Loop quantum gravity and light propagation

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Within loop quantum gravity we construct a coarse-grained approximation for the Einstein-Maxwell theory that yields effective Maxwell equations in flat spacetime comprising Planck scale corrections. The corresponding Hamiltonian is defined as the expectation value of the electromagnetic term in the Einstein-Maxwell Hamiltonian constraint, regularized in the manner of Thiemann, with respect to a would-be semiclassical state. The resulting energy dispersion relations entail Planck scale corrections to those in flat spacetime. Both the helicity dependent contribution of Gambini and Pullin and, for a value of a parameter of our approximation, that of Ellis and co-workers are recovered. The electric-magnetic asymmetry in the regularization procedure yields nonlinearities only in the magnetic sector which are briefly discussed. Observations of cosmological gamma ray bursts might eventually lead to the needed accuracy to study some of these quantum gravity effects.

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I. INTRODUCTION

It has been recently suggested that quantum gravity effects might be indeed observable [1–3]. Among the diverse phenomena where these observations could take place we can distinguish (see e.g. [4,5]) (i) strain noise induced in gravitational wave detectors, (ii) neutral kaon systems and (iii) the energy dependent time of arrival of photons or neutrinos from distant sources. Experimental sensitivities in each of the above situations can be argued to be at the edge of what is required to reveal quantum gravity phenomena, namely Planck length sensitivities. In this work we focus on the third possibility. The idea is to look for modified dispersion relations of photons with energy E and momentum $\vec{k}$, of the form

$$c^2\vec{k}^2 = E^2\left[1+\xi\frac{E}{E_{QG}}+\mathcal{O}\left(\frac{E}{E_{QG}}\right)^2\right], \tag{1}$$

where ξ is a numerical factor of order one and E_{QG} is an energy scale of order $\leqslant 10^{19}$ GeV, which signals the need of considering the quantum character of gravity. The above expression leads to the following modification of the speed of light *in vacuum*:

$$v=\frac{\partial E}{\partial k}=c\left[1-\xi\frac{E}{E_{QG}}+\mathcal{O}\left(\frac{E}{E_{QG}}\right)^2\right], \tag{2}$$

which implies a retardation time

$$\Delta t\approx\xi\frac{E}{E_{QG}}\frac{L}{c}, \tag{3}$$

with respect to a signal propagating with speed c. When we consider cosmological distances $L\approx 10^{10}$ ly and an energy scale $E_{QG}\approx 10^{19}$ GeV in Eq. (3), the corresponding values are $\Delta t\approx 10^{-3}$ s ($E\approx 20$ MeV) and $\Delta t\approx 10^{-5}$ s ($E\approx 0.20$ MeV). In order to detect such effects, an experimental time resolution δt at least of order Δt is required. Thus, short and intense bursts traveling large distances would be the best candidates. Recent observations pointing towards the possibility of attaining such conditions are: (i) some gamma ray bursts (GRB) originate at cosmological distances ($\sim 10^{10}$ ly) [6] and (ii) sensitivities δt up to submillisecond scale have been achieved in GRB observations [7] and they are expected to improve in future spatial experiments [8].

It is thus timely to investigate whether candidate quantum gravity theories can account for Eq. (1). Modifications to Maxwell's equations in vacuum, induced by quantum gravity effects, have been calculated by Gambini and Pullin [9]. By considering a semiclassical regime in which the electromagnetic field is a classical object whereas space is described by loop quantum gravity, they obtained the dispersion relations

$$\omega_{\pm}(k)=k(1\mp 2\xi l_P k), \quad k=|\vec{k}|, \tag{4}$$

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where $\pm$ labels the helicity of the photon and $l_P \approx 10^{-33}$ cm is the Planck length. This modification, being helicity dependent, yields parity violation and birefringent effects.

On the other hand, a string theory approach proposed by Ellis *et al.* [10] suggests that a D-brane recoil in the quantum-gravitational foam induces a distortion in the surrounding space, which modifies the photon propagation properties. For a review of this approach see for example Ref. [11]. The dispersion relations obtained via this second procedure are

$$\omega(k) = k\left(1 - \xi \frac{k}{M_D}\right), \tag{5}$$

with $\xi > 0$. They arise from parity conserving corrections to Maxwell's equations which lead to a first order (in $1/M_D$) helicity independent effect in the dispersion relations that is linear in the photon energy. No birefringent effects appear at this order. In this approach, the redshifted difference in the time arrival of two photons with present-day energies E_1 and E_2 has been calculated. For the BATSE data, when the redshifts z of the GRB are known, a small subset of coincident photon pulses corresponding to channel 1 (20–50 keV) and channel 3 (100–300 keV) are fitted and Δt is calculated [10]. No significant effect in the data available is found. On the other hand, none of the pulses studied exhibited microbursts having a short time structure on scales $\leqslant 10^{-2}$ s. Were this the case, the sensitivity of the analysis would be greatly improved. Alternative studies based upon effective perturbative quantum gravity [12], open system techniques [13] and quantum light cone fluctuations [14] have been performed.

The study of cosmological neutrinos could also provide an excellent arena to probe quantum gravity induced propagation effects because space is practically transparent to them, even at very high energies. In fact, the fireball model, which is one of the most popular models of GRB, predicts the generation of $10^{14}-10^{19}$ eV neutrino bursts (NB) [15,16]. The planned Neutrino Burster Experiment (NuBE) will measure the flux of ultrahigh energy neutrinos (>10 TeV) over a ~ 1 km^2 effective area, in coincidence with satellite measured GRB's [17]. It is expected to detect ≈ 20 events per year, according to the fireball model. Other relevant experiment aimed at observing ultra high energy cosmic rays, including neutrinos, is the OWL-Airwatch project which expects to see $\sim 3\times 10^3 - 10^5$ cosmic ray events with energies $> 10^{20}$ eV [18,19]. Notably, this experiment is able to investigate time correlations among high energy neutrinos and gamma-rays. Hence in the foreseeable future, it might be possible to study quantum gravity effects on observed astrophysical neutrinos and photons. At the least, such observations could be used to restrict quantum gravity theories.

Motivated by these interesting possibilities, we have calculated the quantum gravity induced modifications to neutrino propagation in [20], within the loop quantum gravity framework. We obtained corrections to the velocity of propagation which are proportional to (kl_P) together with helicity dependent corrections of order $(kl_P)^2$. The energy dependence in the first case coincides with that found later by Ellis *et al.* using string theory methods [21].

In this work we extend our approach to the case of photons. The corrections obtained within our approximation contain those of [9] and, for a given value of a parameter in our scheme, those of [10]. In addition, we briefly discuss higher order nonlinearities arising in the magnetic sector of the effective Maxwell Hamiltonian.

The organization of the paper is as follows. In Sec. II we recall some basic aspects of loop quantum gravity which are necessary for our analysis. After reviewing Thiemann's regularization of the Hamiltonian constraint of the Einstein-Maxwell theory in Sec. III, we provide a general description of our approximation in Sec. IV. The corrections arising from the electric and magnetic sectors are calculated in Sec. V. Once the effective Hamiltonian is obtained, we derive the modified Maxwell equations together with the corresponding dispersion relations in Sec. VI. Section VII contains a brief analysis of the non-linear effects arising from the magnetic sector. An outline of redshift effects on cosmological photon time delays is given in Sec. VIII. Finally, Sec. IX contains a discussion of the results.

II. LOOP QUANTUM GRAVITY

In this section we summarize the main ingredients defining this approach and also denominated quantum geometry [22]. Among the main results along this approach one finds: (i) well defined geometric operators possessing a discrete spectrum, thus evidencing discreteness of space [22], (ii) a microscopic account for black hole entropy [23] and, more recently, hints on quantum avoidance of a would-be classical cosmological singularity [24]. (For a review on these topics see for example Ref. [25].)

To begin with it is assumed that the spacetime manifold M has topology $\Sigma \times \mathbb{R}$, with Σ a Riemannian 3-manifold. Here a co-triad e_a^i is defined, with $a,b,c,\ldots$ being spatial tensor indices and $i,j,k,\ldots$ being $su(2)$ indices. Thus the corresponding three-metric is given by $q_{ab} = e_a^i e_b^i$. In addition, a field K_a^i is defined by $K_{ab} = \mathrm{sgn}[\det(e_c^j)] K_a^i e_b^i$, which is related to the extrinsic curvature K_{ab} of Σ. A canonical pair for the gravitational phase space is $(K_a^i, E_j^b/\kappa)$, where $E_i^a = \frac{1}{2}\epsilon^{abc}\epsilon_{ijk} e_b^j e_c^k$ and κ is Newton's constant. It turns out that such a canonical pair yields a complicated form for the Hamiltonian constraint of general relativity. A convenient canonical pair, making this constraint polynomial, was introduced by Ashtekar [26]. Nevertheless, two severe difficulties to proceed with the quantization remained: (i) the implementation of a diffeomorphism covariant regularization for the density-weight two Hamiltonian constraint hereby obtained and (ii) the extension to non-compact groups of the diffeomorphism covariant techniques already developed for gauge theories with compact groups [27]. In fact, the Ashtekar variables $({}^{\mathbb{C}}A_a^i = \Gamma_a^i - iK_a^i,\ iE_i^a/\kappa)$ [26], with Γ_a^i being the torsion free connection compatible with e_a^i, are complex valued. Namely the gauge group is $SL(2,\mathbb{C})$, which is noncompact.

Some proposals to come to terms with difficulty (ii) were:

to consider real connection variables [28], to implement a Wick transform [29] and to define tractable reality constraints [30]. All of these left open (i). Thiemann subsequently proposed to solve (i) and (ii) by incorporating real connection variables while keeping the density weight one character of the Hamiltonian constraint. He further provided a quantum version of the theory in the pure gravity case, as well as in those cases including the coupling of matter to gravity [31]. His approach is next reviewed, since we rely upon it for our analysis of the electromagnetic case.

Let us start with the following canonical pairs for the Einstein-Maxwell system: $(A_a^i=\Gamma_a^i+K_a^i, E_i^a/\kappa)$ for the gravity sector and $(\underline{A}_a, \underline{E}^a/Q^2)$ for the electromagnetic sector. The latter has gauge group $U(1)$ and Q is the electromagnetic coupling constant, related to the dimensionless fine structure constant by $\alpha_{EM}=Q^2\hbar$. The corresponding contributions to the Hamiltonian constraint are

$$H_{\text{Einstein}}[N]=\int_\Sigma d^3x N\frac{1}{\kappa\sqrt{\det q}}\,\text{tr}(2[K_a,K_b]-F_{ab})\times[E_a,E_b]), \tag{6}$$

$$H_{\text{Maxwell}}[N]=\int_\Sigma d^3x N\frac{q_{ab}}{2Q^2\sqrt{\det q}}[\underline{E}^a\underline{E}^b+\underline{B}^a\underline{B}^b].$$

Here F_{ab} is the curvature of A_a and $\underline{B}^b$ is the magnetic field of the $U(1)$ connection $\underline{A}$. The actual classical configuration space is the space $\mathcal{A}/\mathcal{G}$ of (both) connections modulo their gauge transformations. Indeed, this is what occurs in gauge theories where the fundamental field is a connection. This completes the classical description of the phase space of the theory.

The quantum arena is given as follows [27]. As in any quantum field theory, because of the infinite number of degrees of freedom, an enlargement of the classical configuration space is required. This is far from trivial since the measures defining the scalar product, which are required to provide a Hilbert space, get concentrated on distributional fields and hence lie outside the classical configuration space. The key idea to build up such an enlargement is to make Wilson loop variables (traces of parallel transport operators) well defined. The resulting space $\overline{\mathcal{A}/\mathcal{G}}$ can be thought of as the limit of configuration spaces of lattice gauge theories for all possible *floating* (i.e. not necessarily rectangular) lattices. Hence, geometric structures on lattice configuration space are used to implement a geometric structure on $\overline{\mathcal{A}/\mathcal{G}}$. This enables to define a background independent calculus on it which, in turn, leads to the construction of the relevant measures, the Hilbert space and the regulated operators.

In line with the Dirac procedure for constrained systems, one first ignores the constraints and constructs an auxiliary Hilbert space $\mathcal{H}_{\text{aux}}$, so that the set of elementary real functions on the full phase space is represented by self-adjoint operators in $\mathcal{H}_{\text{aux}}$. It turns out that $\mathcal{H}_{\text{aux}}$ is just $L^2(\overline{\mathcal{A}/\mathcal{G}},\mu_0)$, with μ_0 being a suitable measure that implements the self-adjointness property.

Diffeomorphism constraints are well defined operators on $\mathcal{H}_{\text{aux}}$ yielding no anomalies. There exists a dense subspace Φ of $\mathcal{H}_{\text{aux}}$ so that its topological dual Φ' includes a complete set of solutions to the diffeomorphism constraint. They are characterized by generalized knots (i.e. diffeomorphism invariant classes of graphs). Besides, a diffeomorphism invariant Hilbert space is obtained for such states with an inner product that represents real observables by self-adjoint operators.

Furthermore, $\mathcal{H}_{\text{aux}}$ admits a basis in terms of the so called spin network states. A spin network is a triple $(\alpha,\vec{J},\vec{p})$ consisting of a graph α, a *coloring* defined by a set of irreducible representations $(J_1,\ldots,J_n)$ of $SU(2)$, with J_i corresponding to the edge e_i of α and a set of contractors $(p_1,\ldots,p_m)$. Here, a contractor p_k is just an intertwining operator from the tensor product of representations of the incoming edges at the vertex v_k to the tensor product of representations of the outgoing edges. Compactness of $SU(2)$ makes the vector space of all possible contractors p_k finite, for a given $\vec{J}$ and vertex v_k. An additional non-degeneracy condition is included: j_e is not trivial for any edge e and α is taken to be minimal (i.e. any other α', occupying the same points in Σ as α, can always be built by subdividing the edges of α, but not the other way around).

A spin network state is a C^∞ cylindrical function (a function that depends on the connection at the finite number of edges of a graph) on $\overline{\mathcal{A}/\mathcal{G}}$, constructed from a spin network

$$T_{\alpha,\vec{J},c}[A]:=[\otimes_{i=1}^n J_i(h_{e_i}(A))\bullet\otimes_{k=1}^m p_k], \tag{7}$$

for all $A\in\overline{\mathcal{A}}$, which includes distributional besides smooth connections. $h_{e_i}(A)$ is an element of $SU(2)$ associated with the edge e_i and "$\bullet$" stands for contracting, at each vertex v_k of α, the upper indices of the matrices corresponding to all the incoming edges and the lower indices of the matrices assigned to the outgoing edges, with all the indices of p_k.

Given a pair $\alpha,\vec{J}$ the vector space generated by $T_{\alpha,\vec{J},\vec{p}}$, for all possible contractors associated with $\alpha,\vec{J}$ in the way stated previously, is denoted by $\mathcal{H}_{\text{aux}}^{\alpha,\vec{J}}$. Then

$$\mathcal{H}_{\text{aux}}=\oplus_{\alpha,\vec{J}}\mathcal{H}_{\text{aux}}^{\alpha,\vec{J}}, \tag{8}$$

where $\alpha,\vec{J}$ run over all the pairs consisting of minimal graphs and irreducible nontrivial representation labelings. The sum is orthogonal and the spaces $\mathcal{H}_{\text{aux}}^{\alpha,\vec{J}}$ are finite dimensional. It suffices to define an orthonormal basis within each of them.

Note that the aforementioned construction of $\mathcal{H}_{\text{aux}}$ holds actually for any diffeomorphism covariant theory of connections with compact gauge group. The choice of $SU(2)$ corresponds to the case of gravity described in terms of real connection variables. So the generalization we are interested in to include both gravity and the electromagnetic field is $\mathcal{H}=\mathcal{H}_{\text{aux}}^{SU(2)}\otimes\mathcal{H}_{\text{aux}}^{U(1)}$. The spin network states for the com-

pound system are denoted by $T_{\alpha,[\vec{J},\vec{p}],[\vec{c},\vec{q}]}[A,\underline{A}]$, with $\vec{c},\vec{q}$ labeling the $U(1)$ coloring and contractors, respectively.

To extract physical information we will further need a state describing a flat continuous space Σ at scales much larger than the Planck length, but not necessarily so at distances comparable to Planck length itself. States of this kind were introduced under the name of weave [32] for pure gravity. Flat weave states $|W\rangle$, having a characteristic length $\mathcal{L}$, were first constructed by considering collections of Planck scale circles randomly oriented. If one probes distances $d \gg \mathcal{L}$ the continuous flat classical geometry is regained, while for distances $d \ll \mathcal{L}$ the quantum loop structure of space is manifest. In other words, one expects a behavior of the type $\langle W|\hat{q}_{ab}|W\rangle = \delta_{ab} + O(l_P/\mathcal{L})$. It was soon realized that such states could not yield a nontrivial volume due to the lack of self intersections [33]. Couples of circles, intersecting at a point, were also considered as specific models of weaves to overcome this defect [34]. With the recent advances on the kinematical Hilbert space $\mathcal{H}_{\rm aux}$ it became clear that all proposed weaves were afflicted by two undesirable features. First, they are defined to be peaked at a specific (flat or curved) metric, but not at a connection. This is in contrast with standard semiclassical states in terms of coherent states, for example. Second, the known weave states do not belong either to $\mathcal{H}_{\rm aux}$ or to a dense subspace of it [35]. It may be possible to come to terms with such difficulties by defining coherent states for diffeomorphism covariant gauge theories [36] or by implementing a genuine statistical geometry [37], for instance. Both approaches have recently achieved substantial progress.

Nonetheless, in order to extract physics, there is the alternative possibility of using just the main features that semiclassical states should have, namely peakedness on both geometry and connection together with the property that they yield well defined expectation values of physical operators. An advantage of this alternative is that one may elucidate some physical consequences before the full fledged semiclassical analysis is settled down. Indeed, such an alternative may be considered as complementary, in the sense of hinting at possible features of semiclassical states which could be further elaborated. After its completion, a rigorous semiclassical treatment should tell us whether the results arising from this alternative turn out to hold or not. The weakness of the treatment resides on its generality, since no detailed features of the would-be semiclassical states are used—as opposed, say, to the original weave states—and hence a set of numerical coefficients cannot be calculated. Evaluating them will be the task of the rigorous semiclassical treatment.

On top of the purely gravitational semiclassical states, a generalization is required to include matter fields. For our analysis it will just suffice to exploit the same aspects of peakedness and well defined expectation values, extended to include the case of the electromagnetic field. The semiclassical states here considered will describe flat space and a smooth electromagnetic field living in it. Such a state is denoted by $|W,\underline{\vec{E}},\underline{\vec{B}}\rangle$ and has a characteristic length $\mathcal{L}$. Since no detailed information is used on how the semiclassical state is constructed in terms of, say, a graph, as opposed to weave states, the present approach yields results relying only on the following assumptions: (i) peakedness of the states, (ii) well defined expectation values and (iii) existence of a coarse-grained expansion involving ratios of the relevant scales of the problem: the Planck length l_P, the characteristic length $\mathcal{L}$ and the electromagnetic wavelength λ. States fulfilling such requirements are referred to as *would-be semiclassical states* in the sequel.

III. THE REGULARIZATION

Thiemann has put forward a consistent regularization procedure to define the quantum Hamiltonian constraint of general relativity on $\mathcal{H}_{\rm aux}$, both for pure gravity and matter couplings [31]. The basis of his proposal is the incorporation of the volume operator as a convenient regulator, since its action upon spin network states is finite. We use his regularization for the Einstein-Maxwell theory, which naturally allows the semiclassical treatment here pursued.

Consider the electric part of Eq. (6). The identity $1/\kappa\{A_a^i,V\} = 2\,\mathrm{sgn}(\det e_b^j)e_a^i$ allows to rewrite it as

$$H^E[N] = \frac{1}{2\kappa^2 Q^2}\lim_{\epsilon\to 0}\frac{1}{\epsilon^3}\int_\Sigma d^3x N(x)\frac{\{A_a^i(x),V\}}{2(\det q)^{1/4}(x)}\underline{E}^a(x)$$
$$\times\int_\Sigma d^3y\,\chi_\epsilon(x,y)\frac{\{A_b^i(y),V\}}{2(\det q)^{1/4}(y)}\underline{E}^b(y),$$

$$= \frac{1}{2\kappa^2 Q^2}\lim_{\epsilon\to 0}\int_\Sigma d^3x N(x)\{A_a^i(x),\sqrt{V(x,\epsilon)}\}\underline{E}^a(x)$$
$$\times\int_\Sigma d^3y\,\chi_\epsilon(x,y)\{A_b^i(y),\sqrt{V(y,\epsilon)}\}\underline{E}^b(y), \qquad (9)$$

with $\chi_\epsilon(x,y) = \Pi_{a=1}^3\theta(\epsilon/2 - |x^a - y^a|)$ being the characteristic function of a cube with volume ϵ^3 centered at x and $V(x,\epsilon) := \int d^3y\,\chi_\epsilon(x,y)\sqrt{\det q}(y)$ being the volume of the box as determined by q_{ab}. Remarkably all ϵ dependence resides here. This is possible due to $H_{\rm Maxwell}$ having density weight one and it is achieved at the price of explicitly breaking diffeomorphism covariance. This is harmless as far as diffeomorphism covariance is regained once the regulator is removed. This is the case indeed [31]. Next, let Σ be triangulated into tetrahedra Δ. Hence, the integral over Σ in Eq. (9) is just a sum over the contributions of each tetrahedron Δ.

The form of Eq. (9) suggests to focus upon the term inside each integral. As we will see below, this indeed simplifies the analysis. Let

$$\Theta^i[f] := \int d^3x\, f(x)\underline{E}^a(x)\{A_a^i(x), \sqrt{V(x,\epsilon)}\} = \sum_\Delta \int_\Delta d^3x\, f(x)\underline{E}^a(x)\{A_a^i(x), \sqrt{V(x,\epsilon)}\}$$

$$= \sum_\Delta \int_\Delta f(x)e(x)\wedge\{A^i(x), \sqrt{V(x,\epsilon)}\},$$

$$\Theta^i[f] =: \sum_\Delta \Theta^i_\Delta[f]. \tag{10}$$

Also let us use the dual of $\underline{E}$: $e_{bc} := {}^*\underline{E}_{bc} = \epsilon_{bcd}\underline{E}^d$. For a two-surface S, $\int_S e = \frac{1}{2}\int_S \underline{E}^d \epsilon_{bcd} dx^b \wedge dx^c = \frac{1}{2}\int_S n_d \underline{E}^d \tilde{\epsilon}_{bc} dx^b \wedge dx^c = \int_S \underline{E}^d n_d \tilde{\epsilon}$, $\tilde{\epsilon}$ being the volume two-form. Hence $\Phi^E(S) := \int_S e$ is the flux of $\underline{E}^a$ through S. Recalling that

$$\mathrm{tr}(\tau_i h_{s_L}\{h_{s_L}^{-1}, \sqrt{V(x,\epsilon)}\}) = \mathrm{tr}\left(\tau_i\tau_m \int_0^1 dt\, \dot{s}_L^{-1a}(t)\{A_a^m(s_L^{-1}(t)), \sqrt{V(x,\epsilon)}\}\right) + \cdots$$

$$= -\frac{\delta_{im}}{2}\int_0^1 dt\, \dot{s}_L^{-1a}(t)\{A_a^m(s_L^{-1}(t)), \sqrt{V(x,\epsilon)}\} + \cdots$$

$$\approx -\frac{1}{2}s_L^a(1)\{A_a^i(s_L^{-1}(0)), \sqrt{V(x,\epsilon)}\} \tag{11}$$

and that, for small tetrahedra, $\Phi^E(F_{JK}) \approx \frac{1}{2}\epsilon_{abc}s_J^b(\Delta)s_K^c(\Delta)\underline{E}^a$, it follows

$$f(v)\epsilon^{JKL}\Phi^E(F_{JK})\mathrm{tr}(\tau^i h_{s_{L(\Delta)}}\{h_{s_{L(\Delta)}}^{-1}, \sqrt{V(v(\Delta),\epsilon)}\}) \approx -\frac{1}{4}f(v)\epsilon^{JKL}\epsilon_{abc}s_J^b(\Delta)s_K^c(\Delta)\underline{E}^a s_L^d(\Delta)\{A_d^i(s_l^{-1}(0)), \sqrt{V(x,\epsilon)}\}$$

$$= -\frac{3!}{2}f(v)vol(\Delta)\underline{E}^a\{A_a^i(s_l^{-1}(0)), \sqrt{V(x,\epsilon)}\}$$

$$= -\frac{3!}{2}\int_\Delta f\, e\wedge\{A^i(x), \sqrt{V(x,\epsilon)}\}. \tag{12}$$

We have then

$$\Theta^i_\Delta[f] = -\frac{2}{3!}f(v)\epsilon^{JKL}\Phi^E(F_{JK})\mathrm{tr}(\tau^i h_{s_{L(\Delta)}}\{h_{s_{L(\Delta)}}^{-1}, \sqrt{V(v(\Delta),\epsilon)}\}), \tag{13}$$

where we are denoting by $s_J(\Delta), s_K(\Delta), s_L(\Delta)$ the edges of the tetrahedra Δ having v as common vertex. As stated, F_{JK} is a surface parallel to the face determined by $s_J(\Delta), s_K(\Delta)$ which is transverse to $s_L(\Delta)$.

Hence

$$H^E[N] = \frac{1}{2\kappa^2 Q^2}\lim_{\epsilon\to 0}\sum_{\Delta\Delta'}\Theta^i_\Delta[N]\Theta'^i_{\Delta'}[\chi].$$

Next one replaces $\underline{E}^a$ and $V(x,\epsilon)$ by its quantum counterparts and adapts the triangulation to the graph γ corresponding to the state acted upon, in such a way that at each vertex v of γ and triplet of edges e, e', e'' a tetrahedron is defined with basepoint at the vertex $v(\Delta) = v$ and segments $s_I(\Delta)$, $I = 1,2,3$, corresponding to $s(e), s(e'), s(e'')$, respectively [31]. Here it is assumed that $\epsilon^{IJK}\epsilon_{abc}s_I^a s_J^b s_K^c \geqslant 0$. The arcs connecting the end points of $s_I(\Delta)$ and $s_J(\Delta)$ are denoted $a_{IJ}(\Delta)$, so that a loop $\alpha_{IJ} := s_I \circ a_{IJ} \circ s_J^{-1}$ can be formed. Besides, the face spanned by the segments $s_I(\Delta)$ and $s_J(\Delta)$ is called F_{IJ}.

The action of the regulated operator hereby obtained gets concentrated in the vertices of the graph, as expected from the explicit appearance of the volume operator. In successive steps we replace

$$\hat{\Theta}^i_\Delta[N] = -\frac{2}{3!}\frac{1}{i\hbar}N(v(\Delta))\epsilon^{JKL}\hat{\Phi}^E(F_{JK})\mathrm{tr}(\tau^i h_{s_{L(\Delta)}}[h^{-1}_{s_{L(\Delta)}}, \sqrt{\hat{V}(v(\Delta),\epsilon)}]) \tag{14}$$

$$\hat{\Theta}'^i_{\Delta'}[\chi] = -\frac{2}{3!}\frac{1}{i\hbar}\chi_\epsilon(v(\Delta),v(\Delta'))\epsilon^{MNP}\hat{\Phi}^E(F'_{MN})\mathrm{tr}(\tau^i h_{s_{P(\Delta')}}[h^{-1}_{s_{P(\Delta')}}, \sqrt{\hat{V}(v(\Delta'),\epsilon)}]) \tag{15}$$

to obtain

$$H^E[N] = -\frac{1}{\hbar^2 2\kappa^2 Q^2}\sum_{v\in V(\gamma)}N(v)\left(\frac{2}{3!}\frac{8}{E(v)}\right)^2\sum_{v(\Delta)=v(\Delta')=v}\mathrm{tr}(\tau^i h_{s_{L(\Delta)}}[h^{-1}_{s_{L(\Delta)}}, \sqrt{\hat{V}(v(\Delta),\epsilon)}])\epsilon^{JKL}\hat{\Phi}^E(F_{JK})$$
$$\times\mathrm{tr}(\tau^i h_{s_{P(\Delta')}}[h^{-1}_{s_{P(\Delta')}}, \sqrt{\hat{V}(v(\Delta'),\epsilon)}])\epsilon^{MNP}\hat{\Phi}^E(F'_{MN}). \tag{16}$$

The valence $n(v)$ of the vertex v yields the contribution $E(v)=n(v)[n(v)-1][n(v)-2]/3!$ of the adapted triangulation at each vertex of γ. Also, as $\epsilon\to 0$, $v(\Delta)=v(\Delta')$ are the only contributions left over. The final expression for the electric piece of the Hamiltonian constraint given in [31] is obtained by the explicit action of this operator on cylindrical functions. We refrain from doing that here because the form of the operator (16) is better suited for our approximation given below.

As for the magnetic part of H_{Maxwell} we proceed similarly. Since

$$\underline{h}_s = \mathrm{e}^{-i\int_0^1 dt\dot{s}^a(t)\underline{A}_a(s(t))} = \underline{\mathbb{I}} - i\int_0^1 dt\dot{s}^a(t)\underline{A}_a(s(t)) + \cdots$$

$$(\underline{h}_{\alpha_{JK}} - 1) = -i\int_0^1 dt\,\dot{\alpha}^a_{JK}(t)\underline{A}_a(s(t)) + \cdots = -i\int_{F_{JK}}\underline{B}^a dS_a + \cdots = -i\Phi^B(F_{JK}) + \cdots \approx -i\frac{1}{2}\epsilon_{abc}s^b_J(1)s^c_K(1)\underline{B}^a(v(\Delta)) \tag{17}$$

and

$$f(v)\epsilon^{JKL}(\underline{h}_{\alpha_{JK}} - 1)\mathrm{tr}(\tau_i h_{s_{L(\Delta)}}\{h^{-1}_{s_{L(\Delta)}}, \sqrt{V(x,\epsilon)}\}) \approx i\frac{1}{4}\epsilon^{JKL}\epsilon_{abc}s^b_J s^c_K s^d_L f(v)\underline{B}^a(v)\{A^i_d(v), \sqrt{V(x,\epsilon)}\}$$
$$= i\frac{1}{2}vol(s_J,s_K,s_L)\delta^d_a f(v)\underline{B}^a(v)\{A^i_d(v), \sqrt{V(x,\epsilon)}\}$$
$$= i\frac{3!}{2}vol(\Delta)f(v)\underline{B}^a(v)\{A^i_a(v), \sqrt{V(x,\epsilon)}\}$$
$$= i\frac{3!}{2}\int_\Delta f(x)\underline{B}^I(x)\wedge\{A^i(x), \sqrt{V(x,\epsilon)}\}, \tag{18}$$

we can write

$$H^B[N] = \frac{1}{2\kappa^2 Q^2}\lim_{\epsilon\to 0}\sum_{\Delta\Delta'}\Xi^i_\Delta[N]\Xi'^i_{\Delta'}[\chi], \tag{19}$$

where

$$\Xi^i_\Delta[f] := i\frac{2}{3!}f(v)\epsilon^{JKL}(\underline{h}_{\alpha_{JK}} - 1)\mathrm{tr}(\tau_i h_{s_{L(\Delta)}}\{h^{-1}_{s_{L(\Delta)}}, \sqrt{V(x,\epsilon)}\}). \tag{20}$$

Contrasting $\Xi^i_{I\Delta}$ with $\Theta^i_{I\Delta}$ we notice (i) different numerical factors: $i(2/3!)$ for the former while $-(2/3!)$ for the latter and (ii) to leading order, the magnetic flux $\epsilon^{JKL}(\underline{h}_{\alpha_{JK}} - 1)$ has as its counterpart the electric flux $\epsilon^{JKL}\Phi^E(F_{JK})$.

The quantum version of the above operators is obtained using

$$\hat{\Xi}^i_\Delta[f] := i\frac{2}{3!}\frac{1}{i\hbar}f(v)\epsilon^{JKL}(\underline{h}_{\alpha_{JK(\Delta)}}-1)\mathrm{tr}(\tau_i h_{s_L(\Delta)}[h^{-1}_{s_L(\Delta)},\sqrt{\hat{V}(x,\epsilon)}]).$$

Finally, one gets the regularized magnetic piece of the Hamiltonian constraint as [31]

$$H^B[N] = +\frac{1}{\hbar^2 2\kappa^2 Q^2}\sum_{v\in V(\gamma)}N(v)\left(\frac{2}{3!}\frac{8}{E(v)}\right)^2\sum_{v(\Delta)=v(\Delta')=v}\epsilon^{JKL}\mathrm{tr}(\tau_i h_{s_L(\Delta)}[h^{-1}_{s_L(\Delta)},\sqrt{\hat{V}_v}])(\underline{h}_{\alpha_{JK}(\Delta)}-1)$$
$$\times\epsilon^{MNP}\mathrm{tr}(\tau_i h_{s_P(\Delta')}[h^{-1}_{s_P(\Delta')},\sqrt{\hat{V}_v}])(\underline{h}_{\alpha_{MN}(\Delta')}-1). \tag{21}$$

The electric and magnetic pieces of H_{Maxwell} can be treated in a unified manner in terms of fluxes. To see this recall that for Abelian gauge fields

$$\underline{h}_{\alpha_{JK}(\Delta)} = e^{-i\int_{\alpha_{JK}(\Delta)}dt\dot{s}^a(t)\underline{\hat{A}}_a(s(t))} = e^{-i\hat{\Phi}^B(F_{JK})},$$

where $\hat{\Phi}^B(F_{JK})$ is the flux of the magnetic field through the surface F_{JK}.

Then the full electromagnetic Hamiltonian is

$$H_{\mathrm{Maxwell}}[N] = +\frac{1}{\hbar^2 2\kappa^2 Q^2}\sum_{v\in V(\gamma)}N(v)\left(\frac{2}{3!}\frac{8}{E(v)}\right)^2\sum_{v(\Delta)=v(\Delta')=v}\mathrm{tr}(\tau_i h_{s_L(\Delta)}[h^{-1}_{s_L(\Delta)},\sqrt{\hat{V}_v}])\mathrm{tr}(\tau_i h_{s_P(\Delta')}[h^{-1}_{s_P(\Delta')},\sqrt{\hat{V}_v}])$$
$$\times\epsilon^{JKL}\epsilon^{MNP}[(e^{-i\hat{\Phi}^B(F_{JK})}-1)(e^{-i\hat{\Phi}^B(F'_{MN})}-1)-\hat{\Phi}^E(F_{JK})\hat{\Phi}^E(F'_{MN})]. \tag{22}$$

Let us emphasize the structure of the above regularized Hamiltonian. There is a common gravitational factor included in the $SU(2)$ trace. The basic entities that regularize the electromagnetic part are the corresponding fluxes: one is associated with the magnetic field, which enters through a product of exponential flux factors, while the other is related to the electric field, entering in a bilinear product of fluxes. Thus, in the quadratic field approximation the effective Hamiltonian preserves duality invariance. Nevertheless, the magnetic sector includes higher powers in the field expansion. Hence, nonlinearities in the field equations, inducing duality violations, arise only via the magnetic field.

Before proceeding let us recall that, according to Thiemann's conventions, in flat space we must have

$$H_{\mathrm{Maxwell}} = \int d^3x\frac{1}{2\,Q^2}(\underline{E}^a\underline{E}^a+\underline{B}^a\underline{B}^a), \tag{23}$$

where Q is the electromagnetic coupling constant. The electromagnetic potential is denoted by $\underline{A}_a$ and the corresponding electromagnetic tensor by $\underline{F}_{ab}$. The units are such that the gravitational connection A^i_a has dimensions of $1/L$ (inverse length) and the Newton's constant κ has dimensions of L/M (length over mass). Also we have that $[|\vec{\underline{E}}|/Q^2]=M/L^3$. Taking the dimensions of $\underline{A}_a$ to be $1/L$, according to the corresponding normalization of the Wilson loop, we conclude that $[\vec{\underline{E}}]=[\vec{\underline{B}}]=1/L^2$ and $[Q^2]=1/(ML)$. In our case we also have $[\hbar]=ML$, which in fact leads to $\alpha_{EM}=Q^2\hbar$ to be the dimensionless fine-structure constant, as defined by Thiemann [31].

IV. GENERAL STRUCTURE OF THE CALCULATION

The effective Maxwell Hamiltonian is defined by considering the expectation value of the $U(1)$ gauge sector of the quantum Hamiltonian constraint with respect to $|W,\vec{\underline{E}},\vec{\underline{B}}\rangle$. Inside this expectation value operators are expanded around all relevant vertices of the triangulation in powers of the segments $s^a_L(\Delta)$, which have lengths of order l_P. In this way, a systematic approximation is given involving the scales $l_P \ll \mathcal{L} < \lambda$, where λ is the de Broglie wavelength of the photon. Our corrections to the Maxwell Hamiltonian arise from such an approximation.

We do the full calculation of the magnetic sector, including the non-linear contributions to order l_P^2. Next, to obtain the electric sector, it is enough to consider only the quadratic terms in the magnetic Hamiltonian and make the replacement $B\to E$.

In the case of the magnetic sector, the general form of the expectation value is (recalling that $1/\kappa^2=\hbar^2/l_P^4$)

$$\mathrm{H}^B = -\frac{1}{2\,Q^2}\frac{1}{l_P^4}\sum_{v\in V(\gamma)} N(v)\left(\frac{2}{3!}\frac{8}{E(v)}\right)^2 \sum_{v(\Delta)=v(\Delta')=v} \langle W,\vec{\underline{E}},\vec{\underline{B}}|\hat{\underline{F}}_{p_1q_1}(v)\ldots\hat{\underline{F}}_{p_nq_n}(v)$$
$$\times(\partial^{a_1}\ldots\partial^{a_m}\hat{\underline{F}}_{pq}(v))\hat{T}_{a_1\ldots a_m}{}^{pq\,p_1q_1\ldots p_nq_n}(v,s(\Delta),s(\Delta'))|W,\vec{\underline{E}},\vec{\underline{B}}\rangle. \tag{24}$$

To proceed with the approximation we think of space as made up of boxes, each centered at a given point $\vec{x}$ and with volume $\mathcal{L}^3\approx d^3x$. Each box contains a large number of vertices of the semiclassical state ($\mathcal{L}\gg l_P$), but is considered as infinitesimal in the scale where the space can be regarded as continuous. Also, we assume that the magnetic operators are slowly varying inside the box ($l_P\ll\lambda$), in such a way that for all the vertices inside the box one can write

$$\langle W,\vec{\underline{E}},\vec{\underline{B}}|\ldots\hat{\underline{F}}_{ab}(v)\ldots|W,\vec{\underline{E}},\vec{\underline{B}}\rangle = \mu\,\epsilon_{abc}\underline{B}^c(\vec{x}). \tag{25}$$

Here $\underline{F}_{ab}=\partial_a\underline{A}_b-\partial_b\underline{A}_a$, $\underline{B}^a(\vec{x})$ is the classical magnetic field at the center of the box and μ is a dimensionless constant to be determined in such a way that we recover the standard classical result (23) in the zeroth order approximation. In the next section we show that

$$\mu=\left(\frac{\mathcal{L}}{l_P}\right)^{1+\Upsilon}, \tag{26}$$

with Υ being a parameter defining the leading order contribution of the gravitational connection to the expectation value. Applying to Eq. (24) the procedure just described leads to

$$\begin{aligned}\mathrm{H}^B &= \sum_{\mathrm{Box}} N(\vec{x})\underline{F}_{p_1q_1}(\vec{x})\ldots\underline{F}_{p_nq_n}(\vec{x})\ldots(\partial^{a_1}\ldots\partial^{a_m}\underline{F}_{pq}(\vec{x}))\sum_{v\in\mathrm{Box}} l_P^3\left(\frac{2}{3!}\frac{8}{E(v)}\right)^2\sum_{v(\Delta)=v(\Delta')=v}\left(-\frac{1}{2\,Q^2}\frac{\mu^{n+1}}{l_P^4}\right)\\ &\times\langle W,\vec{\underline{E}},\vec{\underline{B}}|\frac{1}{l_P^3}\hat{T}_{a_1\ldots a_m}{}^{pqp_1q_1\ldots p_nq_n}(v,s(\Delta),s(\Delta'))|W,\vec{\underline{E}},\vec{\underline{B}}\rangle\\ &= \sum_{\mathrm{Box}} N(\vec{x})\underline{F}_{p_1q_1}(\vec{x})\ldots\underline{F}_{p_nq_n}(\vec{x})(\partial^{a_1}\ldots\partial^{a_m}\underline{F}_{pq}(\vec{x}))d^3x\,T_{a_1\ldots a_m}{}^{pqp_1q_1\ldots p_nq_n}(\vec{x})\\ \mathrm{H}^B &= \int d^3x\,N(\vec{x})\underline{F}_{p_1q_1}(\vec{x})\ldots\underline{F}_{p_nq_n}(\vec{x})(\partial^{a_1}\ldots\partial^{a_m}\underline{F}_{pq}(\vec{x}))T_{a_1\ldots a_m}{}^{pqp_1q_1\ldots p_nq_n}(\vec{x}).\end{aligned} \tag{27}$$

The box-averaged tensor $T_{a_1\ldots a_m}{}^{pqp_1q_1\ldots p_n}(\vec{x})$, defined by

$$T_{a_1\ldots a_m}{}^{pqp_1q_1\ldots p_nq_n}(\vec{x}) = \sum_{v\in\mathrm{Box}}\left(\frac{2}{3!}\frac{8}{E(v)}\right)^2\sum_{v(\Delta)=v(\Delta')=v}\left(-\frac{1}{2\,Q^2}\frac{\mu^{n+1}}{l_P^4}\right)$$
$$\times\langle W,\vec{\underline{E}},\vec{\underline{B}}|\frac{1}{l_P^3}\hat{T}_{a_1\ldots a_m}{}^{pqp_1q_1\ldots p_nq_n}(v,s(\Delta),s(\Delta'))|W,\vec{\underline{E}},\vec{\underline{B}}\rangle, \tag{28}$$

is constructed from flat space tensors like δ_{ab}, ϵ_{abc}. In this way we are demanding covariance under rotations at the scale $\mathcal{L}$.

When averaging inside each box, the scaling of the expectation values of the gravitational operators is estimated according to

$$\langle W,\vec{\underline{E}},\vec{\underline{B}}|\ldots A_{ia}\ldots|W,\vec{\underline{E}},\vec{\underline{B}}\rangle\approx\ldots\frac{1}{\mathcal{L}}\left(\frac{l_P}{\mathcal{L}}\right)^{\Upsilon}\ldots, \tag{29}$$

$$\langle W,\vec{\underline{E}},\vec{\underline{B}}|\ldots\sqrt{V_v}\ldots|W,\vec{\underline{E}},\vec{\underline{B}}\rangle\approx\ldots l_P^{3/2}\ldots, \tag{30}$$

respectively. In our previous work [20] we have set $\Upsilon=0$ on the basis that the coarse graining approximation, defined by the scale $\mathcal{L}$, does not allow for the connection to be probed below $1/\mathcal{L}$. On the other hand, by adopting naive kinematical coherent states for representing semiclassical states, one would set $\Upsilon=1$ for two reasons: (i) to guarantee that Eq. (29) yields just zero in the limit $\hbar\to 0$, in agreement with a flat connection and (ii) because such an scaling would saturate the Heisenberg uncertainty relation [38]. Nonetheless, physical semiclassical states may imply a leading order contribution with $\Upsilon\neq 0,1$, thus we choose to consider Υ as a free parameter here.

In order to make the transition to the electric sector it is convenient to express the effective Hamiltonian H^B in terms of the magnetic field, which amounts to a redefinition of the expression (27) in the form

$$\mathrm{H}^B=\int d^3x\, N(\vec{x})\underline{B}_{r_1}(\vec{x})\ldots\underline{B}_{r_n}(\vec{x})(\partial^{a_1}\ldots\partial^{a_m}\underline{B}_r(\vec{x}))R_{a_1\ldots a_m}{}^{rr_1\ldots r_n}(\vec{x}). \tag{31}$$

The relation between the box-averaged tensors R and T is

$$T_{a_1\ldots a_m}{}^{pqp_1q_1\ldots p_nq_n}(\vec{x})=\epsilon^{pq}{}_r\epsilon^{p_1q_1}{}_{r_1}\ldots\epsilon^{p_nq_n}{}_{r_n}R_{a_1\ldots a_m}{}^{rr_1\ldots r_n}(\vec{x}). \tag{32}$$

By expanding Eq. (22) to different powers in $s_I^a(\Delta)$ one can systematically determine all possible contributions to the effective electromagnetic Hamiltonian at a given order in l_P.

V. THE CALCULATION

In this section we provide the details of the calculation of the Maxwell effective Hamiltonian up to order l_P^2. Let us start with the magnetic sector.

The two main ingredients in Eq. (22) which contribute to the expansion in powers of the segments $s_I(\Delta)$ are (i) the trace factors involving the gravitational operators and (ii) the magnetic flux through each surface $F_{IJ}(\Delta)$.

First we calculate the flux of the magnetic field through the surface F_{IJ}. A convenient way to do this is via the Stokes theorem

$$\begin{aligned}\Phi^B(F_{IJ})&=\int_{F_{IJ}}\underline{B}_a n^a d^2x=\int_{F_{IJ}}(\nabla\times\underline{\vec{A}})_a n^a d^2x\\&=\int_{\alpha_{IJ}}dt\,\dot{s}^a(t)\underline{A}_a(t)\\&=\int_{\vec{v}}^{\vec{v}+\vec{s}_I}\underline{A}_a dx^a+\int_{\vec{v}+\vec{s}_I}^{\vec{v}+\vec{s}_J}\underline{A}_a dx^a\\&\quad+\int_{\vec{v}+\vec{s}_J}^{\vec{v}}\underline{A}_a dx^a.\end{aligned} \tag{33}$$

Here the notation is $\vec{s}_I=\{s_I^a\}$ and analogously for $\vec{v}$. We are using straight line trajectories joining the vertices of the corresponding triangle.

The basic building block in Eq. (33) is

$$\begin{aligned}\int_{\vec{v}_1}^{\vec{v}_2}\underline{A}_a(\vec{x})dx^a&=\int_0^1\underline{A}_a[\vec{v}_1+t(\vec{v}_2-\vec{v}_1)](\vec{v}_2-\vec{v}_1)^a dt\\&=\int_0^1\underline{A}_a(\vec{v}_1+t\,\vec{\Delta})\Delta^a dt\\&=\left(1+\frac{1}{2!}\Delta^b\partial_b\right.\\&\quad\left.+\frac{1}{3!}(\Delta^b\partial_b)^2+\ldots\right)\Delta^a\underline{A}_a(v),\end{aligned} \tag{34}$$

with $\Delta^a=(\vec{v}_2-\vec{v}_1)^a$. The infinite series in parentheses is

$$F(x)=1+\frac{1}{2!}x+\frac{1}{3!}x^2+\frac{1}{4!}x^3+\ldots=\frac{e^x-1}{x}, \tag{35}$$

yielding

$$\int_{\vec{v}_1}^{\vec{v}_2}\underline{A}_a(\vec{x})dx^a=F(\Delta^a\partial_a)(\Delta^a\underline{A}_a(\vec{v}_1)). \tag{36}$$

In the following we employ the notation $\Delta^a V_a=\vec{\Delta}\cdot\vec{V}$. Using the above result in the three integrals appearing in Eq. (33) and after some algebra, we obtain

$$\begin{aligned}\Phi^B(F_{IJ})&=F_1(\vec{s}_I\cdot\nabla,\vec{s}_J\cdot\nabla)s_J^a s_I^b[\partial_a\underline{A}_b(\vec{v})-\partial_b\underline{A}_a(\vec{v})]\\&=F_1(\vec{s}_I\cdot\nabla,\vec{s}_J\cdot\nabla)s_J^a s_I^b\epsilon_{abc}\underline{B}^c(v),\end{aligned} \tag{37}$$

where the gradient acts upon the coordinates of $\vec{v}$. The function F_1 is

$$\begin{aligned}F_1(x,y)&=\frac{y(\mathrm{e}^x-1)-x(\mathrm{e}^y-1)}{xy(y-x)}\\&=-\sum_{n=1}^{\infty}\frac{1}{(n+1)!}\frac{x^n-y^n}{x-y}.\end{aligned} \tag{38}$$

Let us emphasize that $F_1(x,y)$ is just a power series in the variables x and y. Expanding to fourth order in the segments s_I^a we obtain

$$\begin{aligned}\Phi^B(F_{IJ})=&\left(1+\frac{1}{3}(s_I^c+s_J^c)\partial_c+\frac{1}{12}(s_I^c s_I^d+s_I^c s_J^d\right.\\&\left.+s_J^c s_J^d)\partial_c\partial_d+\ldots\right)\frac{1}{2}s_I^a s_J^b\epsilon_{abc}\underline{B}^c(v).\end{aligned} \tag{39}$$

Notice that the combination

$$\frac{1}{2}s_I^a s_J^b\epsilon_{abc}=\mathcal{A}\,n_c \tag{40}$$

is just the oriented area of the triangle with vertex v and

sides s_I^c, s_J^c, joining at this vertex, having value $\mathcal{A}$ and unit normal vector n_c.

In order to make the bookkeeping clear, let us denote by T the combination arising from Eq. (22) whose expectation value we are calculating

$$T=-\frac{1}{2\,Q^2}\frac{1}{l_P^4}\hat{w}_{i\,L\Delta}\hat{w}_{i\,P\Delta'}\epsilon^{JKL}\epsilon^{MNP}\times(e^{-i\hat{\Phi}^B(F_{JK})}-1)(e^{-i\hat{\Phi}^B(F'_{MN})}-1). \tag{41}$$

Here

$$\hat{w}_{i\,L\Delta}=\mathrm{tr}(\tau_i h_{s_L(\Delta)}[h^{-1}_{s_L(\Delta)},\sqrt{\hat{V}_v}]). \tag{42}$$

Some remarks are in order before we proceed further. Our final goal is to obtain a power expansion of Eq. (41) up to order l_P^2. Since, as we will show in the sequel, the normalization factor converting magnetic operators inside the semiclassical expectation value into classical fields outside the expectation value is proportional to $(l_P)^{-1}$, we have to take some care regarding the expansion of the given quantities in powers of s_I^a. A detailed power counting analysis in the expression (41) shows that the term $\hat{w}_{i\,L(\Delta)}$ in Eq. (42) is to be expanded up to order s^3, while each magnetic factor $(e^{-i\hat{\Phi}^B(F_{JK})}-1)$ is required to have the following properties: the terms proportional to $\underline{F}$ are to be expanded up to order s^4, those proportional to $\underline{F}^2$ up to order s^5 and finally those proportional to $\underline{F}^3$ up to order s^6. This will lead to the following contributions in T: the terms proportional to $\underline{F}^2$ include the expansion up to order s^8, the terms proportional to $\underline{F}^3$ include the expansion up to order s^9 and the terms proportional to $\underline{F}^4$ include the expansion up to order s^{10}. The final result is that the semiclassical expectation value of the magnetic contribution T will be proportional to l_P^3, which is incorporated in the volume element, times corrections up to order l_P^2.

Let us now continue with the calculation of the contribution to Eq. (41) due to the magnetic flux by writing the expansion

$$(e^{-i\hat{\Phi}^B(F_{JK}(\Delta))}-1)=\sum_{n=1}^{\infty}\frac{(-i)^n}{n!}(\hat{\Phi}^B(F_{JK}))^n=M_{1JK(\Delta)}+M_{2JK(\Delta)}+M_{3JK(\Delta)}+\mathcal{O}(s^7\underline{F}^3) \tag{43}$$

where

$$M_{1JK(\Delta)}:=s_K^a s_J^b\frac{i}{2!}\underline{F}_{ab},$$

$$M_{2JK(\Delta)}:=s_K^a s_J^b\frac{i}{3!}(x_J+x_K)\underline{F}_{ab}-s_K^a s_J^b s_K^c s_J^d\frac{1}{8}\underline{F}_{ab}\underline{F}_{cd},$$

$$M_{3JK(\Delta)}:=s_K^a s_J^b\frac{i}{4!}(x_J^2+x_J x_K+x_K^2)\underline{F}_{ab}-s_K^a s_J^b s_K^c s_J^d\times\left[\frac{1}{4\cdot 3!}(x_J+x_K)\underline{F}_{ab}\underline{F}_{cd}+\frac{1}{4\cdot 3!}\underline{F}_{ab}(x_K+x_J)\underline{F}_{cd}\right]-s_K^a s_J^b s_K^c s_J^d s_K^e s_J^f\frac{i}{4\cdot 3!}\underline{F}_{ab}\underline{F}_{cd}\underline{F}_{ef}, \tag{44}$$

according to the previous analysis. We are using the notation $x_I=\vec{s}_I\cdot\nabla=s_I^a\partial_a$. Let us remark that, contrary to the electric case, the magnetic contribution will incorporate non-linear terms due to the expansion of the exponential in powers of $\underline{\vec{B}}$. This implies that the exact duality symmetry of Maxwell equations in vacuum will be lost due to quantum gravity corrections.

Next let us consider the gravitational contributions to Eq. (22), arising from Eq. (42), which we expand as

$$\hat{w}_{i\,L\Delta}=s_L^a w_{ia}+s_L^a s_L^b w_{iab}+s_L^a s_L^b s_L^c w_{iabc}+\mathcal{O}(s^4 w), \tag{45}$$

with

$$w_{ia}=\frac{1}{2}[A_{ia},\sqrt{V}],\quad w_{iab}=\frac{1}{8}\epsilon_{ijk}[A_{ja},[A_{kb},\sqrt{\hat{V}}]],$$

$$w_{iabc}=-\frac{1}{48}[A_{ja},[A_{jb},[A_{ic},\sqrt{\hat{V}}]]]. \tag{46}$$

The scaling properties of the above gravitational operators under the semiclassical expectation value are

$$\langle W,\underline{\vec{E}},\underline{\vec{B}}|\ldots w_{i\,a_1\ldots a_n}\ldots|W,\underline{\vec{E}},\underline{\vec{B}}\rangle\to\frac{l_P^{3/2}}{\mathcal{L}^n}\left(\frac{l_P}{\mathcal{L}}\right)^{n\gamma}. \tag{47}$$

Let us emphasize that the result (47) is a consequence of the scaling of the expectation value of the connection given by Eq. (29).

For the product $\hat{w}_{i\,L\Delta}\hat{w}_{i\,P\Delta'}$ we need only

$$\hat{w}_{i\,L\Delta}\hat{w}_{i\,P\Delta'}=U_{2LP}+U_{3LP}+U_{4LP}+\mathcal{O}(s^5w^2) \tag{48}$$

with

$$U_{2LP}=s_L^a s_P'^d w_{ia}w_{id},$$

$$U_{3LP}=s_L^a s_P'^d s_P'^e w_{ia}w_{ide}+s_L^a s_L^b s_P'^d w_{iab}w_{id},$$

$$U_{4LP}=s_L^a s_P'^d s_P'^e s_P'^f w_{ia} w_{idef} + s_L^a s_L^b s_P'^d s_P'^e w_{iab} w_{ide} + s_L^a s_L^b s_L^c s_P'^d w_{iabc} w_{id}. \quad (49)$$

Here all the w's are evaluated at a common vertex v.

At this level it is convenient to state the result (no sum over L)

$$s_L^a s_L^b w_{iab} = \frac{1}{8} s_L^a s_L^b \epsilon_{ijk} [A_{ja}, [A_{kb}, \sqrt{\hat{V}}]]$$

$$= \frac{1}{8} s_L^a s_L^b \epsilon_{ijk} (A_{ja} A_{kb} \sqrt{\hat{V}} - A_{ja} \sqrt{\hat{V}} A_{kb} - A_{kb} \sqrt{\hat{V}} A_{ja} + \sqrt{\hat{V}} A_{kb} A_{ja}),$$

$$= 0, \quad (50)$$

which holds due to symmetry properties. This leads to

$$U_{3LP}=0. \quad (51)$$

After taking the expectation value the terms contributing to order l_P^2 in Eq. (41) read

$$T = T_0 + T_1 + T_2 + \mathcal{O}(l_P^3) \quad (52)$$

$$T_0 = -\frac{1}{2Q^2} \frac{1}{l_P^4} \epsilon^{JKL} \epsilon^{MNP} [U_{2LP} M_{1JK} M'_{1MN}], \quad (53)$$

$$T_1 = -\frac{1}{2Q^2} \frac{1}{l_P^4} \epsilon^{JKL} \epsilon^{MNP} U_{2LP} [M_{1JK} M'_{2MN} + M_{2JK} M'_{1MN}], \quad (54)$$

$$T_2 = -\frac{1}{2Q^2} \frac{1}{l_P^4} \epsilon^{JKL} \epsilon^{MNP} [U_{2LP} (M_{1JK} M'_{3MN} + M_{3JK} M'_{1MN} + M_{2JK} M'_{2MN}) + U_{4LP} M_{1JK} M'_{1MN}]. \quad (55)$$

Now we are ready to calculate the different contributions to the magnetic sector of the Hamiltonian (22), which we parametrize in terms of the tensor $R_{a_1 \ldots a_m}^{\quad rr_1 \ldots r_n}$ introduced in Eq. (32).

Recalling that we are only interested in the pieces which are symmetric in the indices $r_1, r_2, \ldots, r_n$, the contribution T_0 produces

$$R_0^{\ r_1 r_2} = \sum_{v \in \mathrm{Box}} \frac{1}{2Q^2} \left(\frac{2}{3!} \frac{8}{E(v)} \right)^2 \times \sum_{v(\Delta)=v(\Delta')=v} \frac{\mu^2}{4 l_P^7} \epsilon^{r_1}{}_{ab} \epsilon^{r_2}{}_{uv} \epsilon^{JKL} \epsilon^{MNP} \times s_K^a s_J^b s_L^c s'^v_M s'^u_N s'^d_P \langle W, \underline{\vec{E}}, \underline{\vec{B}} | w_{ic} w_{id} | W, \underline{\vec{E}}, \underline{\vec{B}} \rangle. \quad (56)$$

In order to simplify the product of vectors s_L^a (s'^p_M) appearing in the sequel and also to exhibit the internal symmetry properties of the quantities involved, it is convenient to keep in mind the relations

$$\epsilon^{KJL} s_K^a s_J^b s_L^c = \det(s) \epsilon^{abc}, \quad \det(s) = \det(s_K^a), \quad \epsilon^{abp} \epsilon_{abq} = 2 \delta^p_q. \quad (57)$$

In this way, Eq. (56) can be rewriten in the simpler form

$$R_0^{\ r_1 r_2} = \sum_{v \in \mathrm{Box}} \frac{1}{2Q^2} \left(\frac{2}{3!} \frac{8}{E(v)} \right)^2 \times \sum_{v(\Delta)=v(\Delta')=v} \frac{\mu^2}{l_P^7} \det(s) \det(s') \langle W, \underline{\vec{E}}, \underline{\vec{B}} | \frac{1}{2} \times \{ w_i^{r_1}, w_i^{r_2} \} | W, \underline{\vec{E}}, \underline{\vec{B}} \rangle. \quad (58)$$

The above equation implies

$$R_0^{\ r_1 r_2} = \frac{1}{2Q^2} \frac{\mu^2}{l_P^7} l_P^6 \frac{l_P^3}{\mathcal{L}^2} \left(\frac{l_P}{\mathcal{L}} \right)^{2\mathrm{Y}} \delta^{r_1 r_2} = \frac{1}{2Q^2} \delta^{r_1 r_2}, \quad (59)$$

which reproduces the zeroth-order magnetic contribution (23) with the choice

$$\mu = \left(\frac{\mathcal{L}}{l_P} \right)^{1+\mathrm{Y}}. \quad (60)$$

Now let us consider the correction arising from T_1, which leads to the following contribution in the effective Hamiltonian

$$\mathrm{H}_{11}^B = \sum_{\mathrm{Box}(\vec{x})} \sum_{v \in \mathrm{Box}} \frac{1}{2Q^2} \left(\frac{2}{3!} \frac{8}{E(v)} \right)^2 l_P^3 \times \sum_{v(\Delta)=v(\Delta')=v} -\frac{1}{l_P^7} \epsilon^{JKL} \epsilon^{MNP} s_L^c s'^d_P s_K^a s_J^b \langle W, \underline{\vec{E}}, \underline{\vec{B}} | \times \{ w_{ic}, w_{id} \} \frac{i}{2} \underline{F}_{ab} \left[s'^v_N s'^q_M \frac{i}{3!} (x'_M + x'_N) \underline{F}_{vq} - s'^v_N s'^q_M s'^r_N s'^t_M \frac{1}{8} \underline{F}_{vq} \underline{F}_{rt} \right] | W, \underline{\vec{E}}, \underline{\vec{B}} \rangle. \quad (61)$$

In the above expression we have interchanged the summations over Δ and Δ' in order to rewrite $w_{ic} w_{id}$ as

$\frac{1}{2}\{w_{ic},w_{id}\}$. We further separate the above contribution to H_{11}^B in two parts: (i) the first contains two powers in the magnetic field and leads to $R_{11a_1}{}^{rr_1}$ and (ii) the second one contains three powers in the magnetic field and leads to the completely symmetric tensor $R_{11}{}^{r_1r_2r_3}$. Since we have no symmetric tensor with three indices at our disposal, the latter contribution is zero. Thus we concentrate in the first one

$$R_{11}{}^{a_1rr_1}=\sum_{v\in \mathrm{Box}}\frac{1}{2\,Q^2}\left(\frac{2}{3!}\frac{8}{E(v)}\right)^2 \times\sum_{v(\Delta)=v(\Delta')=v}\frac{\mu^2}{6\,l_P^7}\epsilon^{JKL}\epsilon^{MNP}s_L^c s_P'^d s_K^a s_J^b s'^v{}_N \times s'^q{}_M s'^{a_1}{}_M\epsilon^{r_1}{}_{ab}\epsilon^r{}_{vq}\langle W,\underline{\vec{E}},\underline{\vec{B}}|\{w_{ic},w_{id}\}|W,\underline{\vec{E}},\underline{\vec{B}}\rangle. \tag{62}$$

In this case the internal symmetry properties are hard to make explicit. In order to determine whether or not the above contribution is zero we contract Eq. (62) with the only three index tensor at our disposal: $\epsilon^{a_1rr_1}$. The result is

$$\epsilon_{a_1rr_1}R_{11}{}^{a_1rr_1}=-\sum_{v\in \mathrm{Box}}\frac{1}{2\,Q^2}\left(\frac{2}{3!}\frac{8}{E(v)}\right)^2\sum_{v(\Delta)=v(\Delta')=v}\frac{\mu^2}{6\,l_P^7} \times\det(s)\epsilon^{MNP}\langle W,\underline{\vec{E}},\underline{\vec{B}}|\{w_{ic},w_{id}\}|W,\underline{\vec{E}},\underline{\vec{B}}\rangle \times(s_P'^d s'^c{}_M s'^q{}_N s'_{qM}+s_P'^d s'^c{}_N s'^q{}_M s'_{qM}). \tag{63}$$

Inside the expectation value, upper spatial indices have been lowered by the flat metric. By symmetry requirements, the second term in parentheses in Eq. (63) yields zero. Nevertheless, the first one gives the result

$$R_{11}{}^{a_1rr_1}=\kappa_8\frac{\mu^2}{Q^2l_P^7}l_P^7\frac{l_P^3}{\mathcal{L}^2}\left(\frac{l_P}{\mathcal{L}}\right)^{2\Upsilon}\epsilon^{a_1rr_1}, \tag{64}$$

which produces a parity-violating term in the magnetic sector of the effective Hamiltonian.

The next contribution arises from T_2 and can be separated into three pieces:

$$H_{21}^B=\sum_{v\in V(\gamma)}\frac{1}{2\,Q^2}\left(\frac{2}{3!}\frac{8}{E(v)}\right)^2 l_P^3\times\sum_{v(\Delta)=v(\Delta')=v}-\frac{1}{l_P^7}\epsilon^{JKL}\epsilon^{MNP}\langle W,\underline{\vec{E}},\underline{\vec{B}}|U_{2LP}\times(M_{1JK}M'_{3MN}+M_{3JK}M'_{1MN})|W,\underline{\vec{E}},\underline{\vec{B}}\rangle, \tag{65}$$

$$H_{22}^B=\sum_{v\in V(\gamma)}\frac{1}{2\,Q^2}\left(\frac{2}{3!}\frac{8}{E(v)}\right)^2 l_P^3\sum_{v(\Delta)=v(\Delta')=v}-\frac{1}{l_P^7}\times\epsilon^{JKL}\epsilon^{MNP}\langle W,\underline{\vec{E}},\underline{\vec{B}}|U_{2LP}M_{2JK}M'_{2MN}|W,\underline{\vec{E}},\underline{\vec{B}}\rangle, \tag{66}$$

$$H_{23}^B=\sum_{v\in V(\gamma)}\frac{1}{2\,Q^2}\left(\frac{2}{3!}\frac{8}{E(v)}\right)^2 l_P^3\sum_{v(\Delta)=v(\Delta')=v}-\frac{1}{l_P^7}\times\epsilon^{JKL}\epsilon^{MNP}\langle W,\underline{\vec{E}},\underline{\vec{B}}|U_{4LP}M_{1JK}M'_{1MN}|W,\underline{\vec{E}},\underline{\vec{B}}\rangle. \tag{67}$$

Let us start discussing H_{21}^B. After some algebra we obtain

$$H_{21}^B=\sum_{v\in V(\gamma)}\frac{1}{2\,Q^2}\left(\frac{2}{3!}\frac{8}{E(v)}\right)^2 l_P^3\sum_{v(\Delta)=v(\Delta')=v}-\frac{i}{l_P^7}\times\epsilon^{JKL}\epsilon^{MNP}s_L^a s_P'^d s'^x_N s'^q_M\langle W,\underline{\vec{E}},\underline{\vec{B}}|w_{ia}w_{id}\times\Bigg(s_K^r s_J^t\frac{i}{4!}(x_J^2+x_Jx_K+x_K^2)\underline{F}_{rt}-s_K^r s_J^t s_K^u s_J^v\times\left[\frac{1}{4\cdot 3!}(x_J+x_K)\underline{F}_{rt}\underline{F}_{uv}+\frac{1}{4\cdot 3!}\underline{F}_{rt}(x_K+x_J)\underline{F}_{uv}\right]-s_K^r s_J^t s_K^u s_J^v s_K^w s_J^z\frac{i}{4\cdot 3!}\underline{F}_{rt}\underline{F}_{uv}\underline{F}_{wz}\Bigg)\underline{F}_{xq}|W,\underline{\vec{E}},\underline{\vec{B}}\rangle, \tag{68}$$

which naturally splits into the following pieces:

$$H_{211}^B=\sum_{v\in V(\gamma)}\frac{1}{2\,Q^2}\left(\frac{2}{3!}\frac{8}{E(v)}\right)^2 l_P^3\sum_{v(\Delta)=v(\Delta')=v}-\frac{i}{l_P^7}\times\epsilon^{JKL}\epsilon^{MNP}s_L^a s_P'^d s'^x_N s'^q_M\langle W,\underline{\vec{E}},\underline{\vec{B}}|w_{ia}w_{id}\times\left(s_K^r s_J^t\frac{i}{4!}(x_J^2+x_Jx_K+x_K^2)\underline{F}_{rt}\right)\underline{F}_{xq}|W,\underline{\vec{E}},\underline{\vec{B}}\rangle, \tag{69}$$

$$H_{212}^B=\sum_{v\in V(\gamma)}\frac{1}{2\,Q^2}\left(\frac{2}{3!}\frac{8}{E(v)}\right)^2 l_P^3\times\sum_{v(\Delta)=v(\Delta')=v}\frac{i}{l_P^7}\epsilon^{JKL}\epsilon^{MNP}s_L^a s_P'^d s'^x_N s'^q_M\times\langle W,\underline{\vec{E}},\underline{\vec{B}}|w_{ia}w_{id}\times\left(s_K^r s_J^t s_K^u s_J^v\left[\frac{1}{4\cdot 3!}(x_J+x_K)\underline{F}_{rt}\underline{F}_{uv}+\frac{1}{4\cdot 3!}\underline{F}_{rt}(x_K+x_J)\underline{F}_{uv}\right]\right)\underline{F}_{xq}|W,\underline{\vec{E}},\underline{\vec{B}}\rangle, \tag{70}$$

$$\mathrm{H}^{B}_{213}=\sum_{v\in V(\gamma)}\frac{1}{2\,Q^2}\left(\frac{2}{3!}\frac{8}{E(v)}\right)^2 l_P^3 \times\sum_{v(\Delta)=v(\Delta')=v}\frac{i}{l_P^7}\epsilon^{JKL}\epsilon^{MNP} \times s_L^a s_P'^d s_N'^x s_M'^q\langle W,\underline{\vec{E}},\underline{\vec{B}}|w_{ia}w_{id} \times\left(s_K^r s_J^t s_K^u s_J^v s_K^w s_J^z\frac{i}{4\cdot 3!}\underline{F}_{rt}\underline{F}_{uv}\underline{F}_{wz}\right)\underline{F}_{xq}|W,\underline{\vec{E}},\underline{\vec{B}}\rangle. \tag{71}$$

Next we calculate the corresponding R tensors. The first one is

$$R_{211}{}^{a_1a_2rr_1}=\sum_{v\in\mathrm{Box}}\frac{1}{2}\left(\frac{2}{3!}\frac{8}{E(v)}\right)^2 \times\sum_{v(\Delta)=v(\Delta')=v}\frac{2\,\mu^2}{4!Q^2l_P^7}\det(s')\det(s) \times\epsilon_{NKJ}(s^{-1})^{rN}\epsilon^{JKL}s_L^a(2s_J^{a_1}s_J^{a_2}+s_J^{a_1}s_K^{a_2}) \times\langle W,\underline{\vec{E}},\underline{\vec{B}}|w_{ia}w_i^{\ r_1}|W,\underline{\vec{E}},\underline{\vec{B}}\rangle, \tag{72}$$

$$R_{211}{}^{a_1a_2rr_1}=\frac{\mu^2}{Q^2l_P^7}l_P^8\frac{l_P^3}{\mathcal{L}^2}\left(\frac{l_P}{\mathcal{L}}\right)^{2\Upsilon}[\kappa_6\delta^{a_1a_2}\delta^{rr_1} +\kappa_7(\delta^{a_1r}\delta^{a_2r_1}+\delta^{a_2r}\delta^{a_1r_1})],$$

that we have constructed using the most general tensor with four indices, which is symmetrical in a_1 and a_2. The terms proportional to κ_7 contribute with $\underline{B}_a\partial^a\partial^b\underline{B}_b$ to the effective Hamiltonian. Keeping this term will be useful to construct the electric sector later. Notice that we have also introduced the inverse matrix $(s^{-1})^N_m$ such that

$$(s^{-1})^P_m s^m_Q=\delta^P_Q,\quad s^a_M(s^{-1})^M_b=\delta^a_b, \tag{73}$$

which elements scale as l_P^{-1}.

Our next contribution, arising from Eq. (70), is zero because the object (no sum over J and K)

$$s_K^r s_J^t s_K^u s_J^v\underline{F}_{uv}(x_J+x_K)\underline{F}_{rt}, \tag{74}$$

which is symmetrical in JK, appears contracted with ϵ^{JKL}.

Now we consider

$$\mathrm{H}^{B}_{213}=\sum_{\mathrm{Box}(\vec{x})}\underline{B}^b(\vec{x})\underline{B}^c(\vec{x})\underline{B}^f(\vec{x})\underline{B}^e(\vec{x})\sum_{v\in Box}\frac{1}{2\,Q^2} \times\left(\frac{2}{3!}\frac{8}{E(v)}\right)^2 l_P^3\sum_{v(\Delta)=v(\Delta')=v}-\frac{\mu^4}{4\times 3!\,l_P^7}2\det(s') \times[\det(s)]^3\epsilon^{JKL}s_L^a(\epsilon_{UKJ}(s^{-1})^U_b)(\epsilon_{VKJ}(s^{-1})^V_c) \times(\epsilon_{XKJ}(s^{-1})^X_f)\langle W,\underline{\vec{E}},\underline{\vec{B}}|w_{ia}w_{ie}|W,\underline{\vec{E}},\underline{\vec{B}}\rangle, \tag{75}$$

where we have introduced the relation

$$\epsilon_{mbc}s_J^b s_L^c=\det(s)\,\epsilon_{KJL}(s^{-1})^K_m, \tag{76}$$

obtained from Eq. (57). The further properties

$$\epsilon^{LKJ}\epsilon_{UKJ}\epsilon_{VKJ}\epsilon_{XKJ}\neq 0\Rightarrow L=U=V=X,$$

$$\epsilon^{1KJ}\epsilon_{1KJ}\epsilon_{1KJ}\epsilon_{1KJ}=2\,\epsilon^{123}\epsilon_{123}\epsilon_{123}\epsilon_{123}=-2, \tag{77}$$

allow us to rewrite the corresponding tensor in the simpler form

$$R_{213a_1a_2a_3a_4}=\sum_{v\in\mathrm{Box}}\frac{1}{2\,Q^2}\left(\frac{2}{3!}\frac{8}{E(v)}\right)^2 \times\sum_{v(\Delta)=v(\Delta')=v}-\frac{\mu^4}{3!l_P^7}\det(s')[\det(s)]^3 \times s_L^a(s^{-1})^L_{a_1}(s^{-1})^L_{a_2}(s^{-1})^L_{a_3} \times\langle W,\underline{\vec{E}},\underline{\vec{B}}|w_{ia}w_{ia_4}|W,\underline{\vec{E}},\underline{\vec{B}}\rangle. \tag{78}$$

This shows explicitly the symmetry in the indices a_1, a_2, a_3 and leads to

$$R_{213a_1a_2a_3a_4}=\frac{\mu^4}{Q^2l_P^7}l_P^{10}\frac{l_P^3}{\mathcal{L}^2}\left(\frac{l_P}{\mathcal{L}}\right)^{2\Upsilon}\kappa_4(\delta_{a_1a_2}\delta_{a_3a_4}+\delta_{a_1a_3}\delta_{a_2a_4} +\delta_{a_1a_4}\delta_{a_2a_3}), \tag{79}$$

where we have written the most general four-index tensor at our disposal, which is completely symmetric in three indices. This implies the complete symmetry in the four indices. The correction (79) leads to cubic non-linear terms in the equation of motion.

Now we continue with the correction arising from Eq. (66), which reduces to

$$\mathrm{H}^{B}_{22}=\sum_{v\in\mathrm{Box}}\frac{1}{2\,Q^2}\left(\frac{2}{3!}\frac{8}{E(v)}\right)^2 l_P^3 \times\sum_{v(\Delta)=v(\Delta')=v}\frac{4}{(3!)^2l_P^7}\epsilon^{JKL}\epsilon^{MNP}s_L^a s_P'^d\langle W,\underline{\vec{E}},\underline{\vec{B}}| \times w_{ia}w_{id}(s_K^u s_J^y x_J\underline{F}_{uy})(s_N'^r s_M'^s x'_M\underline{F}_{rs})|W,\underline{\vec{E}},\underline{\vec{B}}\rangle, \tag{80}$$

because the terms quadratic in $\underline{F}$ are symmetric in J,K and M,N respectively.

From the above we read

$$R_{22}{}^{a_1a_2rr_1}=\sum_{v\in \mathrm{Box}}\frac{1}{2\,Q^2}\left(\frac{2}{3!}\frac{8}{E(v)}\right)^2 \times\sum_{v(\Delta)=v(\Delta')=v}\frac{4\,\mu^2}{(3!)^2 l_P^7}\det(s)\det(s') \times(-\epsilon_{YMN}\epsilon^{PMN}s'^{d}_{P}s'^{a_2}_{M})(\epsilon_{XJK}\epsilon^{LJK}s^{a}_{L}s^{a_1}_{J}) \times(s^{-1})^{Xr_1}(s^{-1})^{Yr}\langle W,\underline{\vec{E}},\underline{\vec{B}}|w_{ia}w_{id}|W,\underline{\vec{E}},\underline{\vec{B}}\rangle. \tag{81}$$

Here the internal symmetry properties are rather obscure. Nevertheless, the symmetry properties induced by the classical magnetic field factors imply that the above tensor must be proportional to the most general tensor with four indices which is symmetric in $a_1\ a_2$. Then we have

$$R_{22}{}^{a_1a_2rr_1}=\frac{\mu^2}{Q^2 l_P^7}l_P^8\frac{l_P^3}{\mathcal{L}^2}\left(\frac{l_P}{\mathcal{L}}\right)^{2\Upsilon}[\kappa_9\delta^{a_1a_2}\delta^{rr_1}+\kappa_{10}(\delta^{a_1r}\delta^{a_2r_1} +\delta^{a_2r}\delta^{a_1r_1})]. \tag{82}$$

This contribution is of the same kind as the one given by $R_{211}{}^{a_1a_2rr_1}$.

Finally we are left with

$$\mathrm{H}^{B}_{23}=\sum_{\mathrm{Box}(\vec{x})}\underline{B}_{r_1}(\vec{x})\underline{B}_{r}(\vec{x})\sum_{v\in\mathrm{Box}(\vec{x})}\frac{1}{2\,Q^2}\left(\frac{2}{3!}\frac{8}{E(v)}\right)^2 l_P^3 \times\sum_{v(\Delta)=v(\Delta')=v}\frac{\mu^2}{4\,l_P^7} \times\epsilon^{JKL}\epsilon^{MNP}s^{u}_{K}s^{v}_{J}s'^{x}_{N}s'^{y}_{M}(s^{d}_{L}s'^{a}_{P}s'^{b}_{P}s'^{c}_{P})\epsilon_{uv}{}^{r_1}\epsilon_{xy}{}^{z} \times\langle W,\underline{\vec{E}},\underline{\vec{B}}|\{w_{id},w_{iabc}\}|W,\underline{\vec{E}},\underline{\vec{B}}\rangle, \tag{83}$$

which leads to

$$R_{23}{}^{rr_1}=\sum_{v\in\mathrm{Box}(\vec{x})}\frac{1}{2\,Q^2}\left(\frac{2}{3!}\frac{8}{E(v)}\right)^2\sum_{v(\Delta)=v(\Delta')=v}-\frac{\mu^2}{l_P^7} \times\det(s')\det(s)(s^{-1})^{rP}(s'^{a}_{P}s'^{b}_{P}s'^{c}_{P}) \times\langle W,\underline{\vec{E}},\underline{\vec{B}}|\{w_i^{r_1},w_{iabc}\}|W,\underline{\vec{E}},\underline{\vec{B}}\rangle. \tag{84}$$

Taking the symmetric part, we have

$$R_{23}{}^{rr_1}=\kappa_{11}\frac{\mu^2}{Q^2 l_P^7}l_P^8\frac{l_P^3}{\mathcal{L}^4}\left(\frac{l_P}{\mathcal{L}}\right)^{4\Upsilon}\delta^{r\,r_1}. \tag{85}$$

Adding all previous contributions, we obtain the magnetic sector of the effective Hamiltonian, up to order l_P^2,

$$\mathrm{H}^{B}=\frac{1}{Q^2}\int d^3x\left\{\left[1+\theta_7\left(\frac{l_P}{\mathcal{L}}\right)^{2+2\Upsilon}\right]\frac{1}{2}\underline{\vec{B}}^2+l_P^2(\theta_2\underline{B}^a\partial_a\partial_b\underline{B}^b +\theta_3\underline{B}^a\nabla^2\underline{B}_a)+\theta_8 l_P\underline{\vec{B}}\cdot(\nabla\times\underline{\vec{B}})+\theta_4\mathcal{L}^2 l_P^2\left(\frac{\mathcal{L}}{l_P}\right)^{2\Upsilon}(\underline{\vec{B}}^2)^2 +\ldots\right\}. \tag{86}$$

The numbers θ_i are linear combinations of the corresponding κ_j appearing in the tensors R. The correspondences are

$$\kappa_7,\kappa_{10}\to\theta_2,\quad \kappa_6,\kappa_9\to\theta_3,\quad \kappa_4\to\theta_4,\quad \kappa_{11}\to\theta_7,\quad \kappa_8\to\theta_8. \tag{87}$$

As it is pointed out after Eq. (22), the electric sector of the effective Hamiltonian can be obtained by changing $\vec{B}$ into $\vec{E}$ in the quadratic contribution of the corresponding magnetic sector. In this way the complete electromagnetic effective Hamiltonian becomes

$$\mathrm{H}^{EM}=\frac{1}{Q^2}\int d^3x\left\{\left[1+\theta_7\left(\frac{l_P}{\mathcal{L}}\right)^{2+2\Upsilon}\right]\frac{1}{2}(\underline{\vec{B}}^2+\underline{\vec{E}}^2) +\theta_3 l_P^2(\underline{B}^a\nabla^2\underline{B}_a+\underline{E}^a\nabla^2\underline{E}_a)+\theta_2 l_P^2\underline{E}^a\partial_a\partial_b\underline{E}^b +\theta_8 l_P[\underline{\vec{B}}\cdot(\nabla\times\underline{\vec{B}})+\underline{\vec{E}}\cdot(\nabla\times\underline{\vec{E}})] +\theta_4\mathcal{L}^2 l_P^2\left(\frac{\mathcal{L}}{l_P}\right)^{2\Upsilon}(\underline{\vec{B}}^2)^2+\ldots\right\}, \tag{88}$$

up to order l_P^2.

VI. MODIFIED MAXWELL EQUATIONS AND DISPERSION RELATIONS IN VACUUM

Since no confusion arises in the sequel, we eliminate the underline in all electromagnetic quantities. From the effective Hamiltonian (88) we obtain the equations of motion

$$A(\nabla\times\vec{B})-\frac{1}{c}\frac{\partial\vec{E}}{\partial t}+2l_P^2\theta_3\nabla^2(\nabla\times\vec{B})-2\theta_8 l_P\nabla^2\vec{B} +4\theta_4\mathcal{L}^2\left(\frac{\mathcal{L}}{l_P}\right)^{2\Upsilon}l_P^2\nabla\times(\vec{B}^2\vec{B})=0, \tag{89}$$

$$A(\nabla\times\vec{E})+\frac{1}{c}\frac{\partial\vec{B}}{\partial t}+2l_P^2\theta_3\nabla^2(\nabla\times\vec{E}) -2\theta_8 l_P\nabla^2\vec{E}=0, \tag{90}$$

where

$$A=1+\theta_7\left(\frac{l_P}{\mathcal{L}}\right)^{2+2\Upsilon}. \tag{91}$$

The above equations are supplemented by the condition $\nabla\cdot\vec{B}=0$, together with the constraint $\nabla\cdot\vec{E}=0$, appropriate for vacuum.

Next we calculate the dispersion relations arising from the modified Maxwell equations (89) and (90). Neglecting the non-linear part and introducing the plane wave ansatz

$$\vec{E}=\vec{E}_0 e^{i(\vec{k}\cdot\vec{x}-\omega t)},\quad \vec{B}=\vec{B}_0 e^{i(\vec{k}\cdot\vec{x}-\omega t)},\quad k=|\vec{k}|, \tag{92}$$

we get

$$\vec{E}_0\cdot\vec{k}=0,\quad \vec{B}_0\cdot\vec{k}=0, \tag{93}$$

$$(\vec{k}\times\vec{E}_0)[A-2\,\theta_3(l_P k)^2]-2\,i\,\theta_8 l_P k^2\vec{E}_0-\frac{\omega}{c}\vec{B}_0=0, \tag{94}$$

$$(\vec{k}\times\vec{B}_0)[A-2\,\theta_3(l_P k)^2]-2\,i\,\theta_8 l_P k^2\vec{B}_0+\frac{\omega}{c}\vec{E}_0=0, \tag{95}$$

which imply the following dispersion relations:

$$\omega=ck\left[1+\theta_7\left(\frac{l_P}{\mathcal{L}}\right)^{2+2\Upsilon}-2\,\theta_3(kl_P)^2\pm 2\,\theta_8(kl_P)\right]. \tag{96}$$

The $\pm$ signs correspond to the different polarizations of the photon. From the above we obtain the speed of the photon

$$\frac{v}{c}=\frac{1}{c}\frac{d\omega}{dk}\bigg|_{\mathcal{L}=1/k} =1\pm 4\,\theta_8(kl_P)-6\,\theta_3(kl_P)^2+\theta_7(kl_P)^{2+2\Upsilon}+\ldots. \tag{97}$$

The last expression gives v expanded to leading order in l_P where we have estimated $\mathcal{L}$ as $1/k$, which is its maximal value. Clearly Eq. (97) is valid only for momenta satisfying $(l_P k)\ll 1$.

To first order in (kl_P) we have only the helicity dependent correction found already by Gambini and Pullin [9]. As far as the Υ dependent terms we have either a quadratic ($\Upsilon=0$) or a quartic ($\Upsilon=1$) correction. The only possibility to have a first order helicity independent correction amounts to setting $\Upsilon=-1/2$ which corresponds to that of Ellis *et al.* [10]. However, we do not have an interpretation for such a value of Υ.

VII. EFFECT OF NONLINEAR TERMS

In this section we explore some implications of the non-linear term in the Maxwell equations induced by the quantum gravity corrections to order l_P^2.

Following Ref. [39] we study the propagation of waves in the presence of a constant magnetic field $\vec{B}_0$. To do this, let us write

$$\vec{B}=\vec{B}_0+\vec{b} \tag{98}$$

and consider only the contribution of the non-linear term in the effective Hamiltonian. Namely, let us set $\theta_3=\theta_7=\theta_8=0$. After linearizing in $\vec{b}$, the field equations reduce to

$$\nabla\times\vec{b}-\frac{1}{c}\frac{\partial\vec{E}}{\partial t}+4\,\overline{\theta}_4\mathcal{L}^2 l_P^2\nabla\times[\vec{B}_0^2\vec{b}+2(\vec{B}_0\cdot\vec{b})\vec{B}_0]=0,$$

$$\nabla\cdot\vec{E}=0,\quad \nabla\cdot\vec{b}=0,\quad \nabla\times\vec{E}+\frac{1}{c}\frac{\partial\vec{b}}{\partial t}=0, \tag{99}$$

with $\overline{\theta}_4=(\mathcal{L}/l_P)^{2\Upsilon}\theta_4$. Now we look for plane wave solutions of Eq. (99) with

$$\vec{E}=\vec{E}_0 e^{i(\vec{k}\cdot\vec{x}-\omega t)},\quad \vec{b}=\vec{b}_0 e^{i(\vec{k}\cdot\vec{x}-\omega t)}, \tag{100}$$

obtaining the conditions

$$\vec{E}_0\cdot\vec{k}=0,\quad \vec{b}_0\cdot\vec{k}=0,\quad (\vec{k}\times\vec{E}_0)-\frac{\omega}{c}\vec{b}_0=0, \tag{101}$$

$$(1+4\,\overline{\theta}_4\mathcal{L}^2 l_P^2\vec{B}_0^2)(\vec{k}\times\vec{b}_0) +\frac{\omega}{c}\vec{E}_0+8\,\overline{\theta}_4\mathcal{L}^2 l_P^2(\vec{b}_0\cdot\vec{B}_0)(\vec{k}\times\vec{B}_0)=0. \tag{102}$$

Substituting in Eq. (102) the expression for $\vec{b}_0$ obtained from the third equation in Eq. (101) we are left with

$$\left(\frac{\omega^2}{c^2}-k^2(1+4\,\overline{\theta}_4\mathcal{L}^2 l_P^2\vec{B}_0^2)\right)\vec{E}_0 =-8\,\overline{\theta}_4\mathcal{L}^2 l_P^2[(\vec{k}\times\vec{E}_0)\cdot\vec{B}_0](\vec{k}\times\vec{B}_0). \tag{103}$$

Since $\vec{B}_0$ and $\vec{k}$ determine a plane, it is natural to study separately the propagation of waves with polarization parallel and perpendicular to this plane. We will express the answer in terms of the refraction index

$$n=\frac{k\,c}{\omega}. \tag{104}$$

From Eq. (103) we obtain the following refraction indices ($\hbar=1=c$):

$$n_\parallel=1-2\,\overline{\theta}_4\frac{1}{\omega^2}l_P^2\vec{B}_0^2,$$

$$n_\perp=1-2\,\overline{\theta}_4\frac{1}{\omega^2}l_P^2\vec{B}_0^2(1+2\sin^2\phi), \tag{105}$$

for parallel and perpendicular photon polarization, respectively. Here ϕ is the angle between $\vec{k}$ and $\vec{B}_0$.

These results can be compared with similar effects in quantum electrodynamics [39]

$$n_{\parallel}=1+\theta_{\parallel}\vec{B}_0^2\,\sin^2\phi,$$

$$n_{\perp}=1+\theta_{\perp}\vec{B}_0^2\,\sin^2\phi. \tag{106}$$

where

$$\theta_{\parallel}=\frac{2e^4}{45m^4},\quad \theta_{\perp}=\frac{7e^4}{90m^4}. \tag{107}$$

As expected, quantum gravity induced effects are much smaller than purely quantum electrodynamics effects, but the former present different signatures. In particular, the indices arising from quantum gravity are frequency dependent and also $n_{\parallel}$ is independent of ϕ.

VIII. PHOTON TIME DELAY

Notice that our considerations assumed a coarse-grained flat spacetime rather than a Friedmann-Lamaître-Robertson-Walker (FLRW) model. The latter would seem more appropriate for GRB traveling cosmological distances. In particular, it would be interesting to study the redshift effects in the photon time delays induced by the energy dependent corrections to the velocity. We are able to estimate these effects as follows.

Using the flat FLRW metric let us calculate the present time delay of two photons emitted simultaneously with different momenta and hence different velocities. We set $r=r_1=L$, $t=t_1$ as the emission coordinates and $r=0$, $t=t_0$ as the detection coordinates in the comoving cosmological system $t,r,\theta(t)=0$. The definition of the velocity $V(t)=R(t)(dr/dt)$, with $dr/dt<0$, leads to

$$r_1=\int_0^{r_1}dr=\int_{t_1}^{t_0}\frac{V(t)}{R(t)}dt. \tag{108}$$

Here $R(t)$ is the scale factor in the FLRW metric. Notice that the above equation leads to the standard redshift result for photons moving with $V=c$. On the other hand, quantum gravity corrections predict

$$\frac{V}{c}=1+\theta(l_P k) \tag{109}$$

to leading order. We are interested in discussing the situation where two photons are emitted at r_1 with different velocities V_1 and $V_1+\delta V_1$, and arrive at $r=0$ at times t_0 and $t_0+\delta t_0$ respectively. Then we have

$$r_1=\int_{t_1}^{t_0}\frac{dt}{R(t)}V(t),$$

$$r_1=\int_{t_1}^{t_0+\delta t_0}\frac{dt}{R(t)}[V(t)+\delta V(t)]. \tag{110}$$

Subtracting the two expressions for the fixed coordinate r_1, we obtain

$$\delta t_0=-\frac{R_0}{V(0)}\int_{t_1}^{t_0}\frac{dt}{R(t)}\delta V(t). \tag{111}$$

In the flat FLRW universe we have

$$t=\frac{t_0}{(1+z)^{3/2}},\quad t_0=\frac{2}{3}\frac{1}{H_0},$$

$$dt=-\frac{1}{H_0}\frac{1}{(1+z)^{5/2}}dz,\quad \frac{R(t)}{R_0}=\frac{1}{1+z}, \tag{112}$$

where z is the redshift. This leads to

$$\delta t_0=\frac{1}{cH_0}\int_0^{z_1}\frac{dz}{(1+z)^{3/2}}\delta V(z), \tag{113}$$

where z_1 is the redshift of the source and $\delta V(z)$ is the difference between the velocities of the two photons, arising from the quantum gravity corrections (109). Using the zeroth-order relation for the redshift of the photon momentum,

$$k(z)=\frac{R_0}{R(z)}k_0, \tag{114}$$

we obtain

$$\delta V(z)=\theta\frac{R_0}{R(z)}(\delta k)_0 l_P=\theta(1+z)(\delta k)_0 l_P, \tag{115}$$

where $(\delta k)_0$ is the present day observed momentum difference between the two photons.

Substitution of the above equation in Eq. (113) yields

$$\delta t_0=\left(\frac{2\,\theta}{H_0}\right)(\delta k)_0 l_P[(1+z_1)^{1/2}-1]. \tag{116}$$

The above result differs from the corresponding one obtained in the second and third papers of Ref. [10].

IX. SUMMARY AND DISCUSSION

In this paper we have considered the propagation of photons in a semiclassical background provided by loop quantum gravity. An effective electromagnetic Hamiltonian, given by Eq. (88), was identified with the expectation value of the electromagnetic piece of the Hamiltonian constraint for the Einstein-Maxwell theory with respect to *a would-be semiclassical state*. The state used was assumed to approximate a classical flat metric, a classical flat gravitational connection and a generic classical electromagnetic field, at scales larger than the coarse-grained characteristic length $\mathcal{L}$, where $\mathcal{L}\gg l_P$ (the Planck length). To leading nontrivial order in l_P, photons of wavelength λ, where $\mathcal{L}<\lambda$, acquire Planck scale modified dispersion relations as compared to those in classical flat spacetime. This in turn yields the effective speed of light (97) which involves two types of corrections. One of them is just that of Gambini and Pullin [9] including the

helicity of the photon, whereas the other depends on the scale $\mathcal{L}$. Moreover, the latter type contains a parameter Υ that encodes the scaling of the gravitational connection under the semiclassical expectation value.

When estimating the coarse-grained characteristic length by $\mathcal{L}=1/k$, which is its maximal value, the following values of Υ are prominent: (i) $\Upsilon=0$ can be understood as that the connection cannot be probed below the coarse graining scale $\mathcal{L}$. The corresponding correction scales as $(kl_P)^2$. (ii) $\Upsilon=1$ may be interpreted as the analog of the coherent states analysis, where such states saturate the Heisenberg uncertainty relation inside a box of volume $\mathcal{L}^3$: $\Delta q \sim l_P/\mathcal{L}$, $\Delta A \sim l_P/\mathcal{L}^2$ and $\Delta q \Delta A \sim \kappa\hbar/\mathcal{L}^3$ [38]. Then the correction behaves like $(kl_P)^4$. (iii) Interestingly, a value $\Upsilon=-\frac{1}{2}$ leads to a helicity independent first order correction [i.e. (kl_P)] similar to that of Ellis *et al.* [10]. We do not have an interpretation of this case though.

A prime candidate for testing the effects which are linear in the energy would be the gamma ray bursts that travel cosmological distances and which might be detected with a time resolution beyond 10^{-5} s. This seems possible in future spatial experiments [8].

Moreover, new non-linear terms in the Maxwell equations appear. These terms are not present either in [9] or [10]. We have explored the significance of this contribution to the propagation of photons in a constant strong magnetic field. The corrections obtained in the corresponding refraction indices are much smaller than similar effects in quantum electrodynamics. Nevertheless, quantum gravity corrections have distinct signatures: a main difference is that the speed of photons with polarization parallel to the plane formed by the background magnetic field and the direction of the wave is isotropic.

Our results should be taken as first steps in the exploration of possible observable consequences of quantum gravity. We have given evidence that dispersion relations of the form (1) can have origin in the microstructure of spacetime. It is expected that the recently proposed coherent states for quantum gravity and gauge theories in [36] and/or statistical geometry [37], will come to terms systematically with the unknown numerical coefficients we have left undetermined in our calculation (see also [41]). Interestingly, a quantum field theory in effective spacetimes might possibly emerge along these lines. Yet another avenue in the context of canonical quantization of gravity and gauge theories has recently emerged to understand the semiclassical regime. It is aimed at establishing a relation between Fock space and $\mathcal{H}_{\rm aux}$ [42,43].

Further work remains to be done in the framework here developed. For instance, in the case of inflationary cosmology as well as in the study of the Hawking effect, use is made of scalar fields with nonstandard near Planckian frequency dispersion relations to model the effect of short distance physics on the quantum fields [40]. Indeed a systematic study of the modifications induced by quantum gravity along the lines we have developed could be performed to investigate whether the dispersion relations used in [40] can be accounted for.

Finally, we stress that the dispersion relations we have found, as well as those in [9,10,20,21,40] are Lorentz symmetry violating. This is not necessarily an issue as it has been extensively discussed previously [4,5]. In fact they may alleviate some long standing astrophysical and cosmological problems [44,45]. Remarkably, there has been considerable progress in setting bounds to Lorentz invariance violation [46–48] and to the values of some coefficients in the effective Maxwell equations [49].

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