Fenomenología de Gravitación Cuántica

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Prueba 2

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- Problema 1

Sea S el espacio vectorial generado por las funciones cilíndricas:

$$\Psi_{\Gamma,f}[A] = f(U(A,\gamma_1)....,U(A,\gamma_L))$$

donde $U(A,\gamma)$ es la holonomía de A en γ.

(A) Se define

$$(\Psi_{\Gamma,f}|\Psi_{\Gamma,g}) = \int \mathrm{dU}_1....\mathrm{dU}_L\overline{f(U_1,....U_L)}\, g(U_1,....U_L) \tag{1}$$

donde dU es la medida de Haar de SU(2).

Demuestre que (1) define un producto interior en S.

Sol:

(i)

$$(\Psi_{\Gamma,f}|\Psi_{\Gamma,f}) = \int \mathrm{dU}_1....\mathrm{dU}_L\overline{f(U_1,....U_L)}\, f(U_1,....U_L) \quad \geq 0$$

$$\text{Si} \quad \int \mathrm{dU}_1....\mathrm{dU}_L\overline{f(U_1,....U_L)}\, f(U_1,....U_L) = 0$$
$$\text{implica } f(U_1,....U_L) = 0, \text{esto es } \Psi_{\Gamma,f} = 0$$

(ii)

$$(\Psi_{\Gamma,f}|\Psi_{\Gamma,g}) = \overline{(\Psi_{\Gamma,g}|\Psi_{\Gamma,f})}, \text{dado que dU es real.}$$

(iii)

$$(\Psi_{\Gamma,g}|a\,\Psi_{\Gamma,f} + b\,\Psi_{\Gamma,h}) = a\,(\Psi_{\Gamma,g}|\Psi_{\Gamma,f}) + b(\Psi_{\Gamma,g}|\Psi_{\Gamma,h}), \forall a, b \epsilon C$$

(B) Considere un estado de dos "loops" idénticos, $\alpha_1 = \alpha_2$, definido por

$$\Psi_{[\alpha]}[A] = \mathrm{tr}(U(A,\alpha_1)U(A,\alpha_1)) \tag{2}$$

Encuentre la norma de este estado en el producto interior (1).

Sol:

$$(\Psi_{[\alpha]}[A]|\Psi_{[\alpha]}[A]) = \int dU\ |\mathrm{tr}(U^2)|^2$$

Debemos encontrar:

$$A_{i\,\mathrm{jk}\,l,mnpq} = \int dU\, \overline{U_{ij}U_{kl}}\, U_{mn}\, U_{pq}$$
$$\text{Cambiemos variables}: U \to V_1 \mathrm{UV}_2$$

Por lo tanto:

$$A_{i\,\mathrm{jk}\,l,mnpq} = V_2^{\uparrow}{}_{ja} V_1^{\uparrow}{}_{bi} V_2^{\uparrow}{}_{lc} V_1^{\uparrow}{}_{dk} V_{1\,me} V_{2\,fn} V_{1\,ph} V_{2\,rq} A_{badc,efhr} \tag{3}$$

Esto es:

$$A_{badc,efhr} = \lambda_{1\,bd,eh}\,\delta_{af}\delta_{cr} + \lambda_{2\,bd,eh}\,\delta_{ar}\delta_{cf} \tag{4}$$

Insertando (4) en (3), se tiene:

$$\lambda_{\alpha\,ik,mp} = V_1^{\uparrow}{}_{bi} V_1^{\uparrow}{}_{dk} V_{1\,me} V_{1\,ph}\, \lambda_{\alpha\,b\,d,eh}\ \ \alpha = 1,2 \tag{5}$$

Se tiene:

$$\lambda_{\alpha\,b\,d,eh} = \mu_{\alpha 1}\delta_{eb}\delta_{hd} + \mu_{\alpha 2}\delta_{ed}\delta_{hb}$$

Resumiendo,

$$A_{badc,efhr} = \mu_{11}\delta_{eb}\delta_{hd}\delta_{af}\delta_{cr} + \mu_{12}\delta_{ed}\delta_{hb}\delta_{af}\delta_{cr} + \mu_{21}\delta_{eb}\delta_{hd}\delta_{ar}\delta_{cf} + \mu_{22}\delta_{ed}\delta_{hb}\delta_{ar}\delta_{cf} \tag{6}$$

Contrayendo $b = e, d = h$, obtenemos:

$$4\mu_{11}\delta_{af}\delta_{cr} + 2\mu_{12}\delta_{af}\delta_{cr} + 4\mu_{21}\delta_{ar}\delta_{cf} + 2\mu_{22}\delta_{ar}\delta_{cf} = \delta_{af}\delta_{cr}$$

Esto es:

$$4\,\mu_{11} + 2\mu_{12} = 1\ ,\, 2\,\mu_{21} + \mu_{22} = 0$$

Contrayendo $b = h, d = e$, obtenemos:

$$2\mu_{11}\delta_{af}\delta_{cr} + 4\mu_{12}\delta_{af}\delta_{cr} + 2\mu_{21}\delta_{ar}\delta_{cf} + 4\mu_{22}\delta_{ar}\delta_{cf} = \delta_{ar}\delta_{cf}$$

Esto es,

$$\mu_{11}+2\mu_{12}=0\,,4\mu_{22}+2\mu_{21}=1$$

$$\mu_{12}=-\frac{1}{6},\,\mu_{11}=\frac{1}{3},\,\mu_{21}=-\frac{1}{6},\,\mu_{22}=\frac{1}{3}$$

Por lo tanto:

$$A_{badc,efhr}=\frac{1}{3}(\delta_{eb}\delta_{hd}\delta_{af}\delta_{cr}+\delta_{ed}\delta_{hb}\delta_{ar}\delta_{cf})-\frac{1}{6}(\delta_{ed}\delta_{hb}\delta_{af}\delta_{cr}+\delta_{eb}\delta_{hd}\delta_{ar}\delta_{cf}) \tag{7}$$

Verifiquemos $b=e,a=f$ da:

$$\frac{1}{3}(4\delta_{hd}\delta_{cr}+\delta_{hd}\delta_{cr})-\frac{1}{6}(2\delta_{hd}\delta_{cr}+2\delta_{hd}\delta_{cr})=\delta_{hd}\delta_{cr}$$

Correcto.

Otro test:

$$\epsilon_{mp}\epsilon_{nq}\int dU\,\overline{U_{ij}U_{kl}}\,U_{mn}\,U_{pq}=2\int dU\,\overline{U_{ij}U_{kl}}=\epsilon_{ik}\epsilon_{jl}$$

Se tiene, de (7):

$$\epsilon_{eh}\epsilon_{fr}[\frac{1}{3}(\delta_{eb}\delta_{hd}\delta_{af}\delta_{cr}+\delta_{ed}\delta_{hb}\delta_{ar}\delta_{cf})-\frac{1}{6}(\delta_{ed}\delta_{hb}\delta_{af}\delta_{cr}+\delta_{eb}\delta_{hd}\delta_{ar}\delta_{cf})]=$$
$$\epsilon_{bd}\epsilon_{ac}[\frac{2}{3}-\frac{1}{6}(-1-1)]=\epsilon_{bd}\epsilon_{ac}$$

Correcto.

Finalmente,

$$(\Psi_{[\alpha]}[A]|\Psi_{[\alpha]}[A])=\int dU\;\overline{U_{ij}U_{ji}}U_{kl}U_{lk}=\frac{1}{3}(4+4)-\frac{1}{6}(2+2)=2$$

Por lo tanto $N=\sqrt{2}$

- Problema 2

La acción de Einstein-Hilbert-Cartan está dada por:

$$S[e,\omega]=\tfrac{1}{16\pi G}\int\;\epsilon_{\mathrm{IJKL}}(\tfrac{1}{4}e^I\wedge e^J\wedge R(\omega)^{\mathrm{KL}}-\tfrac{1}{12}\lambda e^I\wedge e^J\wedge e^K\wedge e^L)$$

G es la constante de Newton y λ es la constante cosmológica. $R(\omega)^{\mathrm{KL}}$ es la forma de curvatura correspondiente a la conexión de espín ω.

(A) Encuentre la ecuación de movimiento que se obtiene al variar ω. Escriba esta ecuacion usando la forma de torsion.

Sol:

$$\delta R(\omega)^{\mathrm{KL}} = d\delta\omega^{\mathrm{KL}} + \delta\omega^{K}{}_{J}\wedge\omega^{\mathrm{JL}} + \omega^{K}{}_{J}\wedge\delta\omega^{\mathrm{JL}},$$
$$\delta S[e,\omega] = \int\epsilon_{\mathrm{IJKL}}(-d(e^I\wedge e^J)\wedge\delta\omega^{\mathrm{KL}} - \omega_M{}^L\wedge\delta\omega^{\mathrm{KM}} + \omega^K{}_M\wedge\delta\omega^{\mathrm{ML}}) = 0,$$
$$\epsilon_{\mathrm{IJKM}}(-d(e^I\wedge e^J)) + (\epsilon_{\mathrm{IJKL}}e^I\wedge e^J\wedge\omega^L{}_M - \epsilon_{\mathrm{IJML}}e^I\wedge e^J\wedge\omega^L{}_K) = 0, \omega^{LM} = -\omega^{ML}$$

$$S[e,\omega] = \frac{1}{16\pi G}\int\epsilon_{\mathrm{IJKL}}(\frac{1}{4}e^I\wedge e^J\wedge R(\omega)^{\mathrm{KL}} - \frac{1}{12}\lambda e^I\wedge e^J\wedge e^K\wedge e^L)$$
$$\delta S = \frac{1}{16\pi G}\int\epsilon_{\mathrm{IJKL}}(\frac{1}{2}\delta e^I\wedge e^J\wedge R(\omega)^{\mathrm{KL}} + \frac{1}{4}e^I\wedge e^J\wedge\delta R(\omega)^{\mathrm{KL}} - \frac{1}{3}\lambda\delta e^I\wedge e^J\wedge e^K\wedge e^L)$$

Therefore:

$$\epsilon_{\mathrm{IJKL}}(e^J\wedge R(\omega)^{\mathrm{KL}} - \frac{2}{3}\lambda e^J\wedge e^K\wedge e^L) = 0$$
$$\delta R(\omega)^{\mathrm{KL}} = d\delta\omega^{\mathrm{KL}} + \delta\omega^K{}_J\wedge\omega^{\mathrm{JL}} + \omega^K{}_J\wedge\delta\omega^{\mathrm{JL}}$$

$\int\epsilon_{\mathrm{IJKL}}e^I\wedge e^J\wedge(d\delta\omega^{\mathrm{KL}} + \delta\omega^K{}_M\wedge\omega^{\mathrm{ML}} + \omega^K{}_M\wedge\delta\omega^{\mathrm{ML}}) = \int\epsilon_{\mathrm{IJKL}}(-d(e^I\wedge e^J)\wedge\delta\omega^{\mathrm{KL}} - e^I\wedge e^J\wedge\omega_M{}^L\wedge\delta\omega^{\mathrm{KM}} + e^I\wedge e^J\wedge\omega^K{}_M\wedge\delta\omega^{\mathrm{ML}}) = 0,$
$\epsilon_{\mathrm{IJKM}}(-d(e^I\wedge e^J)) - \frac{1}{2}(\epsilon_{\mathrm{IJKL}}e^I\wedge e^J\wedge\omega_M{}^L - \epsilon_{\mathrm{IJML}}e^I\wedge e^J\wedge\omega_K{}^L) - \frac{1}{2}(\epsilon_{\mathrm{IJLK}}e^I\wedge e^J\wedge\omega^L{}_M - \epsilon_{\mathrm{IJKL}}e^I\wedge e^J\wedge\omega^L{}_K) = 0$

P.D. Last equation implies $T^I = 0$

The action is invariant under local Lorentz rotations in capital latin indices, so last equation must be covariant under local Lorentz. The only Lorents 1- form vectors are e^I, T^J. Therefore, last equation must be linear combinations of $e^I\wedge T^J$. Let us show this point explicitly.

$$\epsilon_{\mathrm{IJKM}}(-d(e^I\wedge e^J)) - \frac{1}{2}(\epsilon_{\mathrm{IJKL}}e^I\wedge e^J\wedge\omega_M{}^L - \epsilon_{\mathrm{IJML}}e^I\wedge e^J\wedge\omega_K{}^L) - \frac{1}{2}(\epsilon_{\mathrm{IJLK}}e^I\wedge e^J\wedge\omega^L{}_M - + \epsilon_{\mathrm{IJKL}}e^I\wedge e^J\wedge\omega^L{}_K) = 0$$
$$\epsilon_{\mathrm{IJKM}}(-T^I\wedge e^J + e^I\wedge T^J) + \epsilon_{\mathrm{IJKM}}(\omega^I_A\wedge e^A\wedge e^J + \omega^J_A\wedge e^I\wedge e^A)$$

$-\frac{1}{2}(\epsilon_{\mathrm{IJKL}}e^I\wedge e^J\wedge\omega_M{}^L - \epsilon_{\mathrm{IJML}}e^I\wedge e^J\wedge\omega_K{}^L) - \frac{1}{2}(\epsilon_{\mathrm{IJLK}}e^I\wedge e^J\wedge\omega^L{}_M - \epsilon_{\mathrm{IJKL}}e^I\wedge e^J\wedge\omega^L{}_K) = 0$

$$K = 0, M = 1$$
$$\omega^2_A\wedge e^A\wedge e^3 + \omega^3_A\wedge e^2\wedge e^A \quad -\omega^3_A\wedge e^A\wedge e^2 - \omega^2_A\wedge e^3\wedge e^A$$

$$-\frac{1}{2}(\epsilon_{\mathrm{IJ}0L}e^I\wedge e^J\wedge\omega_1{}^L - \epsilon_{\mathrm{IJ}1L}e^I\wedge e^J\wedge\omega_0{}^L) - \frac{1}{2}(\epsilon_{\mathrm{IJL}0}e^I\wedge e^J\wedge\omega^L{}_1 - \epsilon_{\mathrm{IJ}1L}e^I\wedge e^J\wedge\omega^L{}_0) =$$
$$2\omega^2_A\wedge e^A\wedge e^3 + 2\omega^3_A\wedge e^2\wedge e^A - (\epsilon_{\mathrm{LIJ}}e^I\wedge e^J\wedge\omega_1{}^L - \epsilon_{\mathrm{IJ}1L}e^I\wedge e^J\wedge\omega_0{}^L) =$$
$2\omega^2_0\wedge e^0\wedge e^3 + 2\omega^3_0\wedge e^2\wedge e^0 + 2\omega^2_1\wedge e^1\wedge e^3 + 2\omega^3_1\wedge e^2\wedge e^1 - (2e^1\wedge e^2\wedge\omega_1{}^3 + 2e^3\wedge e^1\wedge\omega_1{}^2) - 2e^3\wedge e^0\wedge\omega_0{}^2 - 2e^0\wedge e^2\wedge\omega_0{}^3 = 0$

That is:

$$\epsilon_{\mathrm{IJKM}}(-e^J\wedge T^I + e^I\wedge T^J) = 0$$

Taking dual, we get:

$$-e^J\wedge T^I + e^I\wedge T^J = 0$$

Write:

$$T^I = T^I{}_{\mathrm{AB}} e^A \wedge e^B$$

$$-T^I{}_{\mathrm{AB}} e^A \wedge e^B \wedge e^J + T^J{}_{\mathrm{AB}} e^A \wedge e^B \wedge e^I = 0$$

$$-T^I{}_{\mathrm{AB}}\,\epsilon^{\mathrm{ABJL}} + T^J{}_{\mathrm{AB}}\,\epsilon^{\mathrm{ABIL}} = 0$$

$$\tilde{T}^{\mathrm{IJL}} = \tilde{T}^{\mathrm{JIL}}$$

That is $\tilde{T}^{\mathrm{IJL}}$ is symmetric in the two first indices and antisymmetric in the last two.

$$\tilde{T}^{\mathrm{IJL}} = \tilde{T}^{\mathrm{JIL}} = -\tilde{T}^{\mathrm{JLI}} = -\tilde{T}^{\mathrm{LJI}} = \tilde{T}^{\mathrm{LIJ}} = \tilde{T}^{\mathrm{ILJ}} = -\tilde{T}^{\mathrm{IJL}}$$

Therefore, $\tilde{T}^{\mathrm{IJL}} = 0, T^I{}_{\mathrm{AB}} = 0$

(B) Encuentre la ecuación de movimiento que se obtiene al variar e^I. Muestre que esta ecuación es equivalente a la ecuación de Einstein en el vacío en lenguaje tensorial.

Sol:

$$\delta S = \frac{1}{16\pi G}\int \epsilon_{\mathrm{IJKL}}\left(\frac{1}{2}\delta e^I \wedge e^J \wedge R(\omega)^{\mathrm{KL}} + \frac{1}{4} e^I \wedge e^J \wedge \delta R(\omega)^{\mathrm{KL}} - \frac{1}{3}\lambda \delta e^I \wedge e^J \wedge e^K \wedge e^L\right)$$

$$\epsilon_{\mathrm{IJKL}}\left(e^J \wedge R(\omega)^{\mathrm{KL}} - \frac{2}{3}\lambda e^J \wedge e^K \wedge e^L\right) = 0$$

$$\epsilon_{\mathrm{IJKL}}\left(e^J{}_\mu R^{\mathrm{KL}}{}_{\nu\lambda} - \frac{2}{3}\lambda e^J_\mu e^K_\nu e^L_\lambda\right) dx^\mu \wedge dx^\nu \wedge dx^\lambda =$$

$$\left(\epsilon_{\mathrm{IJKL}} e^J{}_\mu e^K{}_\alpha e^L{}_\beta R^{\alpha\beta}{}_{\nu\lambda} - \frac{2}{3}\epsilon_{\mathrm{IJKL}}\lambda e^J_\mu e^K_\nu e^L_\lambda\right) dx^\mu \wedge dx^\nu \wedge dx^\lambda = 0$$

multiply by $e^I{}_\rho$

$$\left(\epsilon_{\rho\mu\alpha\beta} e R^{\alpha\beta}{}_{\nu\lambda} - \frac{2}{3}\lambda e \epsilon_{\rho\mu\nu\lambda}\right) dx^\mu \wedge dx^\nu \wedge dx^\lambda = 0$$

Taking dual:

$$\left(\epsilon_{\rho\mu\alpha\beta} e R^{\alpha\beta}{}_{\nu\lambda} - \frac{2}{3}\lambda e\, \epsilon_{\rho\mu\nu\lambda}\right)\epsilon^{\mu\nu\lambda\eta} = 0$$

$$\epsilon_{\rho\mu\nu\lambda}\epsilon^{\mu\nu\lambda\eta} = -6\delta^\eta{}_\rho$$

$$4R^\eta{}_\rho - 2R\delta^\eta{}_\rho + 4\lambda\delta^\eta{}_\rho = 0$$

- Problema 3

Muestre que la conexión de espín ω se puede escribir de la siguiente forma:

$$\omega_\mu^{IJ} = A_\mu^{IJ} + B_\mu^{IJ}$$

donde A es auto-dual y B es anti-auto-dual.

$$i K_\mu^{IJ} = \pm\frac{1}{2}\epsilon_{MN}{}^{IJ} K_\mu^{MN}\text{ , para auto-dual}(+)\text{, anti-auto-dual}(-)$$

Encuentre A y B como función de ω.

Sol:

$$\frac{1}{2}\epsilon_{MN}{}^{IJ}\omega_\mu^{MN} = i A_\mu^{IJ} - i B_\mu^{IJ}$$

$$i\omega_\mu^{IJ} = i A_\mu^{IJ} + i B_\mu^{IJ}$$

$$A_\mu^{IJ} = \frac{1}{2i}\left(\frac{1}{2}\epsilon_{MN}{}^{IJ}\omega_\mu^{MN} + i\omega_\mu^{IJ}\right) = \frac{1}{2}\left(-i\frac{1}{2}\epsilon_{MN}{}^{IJ}\omega_\mu^{MN} + \omega_\mu^{IJ}\right)$$

$$B_\mu^{IJ} = \frac{1}{2i}\left(-\frac{1}{2}\epsilon_{MN}{}^{IJ}\omega_\mu^{MN} + i\omega_\mu^{IJ}\right) = \frac{1}{2}\left(i\frac{1}{2}\epsilon_{MN}{}^{IJ}\omega_\mu^{MN} + \omega_\mu^{IJ}\right)$$