We consider a set of matter fields ϕ, a column vector, that transform under a unitary local matrix $U(x)$:

$$\phi'(x) = U(x)\phi(x)$$

$\mathcal{L} = \partial_\mu \phi^\uparrow \partial^\mu \phi$ es invariante si U no depende de x.

We introduce the covariant derivative:

$$D_\mu = \partial_\mu - i A_\mu(x)$$

We want that:

$$D'_\mu \phi'(x) = U(x) D_\mu \phi(x)$$

Thus:

$$\begin{aligned} \partial_\mu(U(x)\phi(x)) - i A'_\mu(x) U(x)\phi(x) = \\ U(x)(\partial_\mu \phi(x) - i A_\mu(x)\phi(x)) &= \\ \partial_\mu U(x)\phi(x) + U(x)\partial_\mu \phi(x) - i A_\mu(x) U(x)\phi(x) \end{aligned}$$

i.e.

$$\partial_\mu U(x) - i U(x) A_\mu(x) = -i A'_\mu(x) U(x)$$

$$A'_\mu(x) = U(x)A_\mu(x)U(x)^{-1} + i\partial_\mu U(x)U(x)^{-1}$$

Field Strength:

$$\begin{aligned}
[D_\mu, D_\nu]\phi &= -iF_{\mu\nu}\phi \\
(\partial_\mu - iA_\mu(x))(\partial_\nu - iA_\nu(x))\phi - (\mu \leftrightarrow \nu) &= \\
(\partial_\mu - iA_\mu(x))(\partial_\nu\phi - iA_\nu(x)\phi) - (\mu \leftrightarrow \nu) &= \\
-i\partial_\mu A_\nu(x)\phi - iA_\nu(x)\partial_\mu\phi - iA_\mu(x)\partial_\nu\phi - A_\mu(x)A_\nu(x)\phi \;\; -(\mu \leftrightarrow \nu) &= \\
(-i\partial_\mu A_\nu(x) + i\partial_\nu A_\mu(x) - [A_\mu(x), A_\nu(x)])\phi & \\
F_{\mu\nu} = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) - i[A_\mu(x), A_\nu(x)] &
\end{aligned}$$

$$\begin{aligned}
[D'_\mu, D'_\nu]\phi' &= -iF'_{\mu\nu}\phi' \\
-iU(x)F_{\mu\nu}\phi &= -iF'_{\mu\nu}U(x)\phi \\
F'_{\mu\nu} &= U(x)F_{\mu\nu}U(x)^{-1}
\end{aligned}$$

That is the lagrangian of the YM field is

$$\mathcal{L}_{\mathrm{YM}} = -\frac{1}{4}\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu})$$

Total lagrangian

$$\mathcal{L} = D_\mu\phi^\dagger D^\mu\phi + \mathcal{L}_{\mathrm{YM}}$$

Consider a non local object $u(x,x')$ defined by

$$\frac{dx^\mu}{ds} D^x_\mu u(x,x') = 0, u(x,x) = 1, x^\mu(1) = x', x^\mu(0) = x$$

Here $x^\mu(s)$ is the equation of a curve C that connect x with x'.

Both the differential equation and the boundary condition are invariant under the gauge transformation:

$$u'(x,x') = U(x)u(x,x')U(x')^{-1}$$

Solution of the differential equation:

$$\frac{du}{ds} - i\frac{dx^\mu}{ds} A_\mu u = 0$$

$$u(x,x') = P \exp i \int_0^1 ds \;\; \frac{dx^\mu}{ds} A_\mu =$$

$$P \exp\left(i \int_x^{x'} dx^\mu A_\mu(x) \right)$$

P means path order along the curve C It is the same that time order where the time is s.

This object is called a Wilson-Polyakov loop. If the curve is closed, we can built a gauge invariant non-local object:

$$W_C(A) = \mathrm{Tr}\, P \exp\left(i \oint_C dx^\mu A_\mu(x) \right)$$

These object can be used to study non perturbative properties of gauge theories.

They also fom the basis of Loop Quantum Gravity, a canonical approach to quantize gravity.

In this section we derive the BRST transformation for the non- abelian gauge field.

We follow T. Kugo and S. Uehara, Nuclear Physics B197(1982)378..

$$\delta A_\mu = \partial_\mu \lambda C - i[A_\mu, \lambda C] \;=\; \lambda \delta_{\rm BRST} A_\mu$$
$$\delta_{\rm BRST} A_\mu = \partial_\mu C - i[A_\mu, C]$$

λ is a Grassman parameter and C is the ghost field. We keep the symbol δ for the BRST transformation from now on.

We impose nilpotency of δ.

$$\delta^2 A_\mu = 0 = \partial_\mu \delta C - i(\delta A_\mu C + A_\mu \delta C - \delta C A_\mu + C \delta A_\mu)$$

Ansatz:

$$\delta C = a C^2$$

We study the cancellation of various terms.

Derivative of δC:

$$a \partial_\mu C^2 - i(\partial_\mu C C + C \partial_\mu C) = 0 \quad a = i$$

Non derivative terms:

$$
\begin{aligned}
(-i[A_\mu, C])C + A_\mu \delta C - i\,C^2 A_\mu + C(-i[A_\mu, C]) &= \\
-i(A_\mu C^2 - C A_\mu C) + i\,A_\mu C^2 - i\,C^2 A_\mu - i(C A_\mu C - C^2 A_\mu) &= 0
\end{aligned}
$$

BRST Transformations:

$$
\begin{aligned}
\delta A_\mu &= \partial_\mu C - i[A_\mu, C] \\
\delta C &= i\,C^2 \\
\delta \bar{C} &= i\,B \\
\delta B &= 0
\end{aligned}
$$

To fix the gauge, we add to the action the following term:

$$-i\,\delta\text{Trace}\left\{\left(\partial_\mu A^\mu + \frac{1}{2}\alpha B\right)\bar{C}\right\} =$$

$$-i\,\text{Trace}\left\{(\partial_\mu(\partial_\mu C - i[A_\mu, C])\bar{C} + i\left(\partial_\mu A^\mu + \frac{1}{2}\alpha B\right)B\right\} =$$

$$-i\,\text{Trace}\left\{-\bar{C}(\Box)C + i\,\bar{C}\partial_\mu[A_\mu, C] + i\left(\partial_\mu A^\mu + \frac{1}{2}\alpha B\right)B\right\}$$

Integrating over B we get:

$$\partial_\mu A^\mu + \alpha B = 0$$

$$B = -\frac{1}{\alpha}\partial_\mu A^\mu$$

$$\left(\partial_\mu A^\mu + \frac{1}{2}\alpha B\right)B = -\frac{1}{2\alpha}(\partial_\mu A^\mu)^2$$

$$\mathcal{L}_{\text{GF}} = -i\,\text{Trace}\left\{-\bar{C}(\Box)C + i\,\bar{C}\partial_\mu[A_\mu, C] - \frac{i}{2\alpha}(\partial_\mu A^\mu)^2\right\}$$

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