Action Principle

$$S = \int_{t_i}^{t_f} dt\, L(q_i, \dot{q}_i; t)$$

$$\delta S = 0 \quad \delta q(t_i) = \delta q(t_f) = 0$$

Euler-Lagrange Equations: $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i}$

Coordinates Transformation:

$$q_i' = q_i'(q_i; a_\alpha) \qquad (1)$$

(1) forms a group under function composition.

Let $a_\alpha = 0$ be the identity transformation. Then, to first order in a_α:

$$\delta q_i = q_i'(q_i; a_\alpha) - q_i = \frac{\partial q_i'(q_i; a_\alpha)}{\partial a_\alpha}|_{a_\alpha = 0} a_\alpha \qquad (2)$$

δq_i is the infinitesimal transformation corresponding to (1).

- Assume L is invariant under a group of transformations of the generalized coordinates q_i

Under (2):$\delta L = \frac{\partial L}{\partial q_i}\delta q_i + \frac{\partial L}{\partial \dot{q}_i}\delta \dot{q}_i$. Using the E-L eqs.:$\delta L = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}\delta q_i + \frac{\partial L}{\partial \dot{q}_i}\delta \dot{q}_i = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\delta q_i\right) = 0$

$Q_\alpha = \frac{\partial L}{\partial \dot{q}_i}\frac{\partial q_i'(q_i; a_\alpha)}{\partial a_\alpha}|_{a_\alpha = 0}$ is conserved.

- Actually, all that matters is that the action is invariant. Consider the invariance of the action under time displacements $t' = t + a$. This is true if L does not depends on time explicitly.

- $\delta L = \frac{dL}{dt}a = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\delta q_i\right)$,$\delta q_i = \dot{q}_i a$, $H = \dot{q}_i\frac{\partial L}{\partial \dot{q}_i} - L$ is conserved.

- $p_i = \frac{\partial L}{\partial \dot{q}_i}$, conjugated canonical momentum.
- Legendre transformation :$H = \dot{q}_i p_i - L$, $\delta H = \delta\dot{q}_i p_i + \dot{q}_i \delta p_i - \frac{\partial L}{\partial q_i}\delta q_i - \frac{\partial L}{\partial \dot{q}_i}\delta \dot{q}_i = \dot{q}_i \delta p_i - \frac{\partial L}{\partial q_i}\delta q_i$, $H = H(q,p)$
- Hamilton eqs.:$\frac{\partial H}{\partial p_i} = \dot{q}_i$, $\frac{\partial H}{\partial q_i} = -\frac{\partial L}{\partial q_i} = -\dot{p}_i$.
- Canonical transformations:

$$q'_i = q'_i(q_i, p_i; a_\alpha), p'_i = p'_i(q_i, p_i; a_\alpha) \tag{3}$$

  that leaves Hamilton eqs. invariant.
- Poisson bracket: $\dot{A} = \frac{\partial A}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial H}{\partial q_i} \equiv \{H, A\}$

- q_i, p_i->$\hat{q}_i, \hat{p}_i$.
- $\{A, B\} \rightarrow \frac{i}{\hbar}[\hat{A}, \hat{B}]$
- Canonical commutation relations:$[\hat{q}_i, \hat{p}_j] = i\hbar\delta_{ij}$
- Coordinate representation: $\hat{p}_j = -i\hbar\frac{\partial}{\partial q_j}$
- Schrodinger eq. :$i\hbar\frac{\partial\psi}{\partial t} = \hat{H}\psi$
- If $\hat{H}$ is time independent: $\psi(t, q) = e^{-\frac{i}{\hbar}t\hat{H}}\psi(0, q)$
- Heisenberg representation: Operators are time dependent, wave functions are time independent: $\hat{A}_S\psi_S(t, q) = a\psi_S(t, q)$, $\hat{A}_S e^{-\frac{i}{\hbar}t\hat{H}}\psi(q)_H = a\, e^{-\frac{i}{\hbar}t\hat{H}}\psi(q)_H$, $\hat{A}_H = e^{\frac{i}{\hbar}t\hat{H}}\hat{A}_S e^{-\frac{i}{\hbar}t\hat{H}}$
- Heisenberg equation of motion: $\dot{\hat{A}} = \frac{i}{\hbar}[\hat{H}, \hat{A}]$

- $\hat{q}_H(t)|q,t> = q(t)|q,t>$. Completeness relation $\int dq|q,t><q,t| = \hat{1}$
- Compute $<q',t'|q,t> = <q'\left|e^{-\frac{i}{\hbar}\hat{H}(t'-t)}\right|q>$:

$t'-t = N\varepsilon$, $\varepsilon = \frac{t'-t}{N}$, $e^{-\frac{i}{\hbar}\hat{H}N\varepsilon} = \left(e^{-\frac{i}{\hbar}\hat{H}\varepsilon}\right)^N$,

$<q',t'|q,t> = \int dq_1 \dots dq_N \prod_{n=0}^{n=N} <q_{n+1}\left|e^{-\frac{i}{\hbar}\hat{H}\varepsilon}\right|q_n>$, $e^{-\frac{i}{\hbar}\hat{H}\varepsilon} = 1 - \frac{i}{\hbar}\hat{H}\varepsilon + o(\varepsilon^2)$

- $H = \frac{p^2}{2m} + V(q)$, $<q_{n+1}\left|e^{-\frac{i}{\hbar}\hat{H}\varepsilon}\right|q_n> = \delta(q_{n+1}-q_n)\left(1 - \frac{i}{\hbar}\varepsilon V(q_n)\right) - \frac{i}{\hbar}\varepsilon <q_{n+1}\left|\frac{\hat{p}^2}{2m}\right|q_n>$

$<q_{n+1}|\hat{p}^2|q_n> = \int dp_n <q_{n+1}|\hat{p}|p_n><p_n|\hat{p}|q_n> = \int dp_n\, p_n^2 <q_{n+1}|p_n><p_n|q_n>$

$<q_{n+1}|p_n> = \frac{1}{\sqrt{2\pi\hbar}} e^{\frac{i}{\hbar}p_n q_{n+1}}$

$<q_{n+1}|\hat{p}^2|q_n> = \int \frac{dp_n}{2\pi\hbar} p_n^2 e^{\frac{i}{\hbar}p_n(q_{n+1}-q_n)}$

$$<q_{n+1}\left|e^{-\frac{i}{\hbar}\hat{H}\varepsilon}\right|q_n> = \int \frac{dp_n}{2\pi\hbar}\left(1-\frac{i}{\hbar}\varepsilon V(q_n)-\frac{i}{\hbar}\varepsilon\frac{p_n^2}{2m}\right)e^{\frac{i}{\hbar}p_n(q_{n+1}-q_n)} =$$

$$\int \frac{dp_n}{2\pi\hbar}\,e^{\frac{i}{\hbar}(p_n(q_{n+1}-q_n)-\varepsilon H(q_n,p_n))}$$

$$<q',t'|q,t> = \prod_{n=0}^{n=N}\int dq_n \int \frac{dp_n}{2\pi\hbar}\,e^{\frac{i}{\hbar}(p_n(q_{n+1}-q_n)-\varepsilon H(q_n,p_n))} =$$

$$\int \prod_{n=0}^{n=N} dq_n \frac{dp_n}{2\pi\hbar}e^{\frac{i}{\hbar}\sum_{n=1}^{N}(p_n(q_{n+1}-q_n)-\varepsilon H(q_n,p_n))} = \int_{q(t)=q}^{q(t')=q'} \mathcal{D}q\mathcal{D}p\,e^{\frac{i}{\hbar}\int_t^{t'}d\tau(p\dot{q}-H(q,p))}$$

$$<q',t'|q,t> = \int_{q(t)=q}^{q(t')=q'} \mathcal{D}q\mathcal{D}p\,e^{\frac{i}{\hbar}\int_t^{t'}d\tau(p\dot{q}-H(q,p))}$$

Let us find $\int \frac{dp_n}{2\pi\hbar} e^{\frac{i}{\hbar}\left(p_n(q_{n+1}-q_n)-\varepsilon\frac{p_n^2}{2m}\right)} = B_n$:

$$(q_{n+1}-q_n)-\varepsilon\frac{\bar{p}_n}{m}=0 \quad \bar{p}_n = m\frac{(q_{n+1}-q_n)}{\varepsilon}$$

$$\bar{p}_n(q_{n+1}-q_n)-\varepsilon\frac{\bar{p}_n^2}{2m}=\frac{1}{2}m\varepsilon\left(\frac{q_{n+1}-q_n}{\varepsilon}\right)^2$$

$$B_n = e^{\frac{i}{\hbar}\left(\frac{1}{2}m\varepsilon\left(\frac{q_{n+1}-q_n}{\varepsilon}\right)^2\right)}\frac{1}{2\pi\hbar}\sqrt{\frac{2\pi\hbar m}{i\varepsilon}} = \sqrt{\frac{m}{2\pi\hbar i\varepsilon}}e^{\frac{i}{\hbar}\left(\frac{1}{2}m\varepsilon\left(\frac{q_{n+1}-q_n}{\varepsilon}\right)^2\right)}$$

$$<q', t'|q, t> = \prod_{n=0}^{n=N} \int d\,q_n\sqrt{\frac{m}{2\pi\hbar i\varepsilon}}e^{\frac{i}{\hbar}\left(\frac{1}{2}m\varepsilon\left(\frac{q_{n+1}-q_n}{\varepsilon}\right)^2-V(q_n)\varepsilon\right)} =$$

$$\prod_{n=0}^{n=N} d\,q_n\sqrt{\frac{m}{2\pi\hbar i\varepsilon}}e^{\frac{i}{\hbar}\varepsilon\sum_{n=1}^{N}\left(\frac{1}{2}m\left(\frac{q_{n+1}-q_n}{\varepsilon}\right)^2-V(q_n)\right)}$$

$$<q', t'|q, t> = \int_{q(t)=q}^{q(t')=q'} \mathcal{D}q\, e^{\frac{i}{\hbar}\int d\tau L(q,\dot{q})}, L(q,\dot{q}) = \frac{1}{2}m\,\dot{q}^2 - V(q)$$

- In QFT we will use preferently the Lagrangian path integral, since the lagrangian is invariant under Lorentz transformations whereas the Hamiltonian is not.
- The path integral can be derived for more general forms of the Hamiltonian, that contains products of q, p. Naturally, a ordering problem arises. This is reflected in the path integral by choosing which point in the interval (q_n, q_{n+1}) is used to evaluate the operators.
- Weyl ordering prescription is equivalent to mid point evaluation: $<q_{n+1}|\hat{A}|q_n> = A(\frac{q_{n+1}+q_n}{2})$.See Sakita's book for a detailed derivation.
- The ordering prescription can produce observable effects.See J.L. Gervais and A. Jevicki. Sakita, T.D. Lee.

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- $S = \int_V d^4x \mathcal{L}(\varphi_i(x), \partial_\mu \varphi_i(x))$
- Minimal action principle:$\delta S = 0$,$\delta\varphi_i(x) = 0, x\varepsilon\Sigma$, Σ is a closed surface which bound V.
- $\delta S = \int_V d^4x \left(\frac{\partial \mathcal{L}}{\partial \varphi_i} \delta\varphi_i + \frac{\partial \mathcal{L}}{\partial \varphi_{i,\mu}} \partial_\mu \delta\varphi_i \right) = \int_V d^4x \left(\frac{\partial \mathcal{L}}{\partial \varphi_i} \delta\varphi_i - \partial_\mu \Big(\frac{\partial \mathcal{L}}{\partial \varphi_{i,\mu}} \Big) \delta\varphi_i \right) + \oint_\Sigma dS_\mu \frac{\partial \mathcal{L}}{\partial \varphi_{i,\mu}} \delta\varphi_i$
- The surface term vanishes. Since $\delta\varphi_i(x),\ x\varepsilon V$ is arbitrary, we have Euler-Lagrange equations. $\frac{\partial \mathcal{L}}{\partial \varphi_i} = \partial_\mu \Big(\frac{\partial \mathcal{L}}{\partial \varphi_{i,\mu}} \Big)$

- Assume $\mathcal{L}$ is invariant under an infinitesimal transformation of the fields, $\delta\varphi_i(x)$
- $\delta\mathcal{L} = 0 = \frac{\partial\mathcal{L}}{\partial\varphi_i(x)}\delta\varphi_i(x) + \frac{\partial\mathcal{L}}{\partial\varphi_i(x)_{,\mu}}\partial_\mu\delta\varphi_i(x)$
- Using the E-L equations:$\delta\mathcal{L} = 0 = \partial_\mu\frac{\partial\mathcal{L}}{\partial\varphi_i(x)_{,\mu}}\delta\varphi_i(x) + \frac{\partial\mathcal{L}}{\partial\varphi_i(x)_{,\mu}}\partial_\mu\delta\varphi_i(x) = \partial_\mu\Big(\frac{\partial\mathcal{L}}{\partial\varphi_i(x)_{,\mu}}\delta\varphi_i(x)\Big)$
- $j^\mu(x) = \frac{\partial\mathcal{L}}{\partial\varphi_i(x)_{,\mu}}\delta\varphi_i(x)$, $\partial_\mu j^\mu = 0$, Conserved current.
- Under space time translations $x'^\mu = x^\mu + a^\mu$ the action is invariant and

  $\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial x^\mu}a^\mu = \partial_\mu\Big(\frac{\partial\mathcal{L}}{\partial\varphi_i(x)_{,\mu}}\delta\varphi_i(x)\Big)$, $\delta\varphi_i(x) = \varphi_i(x)_{,\nu}a^\nu$
- Canonical Energy momentum tensor is conserved:

  $T^\mu_\nu(x) = \frac{\partial\mathcal{L}}{\partial\varphi_i(x)_{,\mu}}\varphi_i(x)_{,\nu} - \mathcal{L}\delta^\mu_\nu, T^\mu_{\nu,\mu} = 0$
- $P_\mu = \int\, d^3x T^0_\mu$ is conserved. This the four momentum of the field.