Fenomenología de Gravitación Cuántica

Fiz1410-Primer Semestre 2008

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Prueba 1

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- Problema 1

(A) Considere la métrica de Robertson-Walker:

$$ds^2 = -dt^2 + a^2(t)(\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \text{sen}^2\theta d\phi^2))$$

Usando el lenguaje de formas diferenciales, encuentre:

(a) La conexión de spin.

(b) La forma de curvatura.

Sol:

$$\begin{aligned}
e^0 = dt; e^1 = a(t)\frac{dr}{\sqrt{1-kr^2}}\,; e^2 = a(t)r\,d\theta\,; e^3 = a(t)r\,\text{sen}\,\theta d\phi \\
\omega^0_2 \wedge \frac{dr}{\sqrt{1-kr^2}} + \omega^0_3 \wedge r\,d\theta + \omega^0_4 \wedge r\,\text{sen}\,\theta d\phi = 0 \\
\dot{a}\,dt \wedge \frac{dr}{\sqrt{1-kr^2}} + \omega^1_0 \wedge dt + a(t)(\omega^1_2 \wedge r\,d\theta + \omega^1_3 \wedge r\,\text{sen}\,\theta d\phi) = 0 \\
\dot{a}\,dt \wedge r\,d\theta + a\,dr \wedge d\theta + \omega^2_0 \wedge dt + a(t)(\omega^2_1 \wedge \frac{dr}{\sqrt{1-kr^2}} + \omega^2_3 \wedge r\,\text{sen}\,\theta d\phi) = 0 \\
\dot{a}\,e^0 \wedge \frac{e^3}{a} + \frac{\sqrt{1-kr^2}}{a} e^1 \wedge \frac{e^3}{r} + \frac{e^2}{a\,r} \wedge \frac{e^3}{\text{tg}\,\theta} + \omega^3_0 \wedge e^0 + \omega^3_1 \wedge e^1 + \omega^3_2 \wedge e^2 = 0
\end{aligned}$$

Esto es:

$$\begin{aligned}
\omega^0_1 \wedge e^1 + \omega^0_2 \wedge e^2 + \omega^0_3 \wedge e^3 = 0 \\
\dot{a}\,e^0 \wedge \frac{e^1}{a} + \omega^1_0 \wedge e^0 + \omega^1_2 \wedge e^2 + \omega^1_3 \wedge e^3 = 0 \\
\frac{\dot{a}}{a}\,e^0 \wedge e^2 + \sqrt{1-kr^2}\,e^1 \wedge \frac{e^2}{a\,r} + \omega^2_0 \wedge e^0 + \omega^2_1 \wedge e^1 + \omega^2_3 \wedge e^3 = 0 \\
\dot{a}\,e^0 \wedge \frac{e^3}{a} + \frac{\sqrt{1-kr^2}}{a} e^1 \wedge \frac{e^3}{r} + \frac{e^2}{a\,r} \wedge \frac{e^3}{\text{tg}\,\theta} + \omega^3_0 \wedge e^0 + \omega^3_1 \wedge e^1 + \omega^3_2 \wedge e^2 = 0
\end{aligned}$$

Además:

$$\begin{aligned}
\omega^0_i = -\omega_{0i} = \omega_{i0} = \omega^i_0 \\
\omega^i_j = -\omega^j_i
\end{aligned}$$

Se obtiene:

$$\omega^0{}_1 = \frac{\dot{a}}{a} e^1 = \omega^1{}_0$$
$$\omega^2{}_0 = \frac{\dot{a}}{a} e^2 = \omega^0{}_2$$
$$\omega^2{}_1 = \frac{\sqrt{1-kr^2}}{ar} e^2 = -\omega^1{}_2$$
$$\omega^3{}_0 = \frac{\dot{a}}{a} e^3 = \omega^0{}_3$$
$$\omega^3{}_1 = \frac{\sqrt{1-kr^2}}{ar} e^3 = -\omega^1{}_3$$
$$\omega^3{}_2 = \frac{1}{ar\,\mathrm{tg}\,\theta} e^3 = -\omega^2{}_3$$

The curvature 2-forma es:

$$R^0{}_0 = 0\,;\, R^0{}_1 = \frac{d^2 ln(a)}{dt^2} e^0 \wedge e^1 - \frac{\dot{a}}{a}\omega^1{}_0 \wedge e^0 = \frac{d^2 ln(a)}{dt^2} e^0 \wedge e^1 + (\frac{\dot{a}}{a})^2 e^0 \wedge e^1;$$

$R^0{}_2 = \frac{d^2 ln(a)}{dt^2} e^0 \wedge e^2 - \frac{\dot{a}}{a}(\omega^2{}_0 \wedge e^0 + \omega^2{}_1 \wedge e^1 + \omega^2{}_3 \wedge e^3) + \omega^0{}_1 \wedge \omega^1{}_2 + \omega^0{}_3 \wedge \omega^3{}_2 = \frac{d^2 ln(a)}{dt^2} e^0 \wedge e^2 - \frac{\dot{a}}{a}(\frac{\dot{a}}{a} e^2 \wedge e^0 + \frac{\sqrt{1-kr^2}}{ar} e^2 \wedge e^1) - \frac{\dot{a}}{a} e^1 \wedge \frac{\sqrt{1-kr^2}}{ar} e^2 = \frac{d^2 ln(a)}{dt^2} e^0 \wedge e^2 + (\frac{\dot{a}}{a})^2 e^0 \wedge e^2;$

$R^0{}_3 = \frac{d^2 ln(a)}{dt^2} e^0 \wedge e^3 - \frac{\dot{a}}{a}(\omega^3{}_0 \wedge e^0 + \omega^3{}_1 \wedge e^1 + \omega^3{}_2 \wedge e^2) + \omega^0{}_1 \wedge \omega^1{}_3 + \omega^0{}_2 \wedge \omega^2{}_3 = \frac{d^2 ln(a)}{dt^2} e^0 \wedge e^3 - \frac{\dot{a}}{a}(\frac{\dot{a}}{a} e^3 \wedge e^0 + \frac{\sqrt{1-kr^2}}{ar} e^3 \wedge e^1 + \frac{1}{ar\,\mathrm{tg}\,\theta} e^3 \wedge e^2) - \frac{\dot{a}}{a} e^1 \wedge \frac{\sqrt{1-kr^2}}{ar} e^3 - \frac{\dot{a}}{a} e^2 \wedge \frac{1}{ar\,\mathrm{tg}\,\theta} e^3 = \frac{d^2 ln(a)}{dt^2} e^0 \wedge e^3 + (\frac{\dot{a}}{a})^2 e^0 \wedge e^3;$

$$R_{ab} = d\omega_{ab} + \omega_{ac} \wedge \omega_{db}\eta^{cd} = -d\omega_{ba} - \omega_{bd}\eta^{cd} \wedge \omega_{ca} = -R_{ba}$$

Por lo tanto:

$$R_{00} = 0; R_{10} = (\frac{\dot{a}}{a})^2 e^0 \wedge e^1; R_{20} = (\frac{\dot{a}}{a})^2 e^0 \wedge e^2; R_{30} = (\frac{\dot{a}}{a})^2 e^0 \wedge e^3; R_{11} = 0$$

$R_{21} = R^2{}_1 = \frac{d}{dr}\frac{\sqrt{1-kr^2}}{r} \cdot \frac{\sqrt{1-kr^2}}{a^2} e^1 \wedge e^2 - \frac{\dot{a}}{a^2}\frac{\sqrt{1-kr^2}}{r} e^0 \wedge e^2 - \frac{\sqrt{1-kr^2}}{ar}(\omega^2{}_0 \wedge e^0 + \omega^2{}_1 \wedge e^1 + \omega^2{}_3 \wedge e^3) + \omega^2{}_0 \wedge \omega^0{}_1 + \omega^2{}_3 \wedge \omega^3{}_1 = \frac{d}{dr}\frac{\sqrt{1-kr^2}}{r} \cdot \frac{\sqrt{1-kr^2}}{a^2} e^1 \wedge e^2 - \frac{\dot{a}}{a^2}\frac{\sqrt{1-kr^2}}{r} e^0 \wedge e^2 - \frac{\sqrt{1-kr^2}}{ar}(\frac{\dot{a}}{a} e^2 \wedge e^0 + \frac{\sqrt{1-kr^2}}{ar} e^2 \wedge e^1) + \frac{\dot{a}}{a} e^2 \wedge \frac{\dot{a}}{a} e^1 = (\frac{d}{dr}\frac{\sqrt{1-kr^2}}{r} \cdot \frac{\sqrt{1-kr^2}}{a^2} + (\frac{\sqrt{1-kr^2}}{ar})^2 - (\frac{\dot{a}}{a})^2) e^1 \wedge e^2$

- Problema 2

Sea A una 1-forma con valores en el álgebra de Lie de un grupo G. Dada la transformación de gauge de A, con $g \epsilon G$:

$A' = g^{-1} A g \; + g^{-1} dg$

(a) Mostrar que:

$F = dA + A \wedge A$ transforma como

$F' = g^{-1} F g$

Suponga que, tanto A como g son matrices de $n \times n$.

Sea

$$S = \kappa \int \mathrm{tr}(F \wedge \star F)$$

(b)Mostrar que S es invariante bajo la transformación de gauge de A.

(c) Encontrar las ecuaciones de movimiento para la 1-forma A.

La métrica de Minkowski es: $-+++$.

(d) Derive la identidad de Bianchi para F y pruebe que

$$\mathrm{tr}(F^n) = \mathrm{tr}(F \wedge F \ldots \wedge F)$$

es cerrada.

Sol:

(c)

$$\delta F = d(\delta A) + \delta A \wedge A + A \wedge \delta A$$

$\delta S = \kappa\delta(F, F) = 2\kappa(\delta F, F) = 2\kappa(d(\delta A) + \delta A \wedge A + A \wedge \delta A, F) = 2\kappa((\delta A, \delta F) + (\delta A \wedge A + A \wedge \delta A, F) = 2\kappa \int \mathrm{tr}(\delta A \wedge \star \delta F + \delta A \wedge A \wedge \star F + A \wedge \delta A \wedge \star F) = 2\kappa \int \mathrm{tr}(\delta A \wedge \star \delta F + \delta A \wedge A \wedge \star F - \delta A \wedge \star F \wedge A)$

$$A_{\mathrm{ij}} \wedge B_{\mathrm{jk}} \wedge C_{\mathrm{ki}} = -B_{\mathrm{jk}} \wedge C_{\mathrm{ki}} \wedge A_{\mathrm{ij}}$$

Por lo tanto, la ecuación de movimiento es:

$$\star\delta F + A \wedge \star F - \star F \wedge A = 0, \mathrm{or}$$

$$\delta F + (-1)^{n-1} \star (A \wedge \star F - \star F \wedge A) = 0$$

(d)

$d^2 A = 0 = dF - dA \wedge A + A \wedge dA = dF - (F - A \wedge A) \wedge A + A \wedge (F - A \wedge A) = dF - F \wedge A + A \wedge F$

Encontremos

$$\begin{aligned} d\,\mathrm{tr}(F^n) = \mathrm{tr}(dF \wedge F^{n-1}) + \mathrm{tr}(F \wedge dF \wedge F^{n-2}) + \ldots = 0, n\,\mathrm{par} \\ = \mathrm{tr}(F^{n-1} dF), n\,\mathrm{impar} \end{aligned}$$

Para el caso n impar:

$$\mathrm{tr}(F^{n-1} dF) = \mathrm{tr}(F^{n-1} \wedge F \wedge A) - \mathrm{tr}(F^{n-1} \wedge A \wedge F) = \mathrm{tr}(F^n \wedge A) - \mathrm{tr}(F^n \wedge A) = 0$$

- Problema 3

Dado un sistema con n grados de libertad moviéndose bajo la restricción de r vínculos de Segunda Clase $\chi_a, a = 1, \ldots r$. Si H_c es el Hamiltoniano canónico del sistema:

(a) Encuentre el Hamiltoniano total.

(b) Defina el corchete de Dirac del sistema $(A, B)_D$. Muestre que:

(i) $(A,B)_D = -(B,A)_D$

(ii) $(A,\mathrm{BC})_D = (A,B)_D C + B(A,C)_D$

(iii) $(A,B+C)_D = (A,B)_D + (A,C)_D$

(iv) $(A,\alpha) = 0$, α constante.

(v) $(A,(B,C)_D)_D + (B,(C,A)_D)_D + (C,(A,B)_D)_D = 0$ Identidad de Jacobi.

(c) Una partícula de masa m se mueve libremente en un círculo de radio R . Describa el movimiento de la partícula usando coordenadas cartesianas (x,y)en el plano:

(i) Encuentre el Hamiltoniano canónico H_c

(ii) Encuentre todos los vínculos y diga cuales son de Primera y Segunda Clase.

(iii) Calcule el corchete de Dirac.

(iv) Encuentre las ecuaciones canónicas en coordenadas cartesianas.

Sol:

(a)

$$0 = (H_c, \chi_b) + u_a(\chi_a, \chi_b); C_{ab} = (\chi_a, \chi_b)$$
$$u_a = -(H_c, \chi_b) C_{ba}^{-1}$$
$$H_T = H_c + u_a \chi_a = H_c - (H_c, \chi_b) C_{ba}^{-1} \chi_a$$

(b)

$$(H_T, G) = (H_c, G) - (H_c, \chi_b) C_{ba}^{-1} (\chi_a, G) = (H_c, G)_D$$

$(i)(A, B)_D = (A, B) - (A, \chi_b) C_{ba}^{-1} (\chi_a, B) = -(B, A) + (B, \chi_b) C_{ba}^{-1} (\chi_a, A)$, C antisimétrica

$= -(B,A)_D$

(ii) $(A,\mathrm{BC})_D = (A,\mathrm{BC}) - (A,\chi_b) C_{ba}^{-1} (\chi_a, \mathrm{BC}) = (A,B)_D C + B(A,C)_D$

(iii) Obvia (iv) Obvia

(c)

$$(i) H_c = \frac{p_x^2 + p_y^2}{2m}$$

$(ii)\phi_1 = x^2 + y^2 - R^2 = 0$; $\phi_2 = x p_x + y p_y = 0$, los dos son de Segunda Clase, porque $(\phi_1, \phi_2) = 2x^2 + 2y^2 = 2R^2$ no es cero

$(iii)(A, B)_D = (A, B) + (A, \phi_1) \frac{1}{2R^2} (\phi_2, B) - (A, \phi_2) \frac{1}{2R^2} (\phi_1, B)$

Verifiquemos: $(\phi_1, B)_D = 0$; $(\phi_2, B)_D = (\phi_2, B) - (\phi_2, B) = 0$

$(iv)\dot{x} = (x, H_c)_D =$