Una manera alternativa de probar invariancia bajo Lorentz es la siguiente: considerar que la transformación del spinor está mediada por una matriz, i.e.

$$\psi(x) \rightarrow \psi'(x') = S(L)\,\psi(x)$$

Así,

$$\begin{aligned}
(i\;\gamma^{\mu}\partial_{\mu} - m)\,\psi(x) &= 0 \\
(i\;\gamma^{\mu}\,L^{\nu}_{\mu}\partial'_{\nu} - m)\,S^{-1}\,\psi'(x') &= \qquad\qquad /S\,\cdot \\
(i\,S\gamma^{\mu}S^{-1}L^{\nu}_{\mu}\partial'_{\nu} - m)\psi'(x') &= \\
(i\;\gamma^{\nu}\partial'_{\nu} - m)\,\psi'(x') &= 0
\end{aligned}$$

La relación impuesta es:

$$S\,\gamma^{\mu}\,S^{-1}\,L^{\nu}_{\mu} \equiv \gamma^{\nu},\, S\,\gamma^{\mu}\,S^{-1} = L^{-1}{}^{\mu}_{\nu}\gamma^{\nu}$$

Se tiene:$S\,\gamma^{\mu}\gamma^{\lambda}\,S^{-1} = L^{-1}{}^{\mu}_{\nu}\gamma^{\nu}L^{-1}{}^{\lambda}_{\alpha}\gamma^{\alpha} = L^{-1}{}^{\mu}_{\nu}L^{-1}{}^{\lambda}_{\alpha}\gamma^{\nu}\gamma^{\alpha}$. «tensor 2 veces covariante»

- Transformación infinitesimal:$\eta_{\alpha\beta} = L^\mu_\alpha L^\nu_\beta \eta_{\mu\nu}$,$L^\mu_\alpha = \delta^\mu_\alpha + \omega^\mu_\alpha$,

$$\eta_{\alpha\beta} = (\delta^\mu_\alpha + \omega^\mu_\alpha)(\delta^\nu_\beta + \omega^\nu_\beta)\eta_{\mu\nu} \;\; = (\delta^\mu_\alpha + \omega^\mu_\alpha)(\eta_{\mu\beta} + \omega_{\mu\beta}) = \eta_{\alpha\beta} + \omega_{\alpha\beta} + \omega_{\beta\alpha}$$

- $\omega_{\beta\alpha} = -\omega_{\alpha\beta}$, 6 componentes l.i.
- $S = 1 + k^{\alpha\beta}\omega_{\alpha\beta}$,

$$\begin{aligned} S\,\gamma^\mu\, S^{-1}\, L^\nu_\mu &\equiv \gamma^\nu \\ (1 + k^{\alpha\beta}\omega_{\alpha\beta})\gamma^\mu(1 - k^{\alpha\beta}\omega_{\alpha\beta})(\delta^\nu_\mu + \omega^\nu_\mu) &= (\gamma^\mu + k^{\alpha\beta}\omega_{\alpha\beta}\gamma^\mu)(\delta^\nu_\mu + \omega^\nu_\mu - k^{\alpha\beta}\omega_{\alpha\beta}\delta^\nu_\mu) = \\ \gamma^\nu + k^{\alpha\beta}\omega_{\alpha\beta}\gamma^\nu + \gamma^\mu\omega^\nu_\mu - \gamma^\nu k^{\alpha\beta}\omega_{\alpha\beta} &= \gamma^\nu \\ [k^{\alpha\beta}, \gamma^\nu]\omega_{\alpha\beta} + \gamma^\mu\omega^\nu_\mu &= 0 \\ \gamma^\mu\omega^\nu_\mu = \omega_{\alpha\beta}\gamma^\beta\eta^{\alpha\nu} &= \frac{1}{2}\omega_{\alpha\beta}(\gamma^\beta\eta^{\alpha\nu} - \gamma^\alpha\eta^{\beta\nu}) \end{aligned}$$

- $[k^{\alpha\beta}, \gamma^\nu] = \frac{1}{2}(-\gamma^\beta\eta^{\alpha\nu} + \gamma^\alpha\eta^{\beta\nu})$
- $k^{\alpha\beta}$ es un tensor dos veces covariante, antisimétrico. Debe ser $k^{\alpha\beta} = a[\gamma^\alpha, \gamma^\beta]$

$$k^{01} = 2a\gamma^0\gamma^1 \;\; [k^{01}, \gamma^0] = -4a\gamma^1 = \frac{1}{2}(-\gamma^1\eta^{00} + \gamma^0\eta^{10}) = -\frac{1}{2}\gamma^1, a = \frac{1}{8}$$

$S^{\alpha\beta}=\frac{i}{4}[\gamma^\alpha,\gamma^\beta]$, $S=1+\frac{1}{2i}S^{\alpha\beta}\omega_{\alpha\beta}$. Un elemento finito del grupo de Lorentz es :$S=e^{\frac{1}{2i}S^{\alpha\beta}\omega_{\alpha\beta}}$

Un conjunto completo de matrices l.i. es:$\Gamma_A=\{1,\gamma^5,\gamma^\mu,\gamma^5\gamma^\mu,\sigma^{\mu\nu}\}$, $\sigma^{\mu\nu}=\frac{i}{2}[\gamma^\mu,\gamma^\nu]$. Toda matriz compleja M de $4x4$ se puede escribir $M=\sum_A m_A\Gamma_A$

Bajo una transformación de Lorentz se tiene:$\psi'(x')=S(L)\,\psi(x)$,$\bar{\psi}'(x')=\bar{\psi}(x)\gamma^0 S(L)^\uparrow\gamma^0$

$\gamma^0 S(L)^\uparrow\gamma^0=1-\frac{1}{8}\gamma^0[\gamma^{\alpha\uparrow},\gamma^{\beta\uparrow}]\gamma^0=1-\frac{1}{8}[\gamma^\alpha,\gamma^\beta]=S^{-1}$

$\bar{\psi}'(x')=\bar{\psi}(x)S(L)^{-1}$

Muestre que:

Ejercicio 1. $\bar{\psi}(x)\psi(x)$ es un escalar de Lorentz

Ejercicio 2. $\bar{\psi}(x)\gamma^5\psi(x)$ es un pseudoescalar de Lorentz

Ejercicio 3. $\bar{\psi}(x)\gamma^\mu\psi(x)$ es un vector de Lorentz

Ejercicio 4. $\bar{\psi}(x)\gamma^5\gamma^\mu\psi(x)$ es un pseudovector de Lorentz

Ejercicio 5. $\bar{\psi}(x)\sigma^{\mu\nu}\psi(x)$ es un tensor de Lorentz antisimétrico.