Klein-Gordon equation

- $E^2 = \vec{p}^{\,2} + m^2$
- $E = i\partial_t; p_j = -i\frac{\partial}{\partial x^j}$
- $-\frac{\partial^2}{\partial t^2}\varphi + \nabla^2\varphi - m^2\varphi = 0$, Klein-Gordon equation
- $\mathcal{L} = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2}m^2\varphi^2$
- Complex field:$\mathcal{L} = \partial_\mu\varphi^*\partial^\mu\varphi - \frac{1}{2}m^2\varphi^\star\varphi$
- $\mathcal{L}$ is invariant under $\varphi \to e^{i\alpha}\varphi$,..
- Conserved current(Noether theorem):$j^\mu = i(\varphi^*\partial^\mu\varphi - \mathrm{cc})$
- Probability density:$j^0 = i(\varphi^*\partial_t\varphi - \mathrm{cc})$. It can be negative!
- Solution(Pauli): j^μ is not a probability current but a charge current. Moreover φ must be a field, not a wave function.
- The K-G equation is the right relativistic equation to describe particles of spin=0.
- We use Sakita's signature: $(+,-,-,-)$

Dirac Equation

- To avoid negative probabilities, Dirac proposes that the equation must be first order in time derivative.
- Special relativity treates space and time equal, so the equation must be first order in spatial derivatives also.
- $H = \alpha.p + \beta m$
- $H^2 = p.p + m^2 = \frac{1}{2}\{\alpha_i, \alpha_j\}p_i p_j + \{\alpha_i, \beta\}p_i + \beta^2 m^2$
- $\frac{1}{2}\{\alpha_i, \alpha_j\} = \delta_{ij}$;$\{\alpha_i, \beta\} = 0$;$\beta^2 = 1$
- Dirac equation:$i\frac{\partial\psi}{\partial t} = -i\alpha.\nabla\psi + \beta m\psi$
- Covariant form:$i\left(\beta\frac{\partial\psi}{\partial t} + \beta\alpha.\nabla\psi\right) - m\psi = 0$
- $\gamma^0 = \beta$; $\gamma^i = \beta\alpha_i$
- $(i\gamma^\mu\partial_\mu - m)\psi = 0$
- $\bar{\psi} = \psi^1\gamma^0$;$\mathcal{L} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi$
- Invariance:$\psi \rightarrow e^{i\alpha}\psi$
- Noether current:$j^\mu = \bar{\psi}\gamma^\mu\psi$: Probability density=$\psi^1\psi \geqslant 0$