec{x})$ es el potencial vector.

Transformación de gauge: $\vec{A}'(\vec{x})=\vec{A}(\vec{x})+\nabla\psi$, no cambia $\vec{B}(\vec{x})$.

$$\vec{\nabla}\times\vec{B}(\vec{x})=\vec{\nabla}\times\left(\vec{\nabla}\times\vec{A}(\vec{x})\right)=-\nabla^2 A+\nabla(\nabla A)=\mu_0\vec{J}(\vec{x})$$

Utilizando la libertad de gauge podemos fijar el gauge de Coulomb $\nabla A=0$. En efecto:

$$0=\nabla A'=\nabla A+\nabla^2\psi$$
$$\nabla^2\psi=-\nabla A \text{ tiene siempre solución para } \psi$$

$$\nabla^2 A=-\mu_0\vec{J}(\vec{x})$$

En el espacio abierto:

$$\vec{A}(\vec{x})=\frac{\mu_0}{4\pi}\int d^3x'\frac{\vec{J}(x')}{|\vec{x}-\vec{x}'|}$$

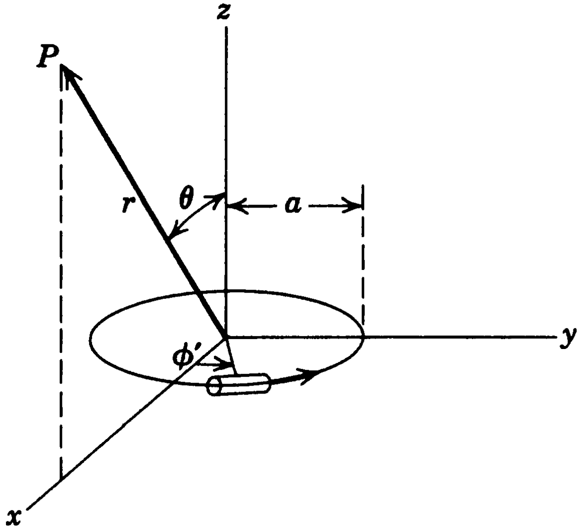

Figura 1.

$$J_\phi = I \operatorname{sen}\theta'\delta(\cos\theta')\frac{\delta(r'-a)}{a}$$

$$J = -J_\phi \operatorname{sen}\phi'\hat{i} + J_\phi \cos\phi'\hat{j}$$

Por simetría, podemos escoger el punto de observación con $\phi = 0$.

$$A_\phi(r,\theta) = \frac{\mu_0}{4\pi}\int d\phi' \int d\theta' \operatorname{sen}\theta' \int dr' r'^2 I \operatorname{sen}\theta' \delta(\cos\theta') \frac{\delta(r'-a)}{a} \frac{\cos\phi'}{|x-x'|} =$$

$$\frac{\mu_0 I a}{4\pi}\int d\phi' \frac{\cos\phi'}{(r^2+a^2-2r\,a\operatorname{sen}\theta\cos\phi')^{1/2}} = \qquad (1)$$

$$\frac{\mu_0 I a}{4\pi}\frac{4}{(r^2+a^2+2r\,a\operatorname{sen}\theta\,)^{1/2}}\left[\frac{(2-k^2)K(k)-2E(k)}{k^2}\right] \quad ,$$

$$k^2 = \frac{4a\,r\operatorname{sen}\theta}{r^2+a^2+2r\,a\operatorname{sen}\theta}$$

Ejercicio: Expandir el potencial vector (1) en armónicos esféricos.

$$\frac{1}{|\vec{x} - \vec{x}\,'|} = \frac{1}{|\vec{x}\,|} + \frac{\vec{x}\,.\vec{x}\,'}{|\vec{x}\,|^3}$$

$$\vec{A}_i(\vec{x}\,) = \frac{1}{|\vec{x}\,|}\int d^3x'\,J_i(x') + \frac{\vec{x}_j}{|\vec{x}\,|^3}\int d^3x'x'_jJ_i(x')$$

$$\int d^3x'\,J_i(x') = \int d^3x'\{(x'_jJ_i(x'))_{,i} - x'_jJ_{i,i}(x')\} = \oint dS_i\,x'_jJ_i(x') = 0$$

$$\int d^3x'\{x'_jx'_kJ_i(x')\}_{,i} = \oint dS_ix'_jx'_kJ_i(x') = 0$$

$$\int d^3x'\{\delta_{ji}x'_kJ_i(x') + x'_j\delta_{ki}J_i(x')\} = \int d^3x'\{x'_kJ_j(x') + x'_jJ_k(x')\}$$

$$\int d^3x'x'_jJ_i(x') = \frac{1}{2}\int d^3x'[x'_jJ_i(x') - x'_iJ_j(x')] = \varepsilon_{jik}\int d^3x'\big(\vec{x}\,'\times\vec{J}\,(\vec{x}\,')\big)_k$$

$$\mathcal{M}(x') = \frac{1}{2}\vec{x}\,'\times\vec{J}\,(\vec{x}\,')$$

es la magnetización.

$$\vec{m}_k = \frac{1}{2}\int d^3x'\big(\vec{x}\,'\times\vec{J}\,(\vec{x}\,')\big)_k$$

es el momento magnético.

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\frac{m \times x}{|x|^3}$$

es el potencial vector del dipolo magnético.

$$B_k(x) = B_k(0) + x.\nabla B_k(0) + ...$$
$$F = \int d^3x\, J(x) \times B(x)$$
$$F_i = \varepsilon_{ijk} \int d^3x\, J_j(x) B_k(x) \sim$$
$$\varepsilon_{ijk} \int d^3x\, J_j(x)[B_k(0) + x.\nabla B_k(0)] =$$
$$\varepsilon_{ijk} \int d^3x\, J_j(x) x_l B_{k,l}(0) = -\varepsilon_{ijk}\varepsilon_{jln} m_n B_{k,l}(0) =$$
$$(\delta_{il}\delta_{kn} - \delta_{\text{in}}\delta_{kl}) m_n B_{k,l}(0) =$$
$$m_k B_{k,i} - m_i B_{k,k} = \nabla_i (m.B)$$

$$N = \int d^3x\, x \times (J(x) \times B(x)) \sim$$
$$\int d^3x\, x \times (J(x) \times B(0))$$
$$N_i = \varepsilon_{ijk} \int d^3x\, x_j \varepsilon_{kln} J_l B_n(0) = (\delta_{il}\delta_{jn} - \delta_{\text{in}}\delta_{\text{jl}}) \int d^3x\, x_j J_l B_n(0) =$$
$$\int d^3x\, x_j J_i B_j(0) - \int d^3x\, x_j J_j B_i(0) =$$

$$\varepsilon_{jik} m_k B_j(0) - 0 =$$
$$(m \times B(0))_i$$

$$\int d^3x \, (x_k \, x_k J_j)_{,j} = \int d^3x \, (2x_j J_j + x_k x_k J_{j,j}) = 2 \int d^3x \, x_j J_j = 0$$

$$\vec{N} = \vec{m} \times \vec{B}(0)$$

Energía Potencial: $\vec{F} = -\vec{\nabla} U$, $U = -\vec{m}.\vec{B}$

$\nabla B = 0$, $B = \nabla \times A$

$$\vec{M}(\vec{x}) = \sum_i N_i < \vec{m}_i >$$

$M(x)$: Magnetización.

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int d^3x' \left[\frac{\vec{J}(x')}{|\vec{x}-\vec{x}\,'|} + \frac{\vec{M}(\vec{x}\,') \times (\vec{x}-\vec{x}\,')}{|x-x'|^3} \right]$$

$$\int d^3x' \left[\frac{\vec{M}(\vec{x}\,') \times (\vec{x}-\vec{x}\,')}{|x-x'|^3} \right] = \int d^3x' \left[\vec{M}(\vec{x}\,') \times \vec{\nabla}' \frac{1}{|x-x'|} \right] =$$

$$\int d^3x' \left[\vec{\nabla}' \times \vec{M}(x') \frac{1}{|x-x'|} \right]$$

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int d^3x' \left[\frac{\vec{J}(x') + \vec{\nabla}' \times \vec{M}(x')}{|\vec{x}-\vec{x}\,'|} \right]$$

Corriente de Magnetización efectiva: $\vec{J}_M(\vec{x}\,') = \vec{\nabla}' \times \vec{M}(x')$.

$$\vec{\nabla} \times \vec{B}(\vec{x}) = \mu_0 \vec{J}(\vec{x}) + \mu_0 \vec{\nabla} \times \vec{M}(x)$$

$$\vec{H} = \frac{1}{\mu_0}\vec{B} - \vec{M}$$

$$\vec{\nabla} \times \vec{H}(\vec{x}) = \vec{J}(\vec{x})$$

$\vec{H}$: Campo magnético

Para materiales diamagnéticos y paramagnéticos isotrópicos :

$$B = \mu H$$

μ:permeabilidad magnética.

Paramagnético: $\mu > \mu_0$

Diamagnético: $\mu < \mu_0$

Ferromagnético: $B = F(H)$

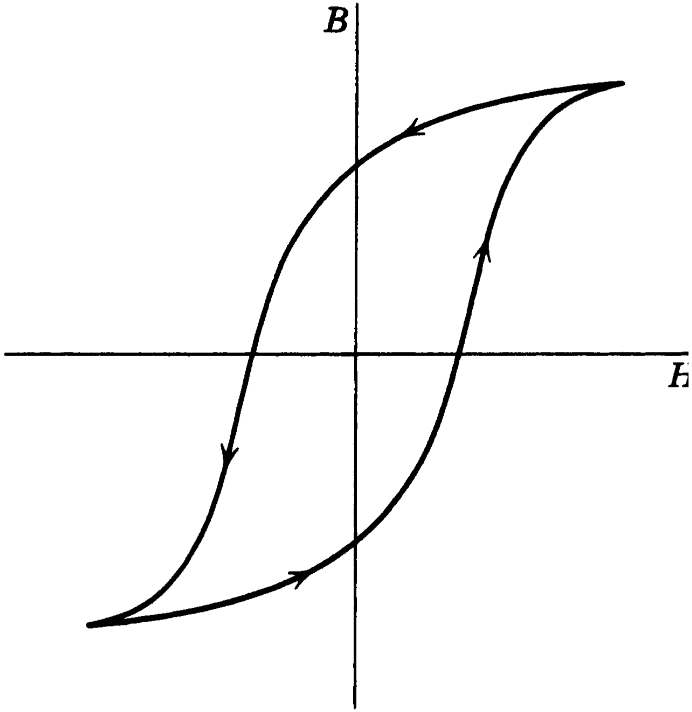

Figura 2. Histéresis

En una superficie S que separa las regiones 1 y 2:

$$\left(\vec{B}_2 - \vec{B}_1\right).\hat{n} = 0$$
$$\hat{n} \times \left(\vec{H}_2 - \vec{H}_1\right) = \vec{K}$$

$\hat{n}$; normal a la superficie que apunta de 1 a 2.

$\vec{K}$: densidad de corriente superficial(no incluye corriente de magnetización).

1. Potencial vector, medios lineales:

$$\vec{B}(\vec{x}) = \vec{\nabla} \times \vec{A}(\vec{x})$$
$$\nabla \times \frac{1}{\mu}\big(\vec{\nabla} \times \vec{A}(\vec{x})\big) = J$$
$$-\nabla^2 A + \nabla(\nabla A) = \mu J \qquad \nabla . A = 0 \, \text{gauge de Coulomb}$$
$$\nabla^2 A = -\mu J$$

2. $\vec{J} = \vec{0}$,$\vec{H} = -\vec{\nabla}\Phi_M$,$\Phi_M$ es el potencial escalar magnético.

$$\vec{\nabla}.\vec{B} = 0, \, \nabla^2 \Phi_M = 0$$

3. Ferromagnetos duros:$J = 0$, M dado.

   a) Potencial escalar:

$$\vec{H} = -\vec{\nabla}\Phi_M, \quad \vec{\nabla}.\vec{B} = 0 = \mu_0 \vec{\nabla}\big(\vec{H} + \vec{M}\big)$$
$$\nabla^2 \Phi_M = -\rho_M, \qquad \rho_M = -\vec{\nabla}.\vec{M}$$

En el espacio abierto:

$$\Phi_M(\vec{x}) = -\frac{1}{4\pi}\int d^3x' \frac{\vec{\nabla}'.\vec{M}(x')}{|\vec{x}-\vec{x}\,'|} = \frac{1}{4\pi}\int d^3x' \vec{M}(x').\vec{\nabla}'\frac{1}{|\vec{x}-\vec{x}\,'|} =$$

$$-\frac{1}{4\pi}\vec{\nabla}\int d^3x' \frac{\vec{M}(x')}{|\vec{x}-\vec{x}\,'|}$$

Si hay una discontinuidad en la magnetización se induce una densidad superficial de magnetización:$\sigma_M = \hat{n}.M$ con lo cual:

$$\Phi_M(\vec{x}) = -\frac{1}{4\pi}\int d^3x' \frac{\vec{\nabla}'.\vec{M}(x')}{|\vec{x}-\vec{x}\,'|} + \frac{1}{4\pi}\oint dS' \frac{\hat{n}.\vec{M}(x')}{|\vec{x}-\vec{x}\,'|} =$$

$\Phi_M \sim_{r\to\infty} \frac{\vec{m}.\vec{x}}{4\pi r^3}$, $\vec{m} = \int \, d^3x' \vec{M}(x')$

b) Potencial vector.

$$\vec{B}(\vec{x}) = \vec{\nabla} \times \vec{A}(\vec{x})$$
$$\vec{\nabla} \times \vec{H}(\vec{x}) = 0 = \vec{\nabla} \times \left(\frac{\vec{B}}{\mu_0} - \vec{M} \right)$$
$$\nabla^2 A = -\mu_0 \vec{\nabla} \times \vec{M}(\vec{x})$$
$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \int d^3x' \frac{\vec{\nabla}' \times \vec{M}(x')}{|\vec{x} - \vec{x}\,'|}$$

Si hay discontinuidades en la magnetización:

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \int d^3x' \frac{\vec{\nabla}' \times \vec{M}(x')}{|\vec{x} - \vec{x}\,'|} + \frac{\mu_0}{4\pi} \oint_S dS' \frac{M(x') \times n'}{|x - x'|}$$

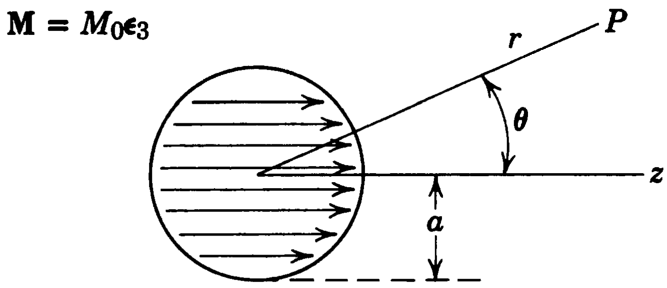

Figura 3.

$$\vec{M} = M_0\,\hat{z}\,\theta(a-r), \sigma_M = \hat{n}.\vec{M} = M_0 \cos\theta$$

$$\Phi_M(\vec{x}) = \frac{1}{4\pi}\oint dS' \frac{\hat{n}.\vec{M}(x')}{|\vec{x}-\vec{x}\,'|} =$$

$$\frac{M_0 a^2}{4\pi} \int d\Omega' \frac{\cos\theta'}{|\vec{x} - \vec{x}\,'|}$$

$$\frac{1}{|\,\mathbf{x} - \mathbf{x}'|} = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{2l+1} \frac{r_<^{\ell}}{r_>^{\ell+1}} Y_{lm}^{\star}(\theta\prime, \phi\prime)\;\; Y_{lm}(\theta, \phi)$$

$$\frac{M_0 a^2}{4\pi} \int d\Omega' \frac{\cos\theta'}{|\vec{x} - \vec{x}\,'|} = \;\; \frac{M_0 a^2}{3} \frac{r_<}{r_>^2} \cos\theta, \;\; r_< = \min\{r, a\}$$

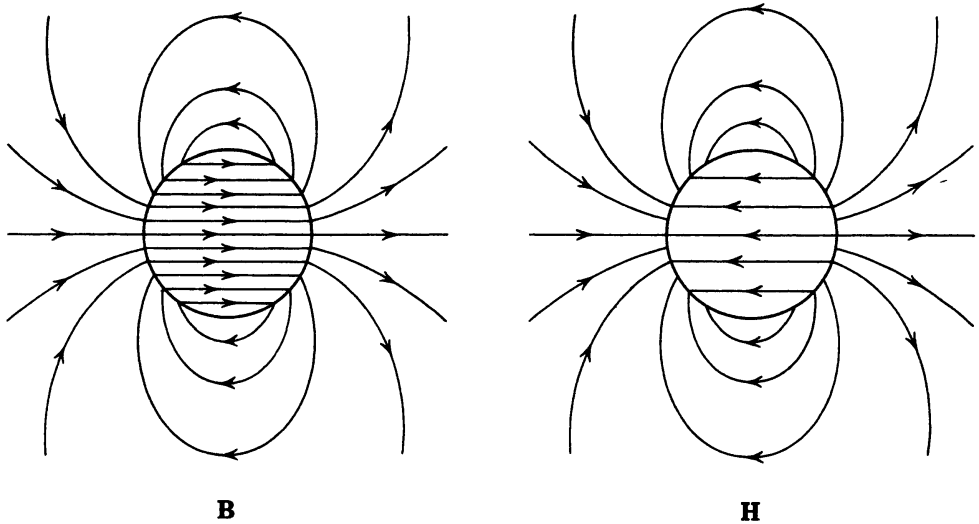

Figura 4.

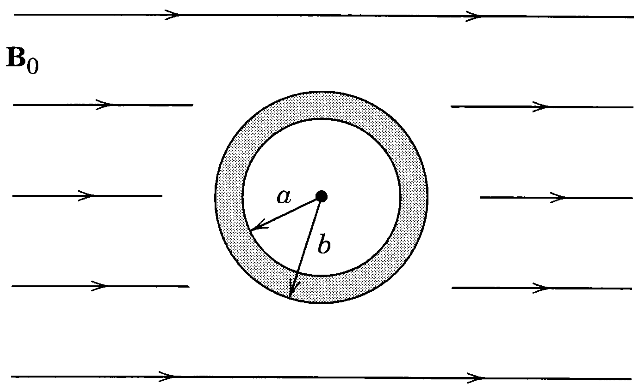

Figura 5.

$$H = -\nabla\Phi_M \quad \nabla B = \mu\nabla H = 0, \quad \nabla^2\Phi_M = 0$$

El problema tiene simetría azimutal con $\vec{B}_0 = B_0\hat{z}$

1)$r > b$, $\Phi_1(r,\theta) = \sum_{\ell=0}^{\infty} \left[A_\ell r^\ell + \alpha_\ell r^{-(\ell+1)} \right] P_\ell(\cos\theta)$, $A_1 = -H_0$, $A_l = 0, l \neq 1$

2) $a < r < b$, $\Phi_2(r,\theta) = \sum_{\ell=0}^{\infty} \left[\beta_\ell r^\ell + \gamma_\ell r^{-(\ell+1)} \right] P_\ell(\cos\theta)$

3) $r < a$, $\Phi(r,\theta) = \sum_{\ell=0}^{\infty} \left[\delta_\ell r^\ell \right] P_\ell(\cos\theta)$

Condiciones de borde en $r = a, b$.

$$\frac{\partial}{\partial\theta}\Phi_1|_{r=b} = \frac{\partial}{\partial\theta}\Phi_2|_{r=b}, \quad \frac{\partial}{\partial\theta}\Phi_3|_{r=a} = \frac{\partial}{\partial\theta}\Phi_2|_{r=a}$$
$$\mu_0\frac{\partial}{\partial r}\Phi_1|_{r=b} = \mu\frac{\partial}{\partial r}\Phi_2|_{r=b}, \quad \mu_0\frac{\partial}{\partial r}\Phi_3|_{r=a} = \mu\frac{\partial}{\partial r}\Phi_2|_{r=a}$$

Todos los coeficientes con $l \neq 1$ se anulan. Para $l = 1$ se tiene:

$$\begin{aligned}
\alpha_1 - b^3\beta_1 - \gamma_1 &= b^3 H_0 \\
2\alpha_1 + \mu' b^3\beta_1 - 2\mu'\gamma_1 &= -b^3 H_0 \\
a^3\beta_1 + \gamma_1 - a^3\delta_1 &= 0 \\
\mu' a^3\beta_1 - 2\mu'\gamma_1 - a^3\delta_1 &= 0
\end{aligned}$$

$\mu' = \frac{\mu}{\mu_0}$.

$$\alpha_1 = \left[\frac{(2\mu'+1)(\mu'-1)}{(2\mu'+1)(\mu'+2) - 2\frac{a^3}{b^3}(\mu'-1)^2}\right](b^3 - a^3)H_0$$

$$\delta_1 = -\left[\frac{9\mu'}{(2\mu'+1)(\mu'+2) - 2\frac{a^3}{b^3}(\mu'-1)^2}\right]H_0$$

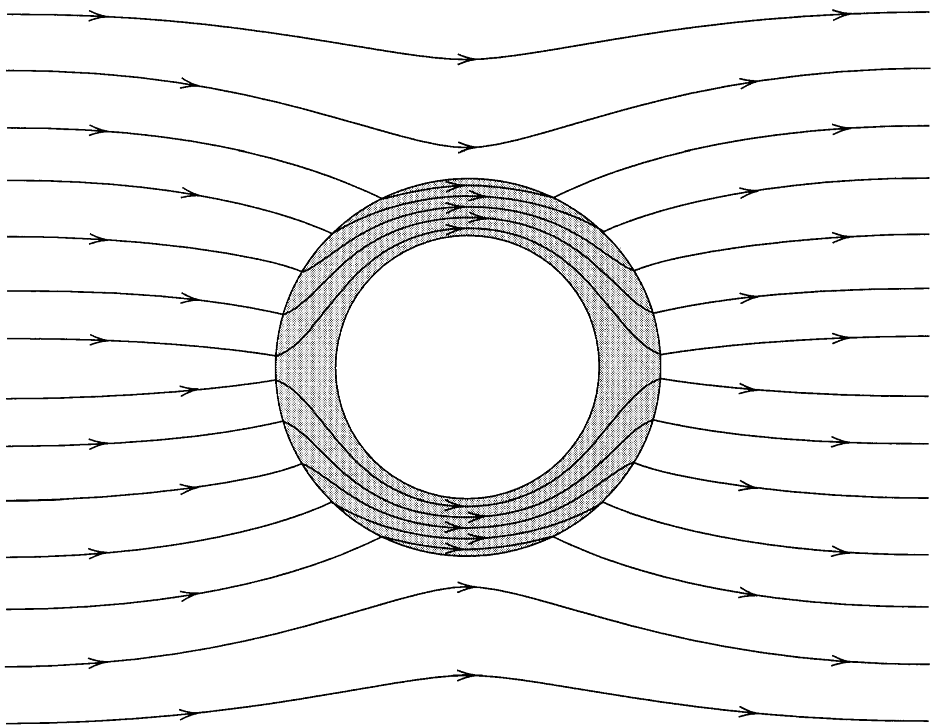

Figura 6.

Para $\mu \gg \mu_0$

$$\alpha_1 = b^3 H_0$$

$$\delta_1 = -\frac{9}{2\mu'}\frac{H_0}{1-\frac{a^3}{b^3}} \rightarrow_{\mu' \rightarrow \infty} 0$$

El campo magnético puede ser muy débil para $r < a$.