General Tensor

Let us consider a general coordinate transformation:

$$x\prime^a = x\prime^a(x), a = 1...., n \tag{1}$$

n is the dimension of space.
We have:

$$dx'^a = \frac{\partial x'^a}{\partial x^b} dx^b \tag{2}$$

In analogy with this we say that V^a is a contravariant vector if under (1) it transforms like (2), i.e.

$$V'^a(x') = \frac{\partial x'^a}{\partial x^b} V^b(x) \tag{3}$$

Similarly, let us consider the gradient of a scalar function. We say that $A(x)$ is a scalar function if:$A\prime(x\prime) = A(x)$. It follows that:

$$\frac{\partial A'(x')}{\partial x'^a} = \frac{\partial A(x)}{\partial x'^a} = \frac{\partial A(x)}{\partial x^b} \frac{\partial x^b}{\partial x'^a} \tag{4}$$

In analogy with this transformation law, we say that U_a is a covariant vector if under a change of variables (1), we have:

$$U'_a(x') = \frac{\partial x^b}{\partial x'^a} U_b(x) \tag{5}$$

1 Tensor Product

Consider two covariant vectors, U_a, V_a. We have:

$$U'_a(x') V'_b(x') = \frac{\partial x^c}{\partial x'^a} \frac{\partial x^d}{\partial x'^b} U_c(x) V_d(x) \tag{6}$$

We say that a set of functions T_{ab} that transforms under (1) as (6) is a covariant tensor of rank 2. That is:

$$T'_{ab}(x') = \frac{\partial x^c}{\partial x'^a} \frac{\partial x^d}{\partial x'^b} T_{cd}(x) \tag{7}$$

We call $S_{ab} = U_a V_b$ the tensorial product of the two covariant vectors U_a and V_b.

Generally we says that a set of functions $T\,{}^{b_1\ldots.b_q}_{a_1\ldots.a_p}$ is a p-covariant and q-contravariant tensor if under (1) we have:

$$T'\,{}^{b_1\ldots.b_q}_{a_1\ldots.a_p}(x') = \frac{\partial x^{c_1}}{\partial x'^{a_1}}\ldots.\frac{\partial x^{c_p}}{\partial x'^{a_p}}\frac{\partial x'^{b_1}}{\partial x^{d_1}}\ldots.\frac{\partial x'^{b_q}}{\partial x^{d_q}}T\,{}^{d_1\ldots.d_q}_{c_1\ldots.c_p}(x) \tag{8}$$

The tensor product of two arbitrary tensors $U\,{}^{b_1\ldots.b_q}_{a_1\ldots.a_p}$, $V\,{}^{d_1\ldots.d_s}_{c_1\ldots c_r}$ defined by

$$U\,{}^{b_1\ldots.b_q}_{a_1\ldots.a_p}(x)V\,{}^{d_1\ldots.d_s}_{c_1\ldots c_r}(x) \tag{9}$$

is a (p+r) covariant and (q+s) contravariant tensor.

2 Contraction

2.1

Consider a mixed tensor $T\,{}^{b}_{a}$, then

$$S = T^a_a$$

is a scalar.
NOTATION(Einstein): Repeated indices in a monomial are summed from 1 to n.
Proof:

$$T'^{a}_{a}(x') = \frac{\partial x'^{a}}{\partial x^{b}} \frac{\partial x^{c}}{\partial x'^{a}} T^{b}_{c}(x) = \delta^{c}_{b} T^{b}_{c}(x) = \ T^{a}_{a}(x)$$

In general the contraction of an upper index with a lower index in a p-covariant and q-contravariant tensor produces a (p-1) covariant and (q-1) contravariant tensor. Notice that contraction of different indices, produces different tensors.

3 Some important tensors

Kronecker delta:δ^{a}_{b} is a one covariant, one contravariant tensor. Moreover it is an invariant tensor, because it has the same components in all coordinate systems.
Proof:

$$\delta'^{b}_{a} = \frac{\partial x'^{b}}{\partial x'^{a}} = \frac{\partial x'^{b}}{\partial x^{c}} \frac{\partial x^{c}}{\partial x^{d}} \frac{\partial x^{d}}{\partial x'^{a}} = \frac{\partial x'^{b}}{\partial x^{c}} \frac{\partial x^{d}}{\partial x'^{a}} \delta^{c}_{d} \tag{10}$$

Symmetric and antisymmetric tensors.

Consider T^c_{ab} a tensor. Then: $S^c_{a\,b} = T^c_{a\,b} + T^c_{b\,a}$ and $A^c_{\ a\,b} = T^c_{a\,b} - T^c_{b\,a}$ are tensors, called the symmetric and antisymmetric component of $T^c_{a\,b}$.
Proof:

$$S'^c_{\ a\,b}\ (x') = T'^c_{\ a\,b}(x') + T'^c_{\ b\,a}(x') = \frac{\partial x'^c}{\partial x^i}\frac{\partial x^j}{\partial x'^a}\frac{\partial x^k}{\partial x'^b}T^i_{\ j\,k}(x) + \frac{\partial x'^c}{\partial x^i}\frac{\partial x^j}{\partial x'^b}\frac{\partial x^k}{\partial x'^a}T^i_{\ j\,k}(x) =$$

$$\frac{\partial x'^c}{\partial x^i}\frac{\partial x^j}{\partial x'^a}\frac{\partial x^k}{\partial x'^b}(T^i_{\ j\,k}(x) + T^i_{\ k\,j}(x)) = \frac{\partial x'^c}{\partial x^i}\frac{\partial x^j}{\partial x'^a}\frac{\partial x^k}{\partial x'^b}S^i_{\ j\,k}(x) \tag{11}$$

The proof is similar for the antisymmetric part.Notice that this operation can be applied to any pair of indices. Iterative application of the operation will produce tensors that will form representations of the permutation group S_n.

4 Pseudotensors

We define a pseudotensor of weight r a set of functions $D^{b_1 \dots . b_q}_{a_1 \dots . a_p}$ that under (1) transform as follows:

$$D'^{\ b_1 \dots . b_q}_{\ a_1 \dots . a_p}\ (x') = J^r \frac{\partial x^{c_1}}{\partial x'^{a_1}} \dots . \frac{\partial x^{c_p}}{\partial x'^{a_p}} \frac{\partial x'^{b_1}}{\partial x^{d_1}} \dots . \frac{\partial x'^{b_q}}{\partial x^{d_q}} D^{\ d_1 \dots . d_q}_{\ c_1 \dots . c_p}\ (x) \tag{12}$$

where J is the jacobian of the transformation (1). i.e.

$$J(x) = \det\left(\frac{\partial x^a}{\partial x'^b}\right) \tag{13}$$

r=0, tensor

r=1, pseudotensor or tensor density

r=-1, tensor capacity

5 The Levi-Civita symbol

$$\varepsilon^{a_1....a_n} = \text{sgn}\begin{pmatrix} 1....n \\ a_1....a_n \end{pmatrix} \tag{14}$$

Here sgn is the sign of the permutation in brackets. If some of the indices are repeated, it gives zero.

Use the determinant identity:

$$J^{-1}\text{sgn}(\sigma) = \sum_{\lambda \varepsilon S_n} \text{sgn}(\lambda) \frac{\partial x'^{\sigma(1)}}{\partial x^{\lambda(1)}} .... \frac{\partial x'^{\sigma(n)}}{\partial x^{\lambda(n)}}$$

or

$$J^{-1}\varepsilon^{a_1....a_n} = \frac{\partial {x'}^{a_1}}{\partial x^{b_1}}....\frac{\partial {x'}^{a_n}}{\partial x^{b_n}}\varepsilon^{b_1....b_n} \tag{15}$$

It follows that $\varepsilon^{a_1....a_n}$ is a pseudotensor of weight 1.
Similarly, we can prove that:

$$J\varepsilon_{a_1....a_n} = \frac{\partial {x'}^{b_1}}{\partial x^{a_1}}....\frac{\partial {x'}^{b_n}}{\partial x^{a_n}}\varepsilon_{b_1....b_n} \tag{16}$$

so $\varepsilon_{a_1....a_n}$is a pseudotensor of weight -1.

6 An important identity

We have that:

$$\varepsilon_{j_1....j_n}\varepsilon^{i_1....i_n} = \begin{vmatrix} \delta^{i_1}_{j_1}....\delta^{i_1}_{j_n} \\ \delta^{i_2}_{j_1}....\delta^{i_2}_{j_n} \\ .... \\ \delta^{i_n}_{j_1}....\delta^{i_n}_{j_n} \end{vmatrix} \tag{17}$$

Proof:

We must have: $i_1 = \sigma(1)....,\ i_n = \sigma(n)$ and $j_1 = \lambda(1),\ ....j_n = \lambda(n)$for some permutations σ and λ, otherwise the determinant vanishes (either two columns or two rows are equal). Then, we get:

$$\begin{vmatrix} \delta^{i_1}_{j_1}....\delta^{i_1}_{j_n} \\ \delta^{i_2}_{j_1}....\delta^{i_2}_{j_n} \\ .... \\ \delta^{i_n}_{j_1}....\delta^{i_n}_{j_n} \end{vmatrix} = \mathrm{sgn}(\sigma)\begin{vmatrix} \delta^{1}_{j_1}....\delta^{1}_{j_n} \\ \delta^{2}_{j_1}....\delta^{2}_{j_n} \\ .... \\ \delta^{n}_{j_1}....\delta^{1}_{j_n} \end{vmatrix} = \mathrm{sgn}(\sigma)\mathrm{sgn}(\lambda)\begin{vmatrix} \delta^{1}_{1}....\delta^{1}_{n} \\ \delta^{2}_{1}....\delta^{2}_{n} \\ .... \\ \delta^{n}_{1}....\delta^{1}_{n} \end{vmatrix} = \mathrm{sgn}(\sigma)\mathrm{sgn}(\lambda) =$$

$$\varepsilon_{\ j_1....j_n}\varepsilon^{\ i_1....i_n}$$

7 Lorentz transformations

$\eta_{\alpha\beta} = L^{\mu}_{\alpha}L^{\nu}_{\beta}\eta_{\mu\nu},\ \eta = L^t\eta L$, $\det L = \pm 1$

Proper transformations are continuosly connected to the identity. We can write:

$L^{\mu}_{\alpha} = \delta^{\mu}_{\alpha} + \omega^{\mu}_{\alpha}$,

$$\eta_{\alpha\beta} = (\delta^{\mu}_{\alpha} + \omega^{\mu}_{\alpha})(\delta^{\nu}_{\beta} + \omega^{\nu}_{\beta})\eta_{\mu\nu} \ = (\delta^{\mu}_{\alpha} + \omega^{\mu}_{\alpha})(\eta_{\mu\beta} + \omega_{\mu\beta}) = \eta_{\alpha\beta} + \omega_{\alpha\beta} + \omega_{\beta\alpha}$$

$\omega_{\beta\alpha} = -\omega_{\alpha\beta}$, 6 l.i componentes

7.1 Lorentz group generators

Boosts:

$$K_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$K_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$K_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Rotations:

$$J_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$J_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

$$J_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Lie Algebra:

$$[A, B] = AB - BA$$

$$[J_i, J_j] = \varepsilon_{ijk} J_k, \quad [K_i, K_j] = -\varepsilon_{ijk} J_k, \quad [J_i, K_j] = \varepsilon_{ijk} K_k \tag{18}$$

Exercise: Verify eq.(18).