Algebra de Grassmann

- Extendemos C agregando la variable ψ
- $[x, \psi] = 0, x\varepsilon C$, $\psi\psi + \psi\psi = 0$
- El producto es asociativo y distributivo:

$$(a + b\psi)(c + d\psi) = ac + ad\psi + bc\psi + bd\psi^2, \quad \psi^2 = 0$$

- $1 + 0\psi = 1$ es la identidad
- $(a + b\psi)(c + d\psi) = ac + ad\psi + bc\psi = 1$,$ad + bc = 0$,$ac = 1$,$c = a^{-1}$,$d = -ba^{-2}$,$a \neq 0$

- Función $f(\psi) = a + b\psi$, «serie de Taylor»
- Derivada:$\frac{df(\psi)}{d\psi} = b$
- $\frac{d}{d\psi}[f(\psi)g(\psi)] = \frac{df(\psi)}{d\psi}g(\psi) + f(-\psi)\frac{dg(\psi)}{d\psi}$
- $\frac{d}{d\psi}\psi = 1 - \psi\frac{d}{d\psi}$
- Integral(Berezin):Imponemos invarianza translacional:

$$\int d\psi f(\psi + \chi) = \int d\psi f(\psi)$$

$$\int d\psi (a + b\psi) = \int d\psi (a + b(\psi + \chi)) \quad -b\chi \int d\psi = 0,$$

$$\int d\psi = 0$$

- Normalización:$\int d\psi \psi = 1$,
- $\int d\psi f(\psi) = b = \frac{df(\psi)}{d\psi}$
- Serie de Taylor:$f(\psi) = f(0) + \frac{df(\psi)}{d\psi}|_0 \psi$

Complejo conjugado

- $(a+b\psi)^\star = a^\star + b^\star\bar{\psi}$,$a^\star = \bar{a}$
- $\{\psi,\bar{\psi}\}=0$, $\left\{\frac{d}{d\psi},\bar{\psi}\right\}=0$
- $F(\bar{\psi},\psi)=F_0+F_1\psi+\bar{F}_1\bar{\psi}+F_2\psi\bar{\psi}$
- $\int d\bar{\psi}d\psi F(\bar{\psi},\psi)=F_2$
- Cambio de variables:$\psi'=a\psi$,$\bar{\psi}'=\bar{a}\bar{\psi}$

$$\int d\bar{\psi}d\psi F(\bar{\psi},\psi)=\int d\bar{\psi}'d\psi' F\left(\frac{\bar{\psi}'}{\bar{a}},\frac{\psi'}{a}\right)J$$

$$F_2=\frac{F_2}{|a|^2}J \quad J=|a|^2$$

J es el jacobiano de la transformación

- El jacobiano ordinario es:$\frac{\partial(\bar{\psi},\psi)}{\partial(\overline{\psi'},\psi')}=\begin{vmatrix} \frac{\partial\bar{\psi}}{\partial\bar{\psi}'} & \frac{\partial\bar{\psi}}{\partial\psi'} \\ \frac{\partial\psi}{\partial\bar{\psi}'} & \frac{\partial\psi}{\partial\psi'} \end{vmatrix}=\begin{vmatrix} \bar{a}^{-1} & 0 \\ 0 & a^{-1} \end{vmatrix}=\frac{1}{|a|^2}$
- Para variables de Grassmann:J=inverso del jacobiano ordinario.

- $\psi_i, i=1,....N$, $\{\psi_i, \psi_j\}=0$
- $x = a_0 + a_{1i1}\psi_i + a_{2i1i2}\psi_{i1}\psi_{i2} + ...a_{\mathrm{Ni1}i2....\mathrm{iN}}\psi_{i1}\psi_{i2}....\psi_{\mathrm{iN}}$. El producto de más de N ψ se anula.
- El producto es asociativo y distributivo.
- $f(\psi) = a_0 + a_{1i1}\psi_i + a_{2i1i2}\psi_{i1}\psi_{i2} + ...a_{\mathrm{Ni1}i2....\mathrm{iN}}\psi_{i1}\psi_{i2}....\psi_{\mathrm{iN}}$,Los coeficientes son totalmente antisimétricos bajo permutación de los índices.
- $\frac{d\psi_i}{d\psi_j}=\delta_{ij}$,$\left\{\frac{d}{d\psi_j}, \psi_i\right\}=0$

  Ejemplo: $f(\psi)=a+b\psi_1\psi_2$,$\frac{df}{d\psi_1}=b\psi_2$,$\frac{df}{d\psi_2}=-b\psi_1$
- Serie de Taylor: $f(\psi)=f(0)+\frac{df(\psi)}{d\psi_i}|_0\psi_i+\frac{1}{2}\frac{d^2f(\psi)}{d\psi_i d\psi_j}|_0\psi_i\psi_j+....$

- Imponemos invarianza translacional:

$$\int d\psi_1......d\psi_N f(\psi+\chi) = \int d\psi_1......d\psi_N f(\psi)$$

$$\int d\psi_1......d\psi_N \frac{df(\psi)}{d\psi_j} = 0 \qquad ,$$

$$\int d\psi_1......d\psi_N \psi_{i1}....\psi_{\mathrm{iK}} = 0, K < N$$

- $\int\ d\psi_1......d\psi_N\psi_N......\psi_1 = \int\ d\psi_1\psi_1....\int\ d\psi_N\psi_N = 1$

Calculemos $B = \int \prod_{i=1}^{N} d\bar{\psi}_i d\psi_i e^{-\bar{\psi}_i A_{ij} \psi_j}$

- $N = 2$, $\int \prod_1^2 d\bar{\psi} d\psi e^{\bar{\psi}_i A_{ij} \psi_j} = \int \prod_1^2 d\bar{\psi} d\psi \big(1 - \bar{\psi}_i A_{ij} \psi_j + \frac{1}{2} \bar{\psi}_i A_{ij} \psi_j \bar{\psi}_a A_{ab} \psi_b + 0\big)$

  $B = \frac{1}{2} \int d\bar{\psi}_1 d\psi_1 d\bar{\psi}_2 d\psi_2 (\bar{\psi}_1 A_{11} \psi_1 + \bar{\psi}_1 A_{12} \psi_2 + \bar{\psi}_2 A_{21} \psi_1 + \bar{\psi}_2 A_{22} \psi_2)(\bar{\psi}_1 A_{11} \psi_1 + \bar{\psi}_1 A_{12} \psi_2 + \bar{\psi}_2 A_{21} \psi_1 + \bar{\psi}_2 A_{22} \psi_2)$

  $$B = \frac{1}{2} \int d\bar{\psi}_1 d\psi_1 d\bar{\psi}_2 d\psi_2 (\bar{\psi}_1 A_{11} \psi_1 \bar{\psi}_2 A_{22} \psi_2 + \bar{\psi}_1 A_{12} \psi_2 \bar{\psi}_2 A_{21} \psi_1 + \bar{\psi}_2 A_{21} \psi_1 \bar{\psi}_1 A_{12} \psi_2 + \bar{\psi}_2 A_{22} \psi_2 \bar{\psi}_1 A_{11} \psi_1) = \int d\bar{\psi}_1 d\psi_1 d\bar{\psi}_2 d\psi_2 (\bar{\psi}_1 A_{11} \psi_1 \bar{\psi}_2 A_{22} \psi_2 + \bar{\psi}_1 A_{12} \psi_2 \bar{\psi}_2 A_{21} \psi_1) = A_{11} A_{22} - A_{12} A_{21}$$

- $B = \det(A)$

- Campos bosónicos:

$$[\hat{a}, \hat{a}^{\uparrow}] = 1$$

- La representación de estados coherentes se define como:$|z> = e^{z\,\hat{a}^{\uparrow}}|0>$
- $|z>$ es un autoestado de $\hat{a}$, $\hat{a}|z> = z|z>$
- Además, $\hat{a}^{\uparrow}\Big|z> = \frac{\partial}{\partial z}|z>$
- El adjunto de $|z>$ es $<z| = <0|e^{\hat{a}\bar{z}}$
- $<z|z'> = <0|e^{\hat{a}\bar{z}}e^{z'\hat{a}^{\uparrow}}|0>$, $\frac{\partial}{\partial \bar{z}} < z|z'> = z' < z|z'>$, $<z\Big|z'> = e^{\bar{z}z'}A(z')$

  Evaluando en $z = 0$, se obtiene $A(z') = 1$
- $<z\Big|z'> = e^{\bar{z}z'}$

Ejercicios:

1. $<z|:h(\hat{a}^{\uparrow}, \hat{a}):|z'> = h(\bar{z}, z')e^{\bar{z}z'}$,:: es el producto ordenado normal. Todos los operadores de destrucción a la derecha de los operadores de creación.

2. Relación de completitud:$\int e^{-\bar{z}z}|z> < z\Big| \frac{d\,\bar{z}\,d\,z}{2\pi} = 1$

- Operadores de creación($\hat{\psi}$) y destrucción ($\hat{\psi}^{\uparrow}$)de fermiones:$\{\hat{\psi},\hat{\psi}^{\uparrow}\}=1$,$\hat{\psi}^{\uparrow 2}=\hat{\psi}^{2}=0$
- La representación de número de ocupación estándar es:$\hat{\psi}^{\uparrow}\hat{\psi}|0>=0$, $\hat{\psi}^{\uparrow}\hat{\psi}|1>=|1>$
- $\hat{\psi}|0>=0$,$\hat{\psi}|1>=|0>$, $\hat{\psi}^{\uparrow}|0>=|1>$, $\hat{\psi}^{\uparrow}|1>=0$
- La relación de completitud es:$|0><0|+|1><1|=1$
- Sea ψun número de Grassmann. Definimos un estado coherente por $|\psi>=|0>+\psi|1>$
- $|\psi>=|0>+\psi\hat{\psi}^{\uparrow}|0>=e^{\psi\hat{\psi}^{\uparrow}}|0>$
- $\hat{\psi}|\psi>=\psi|0>=\psi|\psi>$,$\hat{\psi}^{\uparrow}|\psi>=|1>=\frac{\partial}{\partial\psi}|\psi>$
- Estado coherente adjunto $<\psi|=<0|+<1|\bar{\psi}=<0\Big|e^{\hat{\psi}\bar{\psi}}$
- $<\psi|\hat{\psi}^{\uparrow}=<0|\bar{\psi}=<\psi|\bar{\psi}$, $<\psi|\hat{\psi}=<0|=<\psi|\frac{\partial}{\partial\bar{\psi}}$
- $<\psi|\psi'>=(<0|+<1|\bar{\psi})(|0>+\psi'|1>)=1+\bar{\psi}\psi'=e^{\bar{\psi}\psi'}$

1. $<\psi|:h(\hat{\psi}^{\uparrow},\hat{\psi}):|\psi'>=h(\bar{\psi},\psi')e^{\bar{\psi}\psi'}$

2. $\int |\psi><\psi|e^{-\bar{\psi}\psi}d\bar{\psi}d\psi=1$

Integral Funcional holomórfica

- $\hat{H} =: \psi H(\hat{\psi}^{\uparrow}, \hat{\psi}):$
- El núcleo de Feynman:$K(\bar{\psi}_f, t_f; \psi_i, t_i) = <\bar{\psi}_f|e^{-i\,\hat{H}(t_f - t_i)}|\psi_i>$
- Dividimos en intervalo $t_f - t_i$ en N trozos.$\varepsilon = \frac{t_f - t_i}{N}$,$e^{-i\,\hat{H}(t_f - t_i)} = lím_{N\to\infty}(1 - i\varepsilon\hat{H})^N$
- $K(\bar{\psi}_f, t_f; \psi_i, t_i) = \int \ \ldots \int \ \prod_{m=1}^{N-1} d\bar{\psi}_m d\psi_m < \psi_f|1 - i\varepsilon\hat{H}|\psi_{N-1} > e^{-\bar{\psi}_{N-1}\psi_{N-1}} < \psi_{N-1}|1 - i\varepsilon\hat{H}|\psi_{N-2} > \ldots.$
- $<\psi_{m+1}|1 - i\varepsilon\hat{H}|\psi_m> = (1 - i\varepsilon H(\bar{\psi}_{m+1}, \psi_m))e^{\bar{\psi}_{m+1}\psi_m} \simeq e^{-i\varepsilon H(\bar{\psi}_{m+1}, \psi_m) + \bar{\psi}_{m+1}\psi_m}$
-

$$K(\bar{\psi}_f, t_f; \psi_i, t_i) = \int \ldots \int \prod_{m=1}^{N-1} d\bar{\psi}_m d\psi_m$$
$$e^{\sum_{n=1}^{N}(-i\varepsilon H(\bar{\psi}_{m+1}, \psi_m) + \bar{\psi}_m\psi_{m-1}) - \sum_{n=1}^{N-1}\bar{\psi}_m\psi_m} =$$
$$\int \prod_{m=1}^{N-1} d\bar{\psi}_m d\psi_m e^{\sum_{n=1}^{N}\left(-i\varepsilon H(\bar{\psi}_{m+1}, \psi_m) - \varepsilon\bar{\psi}_m\frac{(\psi_m - \psi_{m-1})}{\varepsilon}\right)e^{\bar{\psi}_f\psi_f}}$$

con $\psi_0 = \psi_i, \psi_N = \psi_f$.

$K(\bar{\psi}_f, t_f; \psi_i, t_i) = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi e^{iS}$,$S = \int_{t_i}^{t_f} dt(i\bar{\psi}\partial_t\psi - H)$

- Campo de Dirac:$\hat{H} = \int d^3x\hat{\psi}^{\uparrow}(x)(-i\alpha.\nabla + \beta m)\hat{\psi}(x)$
- $\{\hat{\psi}^{\uparrow}_{\alpha}(\vec{x}), \hat{\psi}_{\beta}(\vec{x}\,')\} = \delta_{\alpha\beta}\delta(\vec{x} - \vec{x}\,')$
- $S = \int dt d^3x(i\bar{\psi}\partial_t\psi - \bar{\psi}(x)(-i\alpha.\nabla + \beta m)\psi(x))$
- Forma covariante:$\bar{\psi} \to \bar{\psi}\beta$,$S = \int d^4x\,\bar{\psi}(i\gamma^{\mu}\partial_{\mu} - m)\psi$