the Palatini principle based on tetrads and $SO(3,1)$ connections. This route was studied in some detail by Kijowski [127]. Unfortunately, the canonical theory based on such connections has second class constraints (in 3+1 dimensions). When one eliminates these, non-polynomialities are introduced and one is led back to the traditional Hamiltonian formulation [61]. It is remarkable that in 2+1 dimensions one actually can formulate the theory in terms of connections, although historically this was realized later and through a different construction. We will review the 2+1 case later.

In 3+1 dimensions, the only successful attempt to obtain a canonical theory in terms of a connection that yields first class constraints is that due to Ashtekar [51]. It is based on the use of self-dual connections. Not only do the constraints remain first class but they are relatively simple polynomial functions. The price to be paid is that the self-dual connections are complex. In the next subsections we will develop this formalism. The treatment will follow closely the book by Ashtekar [2], we direct the reader to it for extensive details.

7.3.1 Tetradic general relativity

To introduce the new variables, we first need to introduce the notion of tetrads. A tetrad is a vector basis in terms of which the metric of spacetime looks locally flat,

$$g_{ab} = e_a^I e_b^J \eta_{IJ}, \tag{7.23}$$

where $\eta_{IJ} = \text{diag}(-1,1,1,1)$ is the Minkowski metric, and equation (7.23) simply expresses that g_{ab}, when written in terms of the basis e_a^I, is locally flat. If spacetime were truly flat, one could perform such a transformation globally, integrating the basis vectors into a coordinate transformation $e_a^I = \partial x^I/\partial x'^a$. In a curved spacetime these equations cannot be integrated and the transformation to a flat space only works locally, the flat space in question being the "tangent space". From equation (7.23) it is immediate to see that given a tetrad, one can reconstruct the metric of spacetime. One can also see that although g_{ab} has only ten independent components, the e_a^I have sixteen. This is due to the fact that equation (7.23) is invariant under Lorentz transformations on the indices $I, J \ldots$. That is, these indices behave as if existing in flat space. In summary, tetrads have all the information needed to reconstruct the metric of spacetime but there are extra degrees of freedom in them, and this will have a reflection in the canonical formalism.

7.3.2 The Palatini action

We now write the Einstein action in terms of tetrads. We introduce a covariant derivative via $D_a K_I = \partial_a K_I + \omega_{aI}{}^J K_J$. Here $\omega_{aI}{}^J$ is a Lorentz connection (its associated covariant derivative annihilates the Minkowski metric). We define a curvature by $\Omega_{ab}{}^{IJ} = \partial_{[a}\omega_{b]}{}^{IJ} + [\omega_a, \omega_b]^{IJ}$, where $[,]$ is the commutator in the Lorentz Lie algebra. The Ricci scalar of this curvature can be expressed as $e^a_I e^b_J \Omega^{IJ}_{ab}$ (indices I, J are raised and lowered with the Minkowski metric). The Einstein action can be written as

$$S(e,\omega) = \int d^4x \; e \; e^a_I e^b_J \Omega^{IJ}_{ab}, \tag{7.24}$$

where e is the determinant of the tetrad (equal to $\sqrt{-g}$).

We will now derive the Einstein equations by varying this action with respect to e and ω as independent quantities. To take the metric and connection as independent variables in the action principle was first considered by Palatini [128].

As a shortcut to performing the calculation (this derivation is taken from reference [2]), we introduce a (torsion-free) connection compatible with the tetrad via $\nabla_a e^b_I = 0$. The difference between the two connections we have introduced is a field $C_{aI}{}^J$, defined by $C_{aI}{}^J V_J = (D_a - \nabla_a)V_I$. We can compute the difference between the curvatures (R^{IJ}_{ab} is the curvature of ∇_a), $\Omega_{ab}{}^{IJ} - R_{ab}{}^{IJ} = \nabla_{[a}C_{b]}{}^{IJ} + C_{[a}{}^{IM}C_{b]M}{}^J$. The reason for performing this intermediate calculation is that it is easier to compute the variation by reexpressing the action in terms of ∇ and $C_a{}^{IJ}$ and then noting that the variation with respect to $\omega_a{}^{IJ}$ is the same as the variation with respect to C^{IJ}_a. The action therefore is

$$S = \int d^4x \; e \; e^a_I e^b_J (R_{ab}{}^{IJ} + \nabla_{[a}C_{b]}{}^{IJ} + C_{[a}{}^{IM}C_{b]M}{}^J). \tag{7.25}$$

The variation of this action with respect to $C_a{}^{IJ}$ is easy to compute: the first term simply does not contain $C_a{}^{IJ}$ so it does not contribute. The second term is a total divergence (notice that ∇ is defined so that it annihilates the tetrad), the last term yields $e^{[a}_M e^{b]}_N \delta^M_{[I}\delta^K_{J]} C_{bK}{}^N$. It is easy to check that the prefactor in this expression is non-degenerate and therefore the vanishing of this expression is equivalent to the vanishing of $C_{bK}{}^N$. So this equation basically tells us that ∇ coincides with D when acting on objects with only internal indices. Thus the connection D is completely determined by the tetrad and Ω coincides with R (some authors refer to this fact as the vanishing of the torsion of the connection). We now compute the second equation, straightforwardly varying with respect to the tetrad. We get (after substituting $\Omega_{ab}{}^{IJ}$ by $R_{ab}{}^{IJ}$ as given by the previous equation of motion)

$$e^c_I R_{cb}{}^{IJ} - \frac{1}{2} R_{cd}{}^{MN} e^c_M e^d_N e^J_b = 0, \tag{7.26}$$

which, after multiplication by e_{Ja} just tells us that the Einstein tensor $R_{ab} - \frac{1}{2} R g_{ab}$ of the metric defined by the tetrads vanishes. We have therefore proved that the Palatini variation of the action in tetradic form yields the usual Einstein equations.

There is a difference between the first order (Palatini) tetradic form of the theory and the usual one. One sees that a solution to the Einstein equations we presented above is simply $e^b_J = 0$. This solution would correspond to a vanishing metric and is therefore forbidden in the traditional formulation since quantities, such as the Ricci or Riemann tensor are not defined for a vanishing metric. However, the first order action and equation of motion are well defined for vanishing triads. We therefore see that strictly speaking the first order tetradic formulation is a "generalization" of general relativity that contains the traditional theory in the case of non-degenerate triads. We will see this subtlety playing a role in subsequent chapters. It should be noticed that the potential of allowing vanishing metrics in general relativity offers new possibilities for some old questions, since one could envisage the formalism "going through", say. the formation of singularities. It also allows for topology change [129].

Is there any advantage in this formulation over the traditional one? The answer is no. If one performs a canonical decomposition of the first order tetradic action, one finds that the momentum canonically conjugate to the connection is quadratic in the tetrads. The factorizability of the momenta leads to new constraints in the theory that turn out to be second class. If one eliminates them through the Dirac procedure one returns to the traditional formulation [61].

7.3.3 The self-dual action

Up to now the treatment has been totally traditional. We will now take a conceptual step that allows the introduction of the Ashtekar variables. We will reconstruct the tetradic formalism of the previous subsection but we will introduce a change. Instead of considering the connection $\omega_a{}^{IJ}$ we will consider its self-dual part with respect to the internal indices and we will call it $A_a{}^{IJ}$, i.e., $iA_a{}^{IJ} = \frac{1}{2}\epsilon_{MN}{}^{IJ} A_a{}^{MN}$. Now, to really be able to do this, the connection must be complex (or one should work in an Euclidean signature). Therefore for the time being we will consider *complex general relativity* and we will then specify appropriately how to recover the traditional real theory. The connection now takes values in the (complex) self-dual subalgebra of the Lie algebra of the Lorentz group.

We will propose as action,

$$S(e,A) = \int d^4x \; e \; e^a_J e^b_K F_{ab}{}^{JK}, \tag{7.27}$$

where $F_{ab}{}^{JK}$ is the curvature of the self-dual connection and it can be checked that it corresponds to the self-dual part of the curvature of the usual connection.

We can now repeat the calculations of the previous subsection for the self-dual case. When one varies the self-dual action with respect to the connection $A_a{}^{IJ}$ one obtains that this connection is the self-dual part of a torsion-free connection that annihilates the triad (if one repeated step by step the previous subsection argument, the self-dual part of $C_a{}^{IJ}$ would vanish). The variation with respect to the tetrad follows along very similar lines except that $\Omega_{ab}{}^{IJ}$ is everywhere replaced by $F_{ab}{}^{IJ}$. The final equation one arrives at again tells us that the Ricci tensor vanishes. Remarkably, the self-dual action leads to the (complex) Einstein equations. This essentially can be explained by the fact that the two actions differ by terms that on-shell are a pure divergence. This implies that the imaginary part of the equations of motion identically vanishes. If one works it out explicitly one finds that this corresponds to the Bianchi identities.

7.3.4 The new canonical variables

As we said before, if one takes the Palatini action principle in terms of tetrads and performs a canonical decomposition, second class constraints appear and one is led back to the traditional formulation. A quite different thing happens if one decomposes the self-dual action. Let us therefore proceed to do the 3+1 split. As we did before, we introduce a vector $t^a = Nn^a + N^a$. Taking the action

$$S(e,A) = \int d^4x \; e \; e^a_I e^b_J F_{ab}{}^{IJ} \tag{7.28}$$

and defining the vector fields $E^a_I = q^a_b e^b_I$ (where $q^a_b = \delta^a_b + n^a n_b$ is the projector on the three-surface), which are orthogonal to n^a, we have

$$S(e,A) = \int d^4x \; (e \; E^a_I E^b_J F_{ab}{}^{IJ} - 2 \; e \; E^a_I e^d_J n_d n^b F_{ab}{}^{IJ}). \tag{7.29}$$

We now define $\tilde{E}^a_I = \sqrt{q} E^a_I$, which is a density on the three-manifold. The determinant of the triad can be written as $e = N\sqrt{q}$. We also introduce the vector in the "internal space" induced by n^a, defined by $n_I = e^d_I n_d$. With these definitions, and exploiting the self-duality of $F_{ab}{}^{IJ}$ to write $F_{ab}{}^{IJ} = -i\frac{1}{2}\epsilon^{IJ}{}_{MN} F_{ab}{}^{MN}$, we get

$$S(e,A) = \int d^4x \; (-\tfrac{i}{2} \underset{\sim}{N} \tilde{E}^a_I \tilde{E}^b_J \epsilon^{IJ}{}_{MN} F_{ab}{}^{MN} - 2Nn^b \tilde{E}^a_I n_J F_{ab}{}^{IJ}). \tag{7.30}$$

The action is now written in canonical form and the conjugate variables can be read off directly. The configuration variable is the self-dual connection A_a . The conjugate momentum is the self-dual part of $-i\tilde{E}^a_J\epsilon^J_{MN}$.

$$\tilde{\pi}^a_{MN} = \tilde{E}^a_{[M}n_{N]} - \frac{i}{2}\tilde{E}^a_I\epsilon^I_{MN}. \tag{7.31}$$

Now, in terms of the canonical variables the Lagrangian takes the form

$$\int_\Sigma d^3x \mathrm{Tr}(-\tilde{\pi}^a\mathcal{L}_tA_a + N^a\tilde{\pi}^bF_{ab} - (A\cdot t)D_a\tilde{\pi}^a - \underset{\sim}{N}\tilde{\pi}^a\tilde{\pi}^bF_{ab}), \tag{7.32}$$

where all references to the internal vector n^I have disappeared. The projection of the spacetime connection on the time-like direction $(A\cdot t)$ is arbitrary and acts as a Lagrange multiplier.

Since n_I is not a dynamical variable it can be gauge fixed. We fix $n^I = (1,0,0,0)$ and therefore $\epsilon^{IJKL}n_L = \epsilon^{IJK0}$. Since A^{IJ}_a and $\tilde{\pi}^a_{IJ}$ are self-dual, they can be determined by their $0I$ components. We may therefore define

$$A^i_a = iA^{0I}_a, \quad \tilde{E}^a_i = \tilde{\pi}^a_{0I}, \tag{7.33}$$

where internal indices i, j refer to the $SO(3)$ Lie algebra. In fact, as is well known the self-dual Lorentz Lie algebra is isomorphic to the (complexified) $SO(3)$ algebra

The new variables satisfy the Poisson bracket relations

$$\{A^i_a(x), \tilde{E}^b_j(y)\} = +i\delta^b_a\delta^i_j\delta^3(x-y). \tag{7.34}$$

The constraints may be read off from the Lagrangian (7.32) and take the form

$$\tilde{\mathcal{G}}^i = D_a\tilde{E}^{ai}, \tag{7.35}$$
$$\tilde{C}_a = \tilde{E}^b_iF^i_{ab}, \tag{7.36}$$
$$\tilde{\tilde{\mathcal{H}}} = \epsilon^{ij}_k\tilde{E}^a_i\tilde{E}^b_jF^k_{ab}, \tag{7.37}$$

and the Hamiltonian is again a linear combination of the constraints.

The last four equations correspond to the usual diffeomorphism and Hamiltonian constraints of canonical general relativity. The first three equations are extra constraints that stem from our use of triads as fundamental variables. These equations, which have exactly the same form as a Gauss law of an $SU(2)$ Yang–Mills theory, are the generators of infinitesimal $SU(2)$ transformations. They tell us that the formalism is invariant under triad rotations, as it should be.

Notice that a dramatic simplification of the constraint equations has occurred. In particular the Hamiltonian constraint is a polynomial function of the canonical variables, of quadratic order in each variable. Moreover, the canonical variables, and the phase space of the theory are exactly those of a (complex) $SU(2)$ Yang–Mills theory. The reduced phase space is actually a subspace of the reduced phase space of a (complex) Yang–Mills theory (the phase space modulo the Gauss law), since general relativity has four more constraints that further reduce its phase space. This resemblance of the formalism to that of a Yang–Mills theory will be the starting point of all the results we will introduce in the rest of the book.

In terms of the new variables, the structure of the constraints is simple enough for the reader to be able to compute the constraint algebra without great effort (this computation can also be carried out with the traditional variables and the results are the same). We only summarize the results here. To express them in a simpler form (and to avoid confusing manipulations of distributions while performing the computations), it is again convenient to smooth out the constraints with arbitrary test fields and to perform some recombinations. We denote

$$\mathcal{G}(N_i) = \int d^3x N_i (\mathcal{D}_a \tilde{E}^a)^i, \tag{7.38}$$

$$\mathcal{C}(\vec{N}) = \int d^3x N^b \tilde{E}^a_i F^i_{ab} - \mathcal{G}(N^a A^i_a), \tag{7.39}$$

$$\mathcal{H}(\underset{\sim}{N}) = \int d^3x \underset{\sim}{N} \epsilon^{ij}{}_k \tilde{E}^a_i \tilde{E}^b_j F^k_{ab}, \tag{7.40}$$

and as before the notation is unambiguous. The constraint algebra then reads

$$\{\mathcal{G}(N_i), \mathcal{G}(N_j)\} = \mathcal{G}([N_i, N_j]), \tag{7.41}$$

$$\{\mathcal{C}(\vec{N}), \mathcal{C}(\vec{M})\} = \mathcal{C}(\mathcal{L}_{\vec{M}} \vec{N}), \tag{7.42}$$

$$\{\mathcal{C}(\vec{N}), \mathcal{G}(N_i)\} = \mathcal{G}(\mathcal{L}_{\vec{N}} N_i), \tag{7.43}$$

$$\{\mathcal{C}(\vec{N}), \mathcal{H}(\underset{\sim}{M})\} = \mathcal{H}(\mathcal{L}_{\vec{N}} \underset{\sim}{M}), \tag{7.44}$$

$$\{\mathcal{G}(N_i), \mathcal{H}(\underset{\sim}{N})\} = 0, \tag{7.45}$$

$$\{\mathcal{H}(\underset{\sim}{N}), \mathcal{H}(\underset{\sim}{M})\} = \mathcal{C}(\vec{K}) - \mathcal{G}(A^i_a K^a), \tag{7.46}$$

where the vector $\vec{K}$ is defined by $K^a = 2\tilde{E}^a_i \tilde{E}^b_i (\underset{\sim}{N} \partial_a \underset{\sim}{M} - \underset{\sim}{M} \partial_a \underset{\sim}{N})$. Here we clearly see that the constraints are first class. The reader should notice, however, that the algebra is not a true Lie algebra, since one of the structure constants (the one defined by the last equation) is not a constant but depends on the fields $\tilde{E}^a_i$ (through the definition of the vector $\vec{K}$).

The new variables are simply related to the traditional Hamiltonian variables:

$$A^i_a = \Gamma^i_a - iK^i_a, \qquad qq^{ab} = \tilde{E}^a_i \tilde{E}^b_i, \tag{7.47}$$

where $K_a^i = K_{ab}E^{bi}$ and Γ_a^i is the spin connection compatible with the triad.

The evolution equations for the canonical variables are obtained taking the Poisson brackets of the variables with the Hamiltonian,

$$\dot{A}_a^i = -i\epsilon^{ijk}\underset{\sim}{N}\tilde{E}_j^b F_{abk} - N^b F_{ab}^i, \tag{7.48}$$

$$\dot{\tilde{E}}_i^a = i\epsilon_i^{jk} D_b(\underset{\sim}{N}\tilde{E}_j^a\tilde{E}_k^b) - 2D_b(N^{[a}\tilde{E}^{b]i}). \tag{7.49}$$

A similar simplification to that introduced in the constraints is evident in the equations of motion.

As we mentioned above, because of the self-duality used in the definition of the canonical variables, these are in general complex. The situation is totally analogous to that introduced when we discussed the harmonic oscillator and Maxwell theory in the Bargmann representation in section 4.5. If we want to recover the classical theory we must take a "section" of the phase space that corresponds to the dynamics of real relativity. This can be done. One gives data on the initial surface that correspond to a real spacetime and the evolution equations will keep these data real through the evolution. Now, strictly speaking, this procedure is not really canonical, since we are imposing these conditions by hand at the end. That does not mean it is not useful*. In fact, one can eliminate the reality conditions and have a canonical theory. However, much of the beauty of the new formulation is lost, in particular the structure of the resulting constraints is basically that of the traditional formalism.

The issue of the reality conditions acquires a different dimension at the quantum level. A point of view that is strongly advocated, and may turn out to be correct, is the following. Start by considering the complex theory and apply the usual steps towards canonical quantization After the space of physical states has been found, when one looks for an inner product. the reality conditions are used in order to choose an inner product that implements them. That is, the reality conditions can be a guideline to finding the appropriate inner product of the theory. One simply requires that the quantities that have to be real according to the reality conditions of the classical theory become self-adjoint operators under the chosen inner product. This solves two difficulties at once, since it allows us to recover the real quantum theory and the appropriate inner product at the same time. This point of view is strictly speaking a deviation from standard Dirac quantization, and works successfully for several model problems [130]. The success or failure in quantum gravity of this approach

* A non-trivial example where it can be worked to the end is the Bianchi II cosmology [132].

is yet to be tested and is one of the most intriguing and attractive features of the formalism. (For a critical viewpoint, see reference [131].)

In terms of the basic variables, the reality conditions are

$$(\tilde{E}^a_i \tilde{E}^{bi})^* = \tilde{E}^a_i \tilde{E}^{bi}, \tag{7.50}$$

$$(\epsilon^{ijk} \tilde{E}^{(a}_i D_c(\tilde{E}^{b)}_k \tilde{E}^c_j))^* = (\epsilon^{ijk} \tilde{E}^{(a}_i D_c(\tilde{E}^{b)}_k \tilde{E}^c_j)). \tag{7.51}$$

This particular form of the reality conditions may be useful to select real initial data for classical evolutions. However, if one wants to impose the conditions as adjointness relations of operators with respect to a quantum inner product, it is clear that one would need to recast the conditions in terms of physical observables, since these are the only quantities defined in the space of physical states. In particular equations (7.50),(7.51) are not well defined in that space.

Up to now we have discussed the theory in vacuum. There is no difficulty in incorporating matter fields in the new variable formulation. The constraints can be made polynomial in a natural fashion for coupling to scalar fields, Yang–Mills fields, and fermions. It is remarkable that Dirac fermions can be introduced only coupled to the self-dual part of the connection. A complete discussion can be found in references [133, 2].

It is immediate to include a cosmological constant in the framework. In the Einstein action the cosmological constant appears as $\int d^4x\sqrt{-g}\Lambda$. This action can be immediately canonically decomposed as

$$S_\Lambda = \int dt \int d^3x \underset{\sim}{N} q \Lambda, \tag{7.52}$$

and this can be written in terms of the new variables noting that the determinant of the three-metric is given by

$$q = \frac{1}{6} \underset{\sim}{\eta}_{abc} \epsilon^{ijk} \tilde{E}^a_i \tilde{E}^b_j \tilde{E}^c_k. \tag{7.53}$$

The only change introduced in the canonical theory is that the Hamiltonian constraint gains an extra term,

$$\mathcal{H}(\underset{\sim}{N}) = \int d^3x \underset{\sim}{N} \epsilon^{ij}{}_k \tilde{E}^a_i \tilde{E}^b_j F^k_{ab} + \frac{\Lambda}{6} \int d^3x \underset{\sim}{N} \underset{\sim}{\eta}_{abc} \epsilon^{ijk} \tilde{E}^a_i \tilde{E}^b_j \tilde{E}^c_k. \tag{7.54}$$

And again, is a polynomial expression. There is no modification to the other constraints, since the entire term in the action is proportional to $\underset{\sim}{N}$.

7.4 Quantum gravity in terms of connections

7.4.1 Formulation

The casting of general relativity as a theory of a connection has important implications at the quantum mechanical level. One can now proceed to quantize the theory exactly like we did in chapter 5, picking a polarization in which wavefunctions are functionals of a connection

$$\Psi[A]. \tag{7.55}$$

The Gauss law will immediately require that these be gauge invariant functions, i.e., functionals in the space of connections modulo gauge transformations. Notice that this is a significant departure from the traditional picture where one considered functionals of a three-metric, or if one imposed the diffeomorphism constraint, of a three-geometry.

As in the Yang–Mills case a representation for the Poisson algebra of the canonical variables considered can be simply achieved by representing the connection as a multiplicative operator and the triad as a functional derivative:

$$\hat{A}_a^i \Psi(A) = A_a^i \Psi(A), \tag{7.56}$$

$$\hat{\tilde{E}}_i^a \Psi(A) = \frac{\delta}{\delta A_a^i} \Psi(A). \tag{7.57}$$

It should be emphasized that a difference with the Yang–Mills case arises since the connection is complex. The wavefunctions considered are holomorphic functions of the connection and the functional derivative treats as independent the connection and its complex conjugate.

We would now like to use this choice in the representation of the canonical algebra to promote the constraint equations to operatorial equations. Since the constraint equations involve operator products, a regularization is needed. This is a fundamental point. Most of the issues one faces when promoting the constraints to wave equations do not have a unique answer unless one has a precise regularization. There is not a complete regularized picture of the theory at present. We will introduce some of the issues in this chapter and will return to them in chapters 8 and 11 as we develop the quantum theory and some of its consequences.

Ignoring for the time being the regularization issue, one can promote the constraints formally to operator equations if one picks a factor ordering. Two factor orderings have been explored: with the triads either to the right or the left of the connections.

7.4.2 Triads to the right and the Wilson loop

If one orders the triads to the right, the constraints become

$$\hat{\tilde{\mathcal{G}}}^i = D_a \frac{\delta}{\delta A_a^i}, \tag{7.58}$$

$$\hat{\tilde{\mathcal{C}}}_a = F_{ab}^i \frac{\delta}{\delta A_b^i}, \tag{7.59}$$

$$\hat{\tilde{\mathcal{H}}} = \epsilon^{ijk} F^i_{ab} \frac{\delta}{\delta A^j_a} \frac{\delta}{\delta A^k_b}. \tag{7.60}$$

This ordering was first considered by Jacobson and Smolin [134] because the Gauss law and the diffeomorphism constraint formally (without a regularization) generate gauge transformations and diffeomorphisms on the wavefuctions.

There is a potential problem when one considers the algebra of constraints. Remember that it is not a true algebra, but as we discussed, the commutator of two Hamiltonians has a structure "constant" that depends on one of the canonical variables, the triad. This means that in this ordering such a "constant" would have to appear to the right of the resulting commutator, which is not expected. In fact, an explicit calculation of the formal commutator shows the triads appear to the right. Therefore, it is not immediate that acting on a solution the commutator of two Hamiltonians vanishes and it has to be checked explicitly.

The simplest solution to the constraints in this representation is

$$\Psi[A] = \text{constant}. \tag{7.61}$$

This state is annihilated by all the constraints formally and it is easy to check that it is also annihilated with simple point-splitting regularizations. This state is less trivial than one may imagine. It has been explored in the context of Bianchi models and it has a quite non-trivial form if transformed into the traditional variables [135].

Jacobson and Smolin set out to find less obvious solutions to the constraint equations in this formalism. If one starts by considering the Gauss law, one would like the wavefunctionals to be invariant under $SU(2)$ gauge transformations. An example of such functionals is the Wilson loop,

$$W(A,\gamma) = \mathrm{Tr}\left(\mathrm{Pexp} \oint_\gamma dy^a A_a(y)\right). \tag{7.62}$$

In fact, as we have seen *any* gauge invariant function of a connection can be expressed as a combination of Wilson loops. In view of this, one can consider Wilson loops as an infinite family of wavefunctions in the connection representation parametrized by a loop $\Psi_\gamma(A) = W(\gamma, A)$ that forms an (overcomplete) basis of solutions to the quantum Gauss law constraint.

What happens to the diffeomorphism constraint? Evidently Wilson loops are not solutions. When a diffeomorphism acts on a Wilson loop, it gives as a result a Wilson loop with the loop displaced by the diffeomorphism performed. Therefore they are not annihilated by the diffeomorphism constraint and cannot become candidates for physical states of quantum gravity. In spite of that, they are worth exploring a bit more.

Remember they form an overcomplete basis in terms of which any physical state should be expandable (since any physical state has to be gauge invariant). We will therefore explore what happens when we act with the Hamiltonian constraint on them. To perform this calculation we only need the formula for the action of a triad on a holonomy along an open path $\gamma_o^{o'}$,

$$\hat{\tilde{E}}^a_i(x)U(\gamma_o^{o'}) = \frac{\delta}{\delta A^i_a(x)}U(\gamma_o^{o'}) = \oint_\gamma dy^a \delta^3(x-y)U(\gamma_o^y)\tau^i U(\gamma_y^{o'}), \quad (7.63)$$

where τ^i are $-i\sqrt{2}/2$ times the Pauli matrices.

The reason why we are considering an open path is to avoid ambiguities when we act with the second derivative. The expression for the action on the Wilson loop we are interested in is obtained in the limit in which o and o' coincide. We now act with a second triad,

$$\begin{aligned}&\frac{\delta}{\delta A^i_a(x)}\frac{\delta}{\delta A^j_b(x)}U(\gamma_o^{o'}) = \\ &\oint_\gamma dy^b \oint_{\gamma_o^y} dz^a \delta(x-y)\delta(x-z)U(\gamma_o^z)\tau^i U(\gamma_z^y)\tau^j U(\gamma_y^{o'}) \\ &+ \oint_\gamma dy^b \oint_{\gamma_y^{o'}} dz^a \delta(x-y)\delta(x-z)U(\gamma_o^y)\tau^j U(\gamma_y^z)\tau^i U(\gamma_z^{o'}). \quad (7.64)\end{aligned}$$

We now take the trace and obtain the action of the Hamiltonian,

$$\begin{aligned}&H(x)\Psi_\gamma[A] = \\ &F^k_{ab}(x)\epsilon_{ijk}\left[\oint_\gamma dy^b \oint_{\gamma_o^y} dz^a \delta(x-y)\delta(x-z)\mathrm{Tr}(\tau^i U(\gamma_z^y)\tau^j U(\gamma_{y\,o}^{\,z}))\right. \\ &\left.+ \oint_\gamma dy^b \oint_{\gamma_y^{o'}} dz^a \delta(x-y)\delta(x-z)\mathrm{Tr}(\tau^j U(\gamma_y^z)\tau^i U(\gamma_{z\,o}^{\,y}))\right], \quad (7.65)\end{aligned}$$

where the notation $U(\gamma_{y\,o}^{\,z})$ denotes the portion of the loop going from y to z through the basepoint o.

If the loop has no kinks or intersections, the portion γ_z^y shrinks to a point due to the presence of the Dirac delta functions and the action of the Hamiltonian can be written as

$$\begin{aligned}&\hat{\mathcal{H}}(x)\Psi_\gamma[A] = \\ &F^k_{ab}(x)\epsilon_{ijk}\left[\oint_\gamma dy^b \oint_\gamma dz^a \delta(x-y)\delta(x-z)\mathrm{Tr}(\tau^i\tau^j U(\gamma_{y\,o}^{\,z}))\right. \\ &+ \oint_\gamma dy^b \oint_\gamma dz^a \delta(x-y)\delta(x-z)\mathrm{Tr}(\tau^j\tau^i U(\gamma_{z\,o}^{\,y})) = \\ &\left.\oint_\gamma dy^b \oint_\gamma dz^a \delta(x-y)\delta(x-z)\mathrm{Tr}(\delta^{ij} U(\gamma_{z\,o}^{\,y}))\right], \quad (7.66)\end{aligned}$$