Quantum gravity and spin-1/2 particle effective dynamics

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Quantum gravity phenomenology opens up the possibility of probing Planck scale physics. Thus, by exploiting the generic properties that a semiclassical state of the compound system fermions plus gravity should have, an effective dynamics of spin-1/2 particles is obtained within the framework of loop quantum gravity. Namely, at length scales much larger than Planck length $\ell_P \sim 10^{-33}$ cm and below the wavelength of the fermion, the spin-1/2 dynamics in flat spacetime includes Planck scale corrections. In particular we obtain modified dispersion relations *in vacuo* for fermions. These corrections yield a time of arrival delay of the spin-1/2 particles with respect to a light signal and, in the case of neutrinos, a novel flavor oscillation. To detect these effects the corresponding particles must be highly energetic and should travel long distances. Hence neutrino bursts accompanying gamma ray bursts or ultrahigh energy cosmic rays could be considered. Remarkably, future neutrino telescopes may be capable of testing such effects. This paper provides a detailed account of the calculations and elaborates on results previously reported in a Letter. These are further amended by introducing a real parameter Υ aimed at encoding our lack of knowledge of scaling properties of the gravitational degrees of freedom.

I. INTRODUCTION

It is commonly accepted that quantum gravity should hold at scales near the Planck length $\ell_P := \sqrt{G_{\text{Newton}} \hbar / c^3}$ $\sim 10^{-33}$ cm or, equivalently, Planck energy $E_P := \hbar c / \ell_P$ $\sim 10^{19}$ GeV. Accordingly, neither astrophysical observations nor ground experiments were considered in the past as a means to directly reveal any quantum gravity effect but only to test indirect consequences. In recent years however this attitude has changed on the basis of potentially testable phenomena probing quantum gravity scenarios in which scales combine to lie not far from experimental resolution. Prominent among such phenomena we find *in vacuo* dispersion relations for gamma ray astrophysics [1–6], laser-interferometric limits on distance fluctuations [7,8], neutrino oscillations [4,9], threshold shift in certain high energy physics processes [10–13], *CPT* violation [14] and clock-comparison experiments in atomic physics [15]. These are the prototypes of the emerging *quantum gravity phenomenology* [16–20].

The present work is aimed at elaborating on effective corrections to propagation *in vacuo* for spin-1/2 particles in the framework of loop quantum gravity as reported in [4]. This framework also has been used previously in studying light propagation [3,5].

In essence, the specific structure of spacetime for a given quantum gravity scenario can be probed by matter propagating and interacting there: what could be considered as flat spacetime macroscopically, might produce microscopic imprints of its detailed structure on the interaction of particles. In particular, dispersion relations of propagating matter could exhibit corrections due to such effects. For particles with energy $E \ll E_P$ and momentum $\vec{p}$ the following *in vacuo* modified dispersion relations were proposed [2]:

$$c^2 \vec{p}^2 = E^2 \left(1 + \xi \frac{E}{E_{QG}} + \mathcal{O}\left(\frac{E}{E_{QG}} \right)^2 \right), \tag{1}$$

where $E_{QG} \lesssim E_P$ and $\xi \sim 1$. In general, such corrections might behave as $(E/E_{QG})^{\Upsilon+1}$, where $\Upsilon \geq 0$, namely not necessarily an integer. This possibility has been considered for photons [5,21] and spin zero particles [21]. In this work we show it applies also to spin-1/2 particles.

According to Eq. (1) the corresponding particle's speed yields a retardation time

$$\Delta t \approx \xi \frac{E}{E_{QG}} \frac{L}{c}, \tag{2}$$

with respect to a speed c signal after traveling a distance L. Interestingly, for gamma ray bursts (GRB's) with E ~ 0.20 MeV, $L \sim 10^{10}$ ly and setting $E_{QG} \sim E_P$, the naive

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estimation Eq. (2) gives $\Delta t \sim 0.01$ ms, barely two orders of magnitude below the sensitivity δt for current observations of GRB's [22,23]. Moreover, it is expected to improve this sensitivity in the foreseeable future [24]. Using an expression analogous to Eq. (2) for the delay of two photons detected with an energy difference ΔE, the observational bound $E_{QG}/\xi \geq 4 \times 10^{16}$ GeV was established in Ref. [25] by identifying events having $\Delta E = 1$ TeV arriving to earth within the time resolution of the measurement $\Delta t = 280$ s, from the active galaxy Markarian 421.

Now, accompanying GRB's there seems to be neutrino bursts (NB) in the range $10^5 - 10^{10}$ GeV according to the so-called fireball model [26,27]. If detected they could provide an excellent means to test quantum gravity effects of the type given by Eq. (2) above. Experiments like Neutrino Burst Experiment (NuBE) might detect ≈ 20 events per year of ultrahigh energy neutrinos ($E > $TeV) coinciding with GRB's [28]. Among other experiments aimed at studying ultrahigh energy cosmic rays including neutrinos we find the OWL-Airwatch project which could detect $\sim 3 \times 10^3 - 10^5$ events ($E > 10^{20}$ eV) per year [29,30]. This experiment also can look for time correlations between high energy neutrinos and GRB's. Complementary to the GRB effect, there is the possibility of looking for neutrino oscillation effect as induced by quantum gravity [4]. An analysis similar to that performed in the case of atmospheric neutrino oscillations [9,31], can help to set bounds on the parameters entering the description as we will suggest below.

In summary, astrophysical observations of photons, neutrinos or cosmic rays could make it possible to test quantum gravity effects or at least to restrict quantum gravity theories.

Indeed, an alternative approach to quantum gravity is based on string theory [32]. On such a basis modified dispersion relations of the type (1) have been regained [33]. The main difference in the case of photons is the absence of helicity dependence that is present in the loop quantum gravity approach. As for the case of spin-1/2 particles both approaches seem to agree in its helicity independence to leading order of the corresponding effect [4,34,35].

Further comments in order here are: delay time effects for traveling particles have also been considered on a different basis, for example an open system approach [36]. Also, perturbative quantum gravity has been considered to obtain effective dispersion relations [37].

Noticeably, the effect considered thus far involves a Lorentz symmetry violation [3,4,34,33], which seems to be in agreement with some astrophysical and cosmological scenarios [11,13,38,39]. In this way, these studies naturally overlap with the systematic approach developed by Colladay and Kostelecky [40] which provides the most general power counting renormalizable extension of the standard model that incorporates both Lorentz and *CPT* violations. This framework has been widely used to set experimental bounds upon the interactions that produce such violations and the observations performed so far cover a wide range of experimental settings [41]. Additional progress in establishing bounds to such symmetry violation can be found in Refs. [6,15,42,43].

Finally, it is interesting to emphasize that Planck scale corrections to either particle propagation or interactions need not necessarily imply violations of Lorentz covariance [44]. Recently the use of non-linear representations of the Lorentz group, leading to what has been called special relativity with two invariant scales, allows for a systematic construction of theories exhibiting these features [45].

The paper is organized as follows. In Sec. II we summarize the basics of loop quantum gravity in the case of fermions plus gravity. Section III is devoted to reviewing the regularization of Thiemann for the corresponding Hamiltonian constraint. Section IV explains general aspects of our approximation scheme whereas Sec. V provides the details of the calculations. In Sec. VI the effective Hamiltonian for noninteracting spin-1/2 particles is obtained. In particular, *in vacuo* dispersion relations are given. Section VII contains some preliminary estimates of the parameter Υ. To conclude, a summary and discussion of our results is presented in Sec. VIII.

II. LOOP QUANTUM GRAVITY

This section provides the basic ingredients of this approach, also known as quantum geometry [46], which we shall use in the sequel. Among the main results along this approach one finds: (i) well defined geometric operators possessing a discrete spectrum, thus evidencing discreteness of space [46]; (ii) a microscopic account for black hole entropy [47] and, more recently, hints on quantum avoidance of a would be classical cosmological singularity [48]. (For a review on these topics see for example Ref. [49].)

To begin with it is assumed that the spacetime manifold M has topology $\Sigma \times \mathbb{R}$, with Σ a Riemannian 3-manifold. Here a cotriad e_a^i is defined, with $a, b, c, \ldots$ being spatial tensor indices and $i, j, k, \ldots$ being $SU(2)$ indices. Thus the corresponding three-metric is given by $q_{ab} = e_a^i e_b^i$. In addition, a field K_a^i is defined by $K_{ab} = \text{sgn}[\det(e_c^j)]K_a^i e_b^i$, which is related to the extrinsic curvature K_{ab} of Σ. A canonical pair for the gravitational phase space is $(K_a^i, E_j^b/\kappa)$, where $E_i^a = \frac{1}{2} \epsilon^{abc} \epsilon_{ijk} e_b^j e_c^k$ and κ is Newton's constant. It turns out that such a canonical pair yields a complicated form for the Hamiltonian constraint of general relativity. A convenient canonical pair, making this constraint polynomial, was introduced by Ashtekar [50]. Nevertheless, two severe difficulties to proceed with the quantization remained: (i) the implementation of a diffeomorphism covariant regularization for the density-weight two Hamiltonian constraint hereby obtained and (ii) the extension to non-compact groups of the diffeomorphism covariant techniques already developed for gauge theories with compact groups [51]. In fact, the Ashtekar variables $({}^{\mathbb{C}}A_a^i = \Gamma_a^i - iK_a^i, \; iE_i^a/\kappa)$ [50], with Γ_a^i being the torsion free connection compatible with e_a^i, are complex valued. Namely the gauge group is $SL(2, \mathbb{C})$, which is noncompact.

Some proposals to come to terms with difficulty (ii) were: to consider real connection variables [52], to implement a Wick transform [53] and to define tractable reality constraints [54]. All of these left open (i). Thiemann subse-

quently proposed to solve (i) and (ii) by incorporating real connection variables while keeping the density weight one character of the Hamiltonian constraint. He further provided a quantum version of the theory in the pure gravity case, as well as in those cases including the coupling of matter to gravity [55]. His approach is next reviewed, since we rely upon it for our analysis of the fermionic case.

As for the fermionic sector a convenient canonical choice is $(\xi, \pi = i\xi^*)$. Here $\xi = (\det q)^{1/4}\eta$ is a half density and η is a Grassmann $SU(2)$ spinor. Half densities are convenient because they do not lead to cumbersome reality conditions at the quantum level and, furthermore, they do not require a complex gravitational connection. However, problems with diffeomorphism covariance can emerge. For this reason a further canonical transformation in the fermionic sector is necessary in order to dedensitize the Grassmann fields through: $\theta(x) := \int_\Sigma d^3y \sqrt{\delta(x,y)}\xi(y)$ and $\xi(x) := \Sigma_{y \in \Sigma} \sqrt{\delta(x,y)}\theta(y)$.

The pieces of the gravity-spin-1/2 system Hamiltonian constraint read

$$H_{\text{Einstein}}[N] = \int_\Sigma d^3x N \frac{1}{\kappa \sqrt{\det q}} \text{tr}((2[K_a, K_b] - F_{ab})[E_a, E_b]),$$

$$H_{\text{spin-1/2}}[N] = \int_\Sigma d^3x N E_i^a \frac{1}{2\sqrt{\det(q)}} \left(i\pi^T \tau_i \mathcal{D}_a \xi + \mathcal{D}_a(\pi^T \tau_i \xi) \right.$$
$$+ \frac{i}{2} K_a{}^i \pi^T \xi + \text{c.c.} \left.\right) + \frac{m}{2\hbar}(\xi^T(i\sigma^2)\xi$$
$$+ \pi^T(i\sigma^2)\pi). \tag{3}$$

Here $\vec{\tau} = -(i/2)\vec{\sigma}$, where $\vec{\sigma} = \{\sigma^i\}$ are the standard Pauli matrices and we have included an explicit mass term. Only one chirality fermion is used $\eta = \frac{1}{2}(1 + \gamma_5)\Psi$, which is equivalent to have a Dirac spinor $\Psi^T = (\eta^T, \mu^T)$ satisfying the Majorana condition. The classical configuration space is then $\mathcal{A}/\mathcal{G}$ of connections modulo gauge transformations, together with that of the fermionic field.

The quantum arena is given as follows [51]. As in any quantum field theory, because of the infinite number of degrees of freedom, an enlargement of the classical configuration space is required. This is far from trivial since the measures defining the scalar product, which are required to provide a Hilbert space, get concentrated on distributional fields and hence lie outside the classical configuration space. The key idea to build up such an enlargement is to make Wilson loop variables (traces of parallel transport operators) well defined. The resulting space $\overline{\mathcal{A}/\mathcal{G}}$ can be thought of as the limit of configuration spaces of lattice gauge theories for all possible *floating* (i.e. not necessarily rectangular) lattices. Hence, geometric structures on lattice configuration space are used to implement a geometric structure on $\overline{\mathcal{A}/\mathcal{G}}$. This enables us to define a background independent calculus on it which, in turn, leads to the construction of the relevant measures, the Hilbert space and the regulated operators.

It turns out that the Hilbert space of gauge invariant functions of gravitational and spinor fields is given by [55]

$$\mathcal{H}_{\text{inv}} = L_2([\bar{\mathcal{A}}_{SU(2)} \otimes \bar{S}]/\bar{\mathcal{G}}, d\mu_{AL}(SU(2)) \otimes d\mu_F). \tag{4}$$

Here $\bar{\mathcal{G}}$ denotes the action of $SU(2)$ on all fields at every point of Σ. $\bar{S}$ denotes the infinite product measurable space $\otimes_x S_x$, S_x being the Grassmann space at the point x. $d\mu_F$ is the corresponding measure on $\bar{S}$. An orthonormal basis using μ_F can be built up. Let the spinor labels be ordered $A = 1,2$. Take $\vec{v}$ a finite set of mutually different points. For each $v \in \vec{v}$ denote by I_v the array $[A_1(v) < \cdots < A_k(v)]$, with $0 \le k \le 2$ and $A_j(v) = 1,2$ for each $1 \le j \le k$. Also set $|I_v| = k$. Thus fermionic vertex functions $F_{\vec{v},\vec{I}}$ can be defined as $F_{\vec{v},\vec{I}} := \Pi_{v \in \vec{v}} F_{v,I}$, $F_{v,I_v} := \Pi_{j=1}^k \theta_{A_j(v)}(v)$ yielding the desired orthonormal basis.

Gauge invariant objects are spin network states defined as follows. Take γ as a piecewise analytic graph with edges e and vertices v which are not necessarily closed. Every edge can be read as outcoming from a vertex by suitably subdividing edges into halves. Given a connection A_a we can compute the holonomy along the edge e: $h_e(A)$. To each e we also associate a spin j_e corresponding to an irreducible representation of $SU(2)$. In addition, we attach to each vertex an integer label n_v, $0 \le n_v \le 2$, and a projector p_v. For a given n_v, one considers the vector subspace of Q_v [the vector space spanned by holomorphic functions of $\theta_A(v)$, spanned by those vectors $F_{I,v}$ such that $|I| = n(v)$]. The projector p_v is a certain $SU(2)$ invariant matrix which projects onto one of the linearly independent trivial representations contained in the decomposition into irreducible representations of the tensor product consisting of (i) the n_v-fold tensor product of fundamental representations of $SU(2)$ associated with the vector subspace of Q_v spanned by the $F_{I,v}, |I| = n(v)$ and (ii) the tensor product of irreducible representations j_e where e runs through the subset of edges of γ starting at v. The resulting gauge invariant states are denoted by

$$T_{\gamma,[\vec{j},\vec{n},\vec{p}]} \tag{5}$$

which extend the definition of the matter free case. They form a basis of $\mathcal{H}_{\text{inv}}$. Although not orthonormal it can be transformed into one that is by suitably decomposing the fermionic dependence into an orthonormal basis of the Q_v.

To extract physical information we will further need a state describing a flat continuous space Σ at scales much larger than the Planck length, but not necessarily so at distances comparable to the Planck length itself. States of this kind were introduced under the name of weave [56] for pure gravity. Flat weave states $|W\rangle$, having a characteristic length $\mathcal{L}$, were first constructed by considering collections of Planck scale circles randomly oriented. If one probes distances $d \gg \mathcal{L}$ the continuous flat classical geometry is regained, while for distances $d \ll \mathcal{L}$ the quantum loop structure of space is manifest. In other words, one expects a behavior of the type $\langle W|\hat{q}_{ab}|W\rangle = \delta_{ab} + O(\ell_P/\mathcal{L})$. It was soon realized that such states could not yield a nontrivial volume due to the lack of self-intersections [57]. Couples of circles, intersecting at a point, were also considered as specific models of weaves to overcome this defect [58]. With the recent ad-

vances on the kinematical Hilbert space $\mathcal{H}_{\text{aux}}$ it became clear that all proposed weaves were afflicted by two undesirable features. First, they are defined to be peaked at a specific (flat or curved) metric, but not at a connection. This is in contrast with standard semiclassical states in terms of coherent states, for example. Second, the known weave states do not belong either to $\mathcal{H}_{\text{aux}}$ or to a dense subspace of it [59]. It may be possible to come to terms with such difficulties by defining coherent states for diffeomorphism covariant gauge theories [60] or by implementing a genuine statistical geometry [61], for instance. Both approaches have recently achieved substantial progress.

Nonetheless, in order to extract physics, there is the alternative possibility of using just the main features that semiclassical states should have. Namely, peakedness on both geometry and connection together with the property that they yield well defined expectation values of physical operators. An advantage of this alternative is that one may elucidate some physical consequences before the full fledged semiclassical analysis is settled down. Indeed, such alternative may be considered as complementary, in the sense of hinting at possible features of semiclassical states which could be further elaborated. After its completion, a rigorous semiclassical treatment should tell us whether the results arising from this alternative turn out to hold or not. The weakness of the treatment resides on its generality, since no detailed features of the would be semiclassical states are used—as opposed, say, to the original weave states—and hence a set of numerical coefficients cannot be calculated. Evaluating them will be the task of the rigorous semiclassical treatment.

On top of the purely gravitational semiclassical states, a generalization is required to include matter fields. For our analysis it will just suffice to exploit the same aspects of peakedness and well defined expectation values, extended to include the case of the fermion field. The semiclassical states here considered will describe flat space and a smooth spin-1/2 field living in it. Such a state is denoted by $|W,\xi\rangle$ and has a characteristic length $\mathcal{L}$. Since no detailed information is used on how the semiclassical state is constructed in terms of, say, a graph, as opposed to weave states, the present approach yields results relying only on the following assumptions: (i) peakedness of the states, (ii) well defined expectation values and (iii) existence of a coarse-grained expansion involving ratios of the relevant scales of the problem: the Planck length ℓ_P, the characteristic length $\mathcal{L}$ and the fermion wavelength λ. States satisfying such requirements are referred to as *would be semiclassical states* in the sequel.

III. THE REGULARIZATION

We will use the regularization method of Thiemann [55] in the following. We focus on the first term of $H_{\text{spin-1/2}}[N]$, Eq. (3). The other terms can be dealt with similarly or, in the case of K_a^i dependence, along the lines of the purely gravitational sector $H_{\text{Einstein}}[N]$ as shown in the sequel. The identity $(1/\kappa)\{A_a^i,V\}=2\,\text{sgn}(\det e_b^j)e_a^i$ yields

$$H_{\text{spin1/2}}^{(1)}[N]=-\frac{i}{2\kappa^2}\int d^3xN(x)$$

$$\times\left(\epsilon^{ijk}\epsilon^{abc}4\frac{\{A_a^i(x),V(x,\delta)\}\{A_b^j(x),V(x,\delta)\}}{\sqrt{q(x)}}\right)$$

$$\times[\pi^T\tau_k\mathcal{D}_c\xi-\text{c.c.}] \tag{6}$$

where δ is small and $V(x,\delta)=\int d^3y\chi_\delta(x,y)\sqrt{q(x)}$ is the volume with respect to the metric q_{ab} of a box centered at x. $\chi_\delta(x,y)=\Pi_{a=1}^3\theta(\delta/2-|x^a-y^a|)$ is the characteristic function of the box. Next one changes from ξ to θ variables and introduces

$$\sum_x f_i^a(x)(\tau_i\mathcal{D}_a\theta)_A(x)\bar\theta_A(x), \tag{7}$$

which regularizes Eq. (6) as we show now. Here f_i^a is a real valued $\text{ad}_{SU(2)}$ vector field. By defining $(\partial_a\theta_A)(x):=\lim_{\epsilon\to0}\partial_{x^a}\theta_A^\epsilon(x)$, and recalling

$$\theta_A(x)=\int d^3y\sqrt{\delta(x,y)}\xi_A(y)=\lim_{\epsilon\to0}\theta_A^\epsilon \tag{8}$$

with $\theta_A^\epsilon=\int d^3y[\chi_\epsilon(x,y)/\sqrt{\epsilon^3}]\xi_A$, one gets

$$(\partial_{x^a}\theta_A)(x)=\int d^3y\frac{\chi_\epsilon(x,y)}{\sqrt{\epsilon^3}}(\partial_{y^a}\theta_A)(y). \tag{9}$$

Now let us divide Σ into boxes of Lebesgue measure ϵ^3 centered at x_n and let Eq. (7) be the limit $\epsilon\to0$ of

$$\sum_n f_i^a(x_n)(\tau_i\mathcal{D}_a\theta^\epsilon)_A(x_n)\bar\theta_A^\epsilon(x_n), \tag{10}$$

which upon expressing θ^ϵ in terms of ξ becomes

$$\int d^3x\int d^3y\left[\sum_n f_i^a(x_n)\frac{\chi_\epsilon(x,x_n)\chi_\epsilon(y,x_n)}{\epsilon^3}\right]$$

$$\times[\tau_i\partial_a\xi(x)+(\omega_a(x_n)\xi(x))]^A\bar\xi_A. \tag{11}$$

Notice that in spite of the density weight of ξ no Christoffel connection is needed since it will drop after the H.c. part is considered. In the limit $\epsilon\to0$, $\chi_\epsilon(x,x_n)/\epsilon^3\to\delta(x,x_n)$ and

$\chi_\epsilon(y, x_n) \rightarrow \delta_{y, x_n}$, so that after x integration and sum over x_n the result is Eq. (6) where f_i^a has to be properly identified.

To proceed with the quantization one keeps fermionic momenta to the right and replaces $\hat{\dot{\theta}}^A$ by $\hbar(\partial/\partial\theta_A)$ in Eq. (7). By further multiplying and dividing by δ^3 the factor $\delta^3\sqrt{q(x)}$ in the denominator is changed to $V(x, \delta)$. This can be absorbed into the Poisson brackets already appearing in Eq. (6). When Poisson brackets are replaced by commutators times $1/i\hbar$ one obtains an operator whose action on a cylindrical function f_γ associated to a graph γ and containing fermionic insertions θ_A at the vertices $v \in V(\gamma)$ gives

$$\hat{H}_{\text{spin-1/2}}^{(1)}[N]f_\gamma = -\frac{\hbar}{2\ell_P^4} \sum_{v \in V(\gamma)} \sum_x N(x)\epsilon^{ijk}\epsilon^{abc}\delta^3$$
$$\times [A_a^i(x), \sqrt{\hat{V}(x,\delta)}][A_b^j(x), \sqrt{\hat{V}(x,\delta)}]$$
$$\times \left[(\tau_k\mathcal{D}_c\theta)_A(v)\frac{\partial}{\partial\theta_A(v)}\delta_{x,v} + \text{H.c.} \right] f_\gamma.$$
(12)

To complete the regularization an adapted triangulation to γ of Σ is introduced. Using $h_s(0,\delta)\theta(s(\delta)) - \theta(s(0)) = \delta\dot{s}^a(0)(\mathcal{D}_a)\theta(s(0))$ and replacing the connection operators by holonomy operators allows one to absorb the δ^3 factor. Also one replaces $\hat{V}(v,\delta) \rightarrow \hat{V}_v$. Because of the presence of $\delta_{x,v}$, Σ_x becomes concentrated in the vertices $\Sigma_{v(\Delta)=v}$. Hence we obtain

$$\hat{H}_{\text{spin-1/2}}^{(1)}[N] = -\frac{m_P}{2\ell_P^3} \sum_{v \in V(\gamma)} N_v \sum_{v(\Delta)=v} \epsilon^{ijk}\epsilon^{IJK}$$
$$\times \text{tr}(\tau_i h_{s_I(\Delta)}[h_{s_I(\Delta)}^{-1}, \sqrt{\hat{V}_v}])$$
$$\times \text{tr}(\tau_j h_{s_J(\Delta)}[h_{s_J(\Delta)}^{-1}, \sqrt{\hat{V}_v}])$$
$$\times \left[[(\tau_k h_{s_K(\delta)}\theta)(s_K(\Delta)(\delta)) - \theta(v)]_A \right.$$
$$\left. \times \frac{\partial}{\partial\theta_A(v)} + \text{H.c.} \right]$$
(13)

which is the operator we shall use below.

At this stage we introduce the notation

$$\hat{w}_{iI\Delta} = \text{tr}(\tau_i h_{s_I(\Delta)}[h_{s_I(\Delta)}^{-1}, \sqrt{V_v}]).$$
(14)

IV. GENERAL STRUCTURE OF THE CALCULATION

The effective Dirac Hamiltonian H^F is obtained by considering the expectation value of the fermionic sector of the quantum Hamiltonian, with respect to $|W, \xi\rangle$. Inside this expectation value, operators are expanded around all relevant vertices of the triangulation in powers of the segments $s_L^a(\Delta)$, having lengths of the order ℓ_P. In this way, a systematic approximation is given involving the scales $\ell_P \ll \mathcal{L}$

$< \lambda$. Here λ is the de Broglie wavelength of the neutrino. Corrections arise at this level. Let us start by taking the would be semiclassical state (WBSC) expectation value of Thiemann's regularized Hamiltonian

$$\text{H}^F = \langle W, \xi | \hat{H}_{\text{spin-1/2}}^{(1)}[N] | W, \xi \rangle$$
$$= +\frac{\hbar}{4\ell_P^4} \sum_{v \in V(\gamma)} N(v)\frac{8}{E(v)}\epsilon^{ijk}\epsilon^{IJK}$$
$$\times \left\{ \langle W, \xi | \left[\frac{\partial}{\partial\theta(v)}\tau_k h_{s_K(\Delta)}\hat{\theta}(v + s_K(\Delta)) \right] \right.$$
$$\times \hat{w}_{iI\Delta}(v)\hat{w}_{jJ\Delta}(v) | W, \xi \rangle - \langle W, \xi | \left[\frac{\partial}{\partial\theta(v)}\tau_k\hat{\theta}(v) \right]$$
$$\left. \times \hat{w}_{iI\Delta}(v)\hat{w}_{jJ\Delta}(v) | W, \xi \rangle - \text{c.c.} \right\}$$
(15)

where $\partial/\partial\theta(v)$ is the fermionic momentum operator.

Our strategy to use the proposed regularization as a computational tool will be to expand the expression (15) around each vertex of the triangulation in powers of the vectors $s_I(\Delta)$. To proceed with the approximation we think of space as made up of boxes of volume $\mathcal{L}^3$, whose center is denoted by $\vec{x}$. Each box contains a large number of vertices of the graph associated with the WBSC, but is considered infinitesimal in the scale where the space can be regarded as continuous, so that we take $\mathcal{L}^3 \approx d^3x$.

Next we discuss how to estimate the average contribution in each box. To begin with we consider that the volume of the box is

$$d^3\vec{x} \approx \sum_{v \in \text{Box}(\vec{x})} \ell_P^3\left(\frac{8}{E(v)} \right)$$
(16)

and define the average $\langle T(\vec{x}) \rangle$ of the quantity $T(v)$ defined in each of the vertices contained in the box, as

$$d^3\vec{x}\langle T(\vec{x}) \rangle = \sum_{v \in \text{Box}(\vec{x})} \ell_P^3\left(\frac{8}{E(v)} \right) T(v).$$
(17)

The WBSC expectation value of the Hamiltonian (15) is of the type

$$\sum_{v \in V(\gamma)} \frac{8}{E(v)}\langle W, \xi | \hat{F}(v)\hat{G}(v) | W, \xi \rangle$$
$$= \sum_{\text{Box}(\vec{x})} F(\vec{x}) \sum_{v \in \text{Box}(\vec{x})} \ell_P^3\frac{8}{E(v)}$$
$$\times \langle W, \xi | \frac{1}{\ell_P^3}\hat{G}(v) | W, \xi \rangle$$
$$= \int_\Sigma F(\vec{x})\left\langle \frac{1}{\ell_P^3}G(\vec{x}) \right\rangle d^3x.$$
(18)

Here, $\hat{F}(v)$ is a fermionic operator which produces the slowly varying function $F(\vec{x})$ within the box; i.e. $\mathcal{L} \ll \lambda$. Furthermore, $\langle (1/\ell_P^3) G(\vec{x}) \rangle$ is the box average of the rapidly varying WBSC expectation values of the gravitational operator $(1/\ell_P^3) \hat{G}(v)$ within the box, whose tensorial and Lie-algebra structure is determined by the tensors characterizing the continuum spacetime we are dealing with, i.e.

$$\overset{0}{e^{ia}}, \tau^k, \partial_b, \epsilon^{cde}, \epsilon^{klm},$$

where $q^{ab} = \overset{0}{e^{ia}} \overset{0}{e_i^{\ b}}$ is the corresponding classical 3-metric. The order of magnitude of these box averaged quantities is estimated according to some prescriptions to be specified in the sequel. The method does not provide exact numerical coefficients which can only be obtained from a detailed knowledge of the semiclassical state.

Since we assume the fermionic fields to be slowly-varying functions inside each box, we demand the following behavior of the fermionic operators inside the WBSC:

$$\langle W, \xi | \dots \hat{\theta}_B(v) \dots \frac{\partial}{\partial \theta^A(v)} \dots | W, \xi \rangle$$

$$= \Theta \left(\dots \xi_B(v) \dots \frac{i}{\hbar} \pi_A(v) \dots \right), \qquad (19)$$

where the normalization constant Θ is to be determined in such a way that the zeroth order approximation reproduces the corresponding classical kinetic energy term in the Hamiltonian. In this way we have

$$H^F = \langle W, \xi | \hat{H}_{\text{spin-1/2}}^{(1)}[N] | W, \xi \rangle$$

$$= + \frac{i\Theta}{4\ell_P^4} \sum_{\text{Box}(\vec{x})} \sum_{v \in \text{Box}(\vec{x})} N(v) \frac{8}{E(v)} \epsilon^{ijk} \epsilon^{IJK}$$

$$\times \left\{ \langle W, \xi | \left[\frac{\partial}{\partial \theta(v)} \tau_k h_{s_K(\Delta)} \theta(v + s_K(\Delta)) \right] \right.$$

$$\times \hat{w}_{iI\Delta}(v) \hat{w}_{jJ\Delta}(v) | W, \xi \rangle - \langle W, \xi | \left[\frac{\partial}{\partial \theta(v)} \tau_k \theta(v) \right]$$

$$\left. \times \hat{w}_{iI\Delta}(v) \hat{w}_{jJ\Delta}(v) | W, \xi \rangle - \text{c.c.} \right\}. \qquad (20)$$

In the above equation we have written only the fermions in its classical version and we have kept them inside the WBSC expectation value just in order to be able to write the above expression in a more compact way. The holonomy $h_{s_K(\Delta)}$ still contains the gravitational connection. In order to have a convenient bookkeeping of the terms involved, we write Eq. (20) as

$$H^F = + \frac{i\Theta}{4\ell_P^4} \sum_{\text{Box}(\vec{x})} \sum_{v \in \text{Box}(\vec{x})} N(v) \frac{8}{E(v)} \epsilon^{ijk} \epsilon^{IJK}$$

$$\times \left\{ \langle W, \xi | \left[\frac{\partial}{\partial \theta(v)} \tau_k \left((s_K^a \hat{\nabla}_a(v)) \right. \right. \right.$$

$$\left. \left. + \frac{1}{2} (s_K^a \hat{\nabla}_a(v))(s_K^b \hat{\nabla}_b(v)) + \cdots \right) \theta(v) \right]$$

$$\left. \times \hat{w}_{iI\Delta}(v) \hat{w}_{jJ\Delta}(v) | W, \xi \rangle - \text{c.c.} \right\} \qquad (21)$$

where the derivative $\hat{\nabla}_a$ includes the covariant derivative $\overset{0}{\nabla}_a(\vec{x})$, which is compatible with the classical metric that we are considering and a piece $\hat{A}_a^i(v)$ producing the quantum corrections associated with the beginning of the continuum at the scale $\mathcal{L}$. Upon taking WBSC expectation value we can make the following substitution:

$$\hat{\nabla}_a(v) \theta(v) \rightarrow (\overset{0}{\nabla}_a(\vec{x}) + \hat{A}_a^i(v) \tau_i) \xi(\vec{x}). \qquad (22)$$

From now on we restrict to a continuum flat metric, in such a way that

$$\overset{0}{\nabla}_a(\vec{x}) = \partial_a. \qquad (23)$$

The partial derivative does not change inside each box, which is consistent with the idea that each box is supposed to represent a point of the continuum. In this way,

$$H^F = + \frac{i\Theta}{4\ell_P^4} \sum_{\text{Box}(\vec{x})} \sum_{v \in \text{Box}(\vec{x})} N(v) \frac{8}{E(v)} \epsilon^{ijk} \epsilon^{IJK}$$

$$\times \left\{ \langle W, \xi | \left[\frac{\partial}{\partial \theta(v)} \tau_k \left(s_K^a \hat{\nabla}_a(v) + \frac{1}{2} s_K^a s_K^b \hat{\nabla}_a(v) \right. \right. \right.$$

$$\left. \left. \times \hat{\nabla}_b(v) + \cdots \right) \theta(v) \right]$$

$$\left. \times \hat{w}_{iI\Delta}(v) \hat{w}_{jJ\Delta}(v) | W, \xi \rangle - \text{c.c.} \right\}. \qquad (24)$$

Our problem now is to parametrize and to estimate the general spinor

$$Y_{(n)}^k(v) = \epsilon^{ijk} \epsilon^{IJK} \langle W, \xi | s_K^{a_1} s_K^{a_2} \dots s_K^{a_n}$$

$$\times \hat{\nabla}_{a_1}(v) \hat{\nabla}_{a_2}(v) \dots \hat{\nabla}_{a_n}(v) \theta(v)$$

$$\times \hat{w}_{iI\Delta}(v) \hat{w}_{jJ\Delta}(v) | W, \xi \rangle \qquad (25)$$

in a given box. Here n denotes the number of covariant derivatives in the expression. Using the above we can write

$$H^F = + \frac{i\Theta}{4\ell_P^4} \sum_{n=1,2,\ldots} \sum_{\text{Box}(x)} \sum_{v \in \text{Box}(x)} \left(N(v) \frac{8}{E(v)} \right)$$

$$\times \pi(v)\tau_k Y_{(n)}^k(v) - \text{c.c.} \quad (26)$$

Before calculating the corresponding box averaged contributions let us obtain the expressions for some of the first quantities $Y_{(n)}^k(v)$. A direct calculation shows

$$Y_{(1)}^k(v) = s_K^a(v)(X^{kK}(v)\partial_a\xi(\vec{x}) + X_a^{m\,kK}(v)\tau_m)\xi(\vec{x}),$$
$$(27)$$

$$Y_{(2)}^k(v) = s_K^a(v)s_K^b(v)X^{kK}(v)\partial_a\partial_b\xi(\vec{x})$$
$$+ s_K^a(v)s_K^b(v)X_a^{m\,kK}(v)\tau_m 2\partial_b\xi(\vec{x})$$
$$+ s_K^a(v)s_K^b(v)X_{ab}^{mn\,kK}(v)\tau_m\tau_n\xi(\vec{x}), \quad (28)$$

$$Y_{(3)}^k(v) = s_K^a s_K^b s_K^c(X^{kK}(v)\partial_a\partial_b\partial_c + 3X_a^{m\,kK}(v)\tau_m\partial_b\partial_c +)\xi(\vec{x})$$
$$+ s_K^a s_K^b s_K^c(3X_{ab}^{mn\,kK}(v)\tau_m\tau_n\partial_c$$
$$+ X_{abc}^{mnp\,kK}(v)\tau_m\tau_n\tau_p)\xi(\vec{x}),$$

where we have explicitly enforced the symmetry implied by the factors $s_K^a s_K^b s_K^c \ldots$ in Eq. (25). The gravitational quantities, which are rapidly varying inside the box are

$$X^{kK}(v) = \epsilon^{ijk}\epsilon^{IJK}\langle W,\xi|\hat{w}_{iI\Delta}(v)\hat{w}_{jJ\Delta}(v)|W,\xi\rangle$$

$$X_a^{m\,kK}(v) = \epsilon^{ijk}\epsilon^{IJK}\langle W,\xi|\hat{A}_a^m(v)\hat{w}_{iI\Delta}(v)\hat{w}_{jJ\Delta}(v)|W,\xi\rangle$$

$$X_{ab}^{mn\,kK}(v) = \epsilon^{ijk}\epsilon^{IJK}\langle W,\xi|\hat{A}_a^m(v)$$
$$\times \hat{A}_b^n(v)\hat{w}_{iI\Delta}(v)\hat{w}_{jJ\Delta}(v)|W,\xi\rangle$$

$$X_{abc}^{mnp\,kK}(v) = \epsilon^{ijk}\epsilon^{IJK}\langle W,\xi|\hat{A}_a^m(v)\hat{A}_b^m(v)\hat{A}_c^p(v)$$
$$\times \hat{w}_{iI\Delta}(v)\hat{w}_{jJ\Delta}(v)|W,\xi\rangle.$$

Next we write the corresponding general expressions

$$Y_{(n)}^k(v) = (s_K^{a_1}s_K^{a_2}\ldots s_K^{a_l})(s_K^{a_{l+1}}s_K^{a_{l+2}}\ldots s_K^{a_n})$$

$$\times \sum_{l=n,n-1,\ldots 1,0}\binom{n}{l}X_{a_{l+1}a_{l+2}\ldots a_n}^{m_{l+1}m_{l+2}\ldots m_n\,kK}(v)$$

$$\times \tau_{m_{l+1}}\tau_{m_{l+2}}\ldots\tau_{m_n}\partial_{a_1}\partial_{a_1}\ldots\partial_{a_l}\xi(\vec{x}) \quad (29)$$

with

$$X_{a_{l+1}a_{l+2}\ldots a_n}^{m_{l+1}m_{l+2}\ldots m_n\,kK}(v) = \epsilon^{ijk}\epsilon^{IJK}\langle W,\xi|\hat{A}_{a_{l+1}}^{m_{l+1}}(v)\hat{A}_{a_{l+2}}^{m_{l+2}}(v)$$

$$\times \ldots \hat{A}_{a_n}^{m_n}(v)$$

$$\times \hat{w}_{iI\Delta}(v)\hat{w}_{jJ\Delta}(v)|W,\xi\rangle. \quad (30)$$

In the above expressions the partition $(1,2,\ldots l)$ and $(l+1,l+2,\ldots n)$ is made in such a way that the indices of the

first set count the number of partial derivatives acting upon the fermion. The indices of the second set count the number of τ matrices involved which is equal to the number of additional connections, i.e. besides those contained in the combination $\hat{w}_{iI\Delta}(v)\hat{w}_{jJ\Delta}(v)$, appearing in Eq. (30).

The box-averaged expression for the Hamiltonian reduces then to

$$H^F = \frac{i\Theta}{4\ell_P^7}\sum_{n=1,2,\ldots}\int d^3\vec{x}\,\pi(\vec{x})\tau_k\langle Y_{(n)}^k(\vec{x})\rangle - \text{c.c.} \quad (31)$$

In more detail this is

$$H^F = + \frac{i\Theta}{4\ell_P^7}\sum_{n=1,2,\ldots}\int d^3\vec{x}\,\pi(\vec{x})\tau_k\Bigg\langle (s_K^{a_1}s_K^{a_2}\ldots s_K^{a_l})$$

$$\times (s_K^{a_{l+1}}s_K^{a_{l+2}}\ldots s_K^{a_n})$$

$$\times \sum_{l=n,n-1,\ldots 1,0}\binom{n}{l}X_{a_{l+1}a_{l+2}\ldots a_n}^{m_{l+1}m_{l+2}\ldots m_n\,kK}(v)\Bigg\rangle$$

$$\times \tau_{m_{l+1}}\tau_{m_{l+2}}\ldots\tau_{m_n}\partial_{a_1}\partial_{a_1}\ldots\partial_{a_l}\xi(\vec{x}). \quad (32)$$

Here it is convenient to define the auxiliary quantities

$$T^{a_1a_2\ldots a_l m_{l+1}m_{l+2}\ldots m_n\,k}$$

$$= \frac{\Theta}{4\ell_P^7}\langle (s_K^{a_1}s_K^{a_2}\ldots s_K^{a_l})(s_K^{a_{l+1}}s_K^{a_{l+2}}\ldots s_K^{a_n})$$

$$\times X_{a_{l+1}a_{l+2}\ldots a_n}^{m_{l+1}m_{l+2}\ldots m_n\,kK}(v)\rangle \quad (33)$$

in such a way that

$$H^F = \sum_{n=1,2,\ldots}\sum_{l=n,n-1,\ldots 1,0}\binom{n}{l}\int d^3\vec{x}\,i\,\pi(\vec{x})\tau_k$$

$$\times T^{a_1a_2\ldots a_l m_{l+1}m_{l+2}\ldots m_n}\tau_{m_{l+1}}\tau_{m_{l+2}}\ldots\tau_{m_n}$$

$$\times \partial_{a_1}\partial_{a_1}\ldots\partial_{a_l}\xi(\vec{x}) \quad (34)$$

$$H^F =: \sum_{n=1,2,\ldots}\sum_{l=n,n-1,\ldots 1,0}H_{nl}^F. \quad (35)$$

The box averages $\langle F(v)\rangle$ are subsequently estimated using the corresponding dimensions in terms of the available tensors in flat space

$$\overset{0}{e}^{ia} = \delta^{ia}, \quad \tau^k, \quad \partial_b, \quad \epsilon^{cde}, \quad \epsilon^{klm},$$

$$q^{ab} = \delta^{ab}, \quad \delta^{ij}, \quad \delta_b^a \quad (36)$$

in a manner described in the next section. In this way we are demanding covariance under rotations at the scale $\mathcal{L}$. As the final input in our prescription we impose that when averag-

ing inside each box, the order of magnitude of the corresponding expectation values of the gravitational operators are estimated according to

$$\langle W,\xi| \ldots, A_{ia}, \ldots, \sqrt{V_v}, \ldots |W,\xi\rangle$$

$$\approx \cdots \frac{1}{\mathcal{L}}\left(\frac{\ell_P}{\mathcal{L}}\right)^{\Upsilon 0} e_{ia}, \ldots, \ell_P^{3/2} \ldots, \quad (37)$$

respectively.

In previous work [4] we have set $\Upsilon = 0$ on the basis that the coarse graining approximation does not allow for the continuum connection to be probed below $1/\mathcal{L}$. On the other hand, by adopting say kinematical coherent states for representing semiclassical states, one would set $\Upsilon = 1$ for two reasons: first, in the limit $\hbar \to 0 (\ell_P \to 0)$ Eq. (37) yields just zero, in agreement with a flat connection, and, second, such an scaling would saturate the Heisenberg uncertainty relation. Nonetheless, physical semiclassical states may lead to a leading order contribution with $\Upsilon \neq 0,1$, thus we leave it open here. Now we have all the ingredients to perform the expansion in powers of s.

Let us consider now the contribution arising from the gravitational operators contained in $\hat{w}_{iI\Delta}(v)$ of Eq. (30). We choose the parametrization

$$\hat{w}_{iI\Delta}(v) = s_I{}^a w_{ia} + s_I{}^a s_I{}^b w_{iab} + \cdots \quad (38)$$

leading to

$$\hat{w}_{iI\Delta}(v)\hat{w}_{jJ\Delta}(v) = s_I{}^a s_J{}^b w_{ia}w_{jb} + s_I{}^a s_J{}^b s_J{}^c w_{ia}w_{jbc}$$

$$+ s_I{}^a s_I{}^b s_J{}^c w_{iab}w_{jc} + O(s^4), \quad (39)$$

where w_{ia} and w_{iab}, which are independent of s, need to be calculated explicitly. The product of $\hat{w}$ starts quadratically in s.

To this end we will need the τ algebra, recalling that $\tau_k = -(i/2)\sigma_k$, with σ_k being the standard Pauli matrices. We have

$$\tau_i \tau_j = -\frac{1}{4}\delta_{ij} + \frac{1}{2}\epsilon_{ijk}\tau_k,$$

$$\tau_i \tau_j \tau_k = -\frac{1}{8}\epsilon_{ijk} - \frac{1}{4}\delta_{ij}\tau_k + \frac{1}{4}\delta_{ik}\tau_j - \frac{1}{4}\delta_{jk}\tau_i,$$

$$\epsilon^{cab}\tau_a \tau_b = \frac{1}{2}\epsilon^{cab}\epsilon_{abm}\tau_m = \tau_c$$

$$\tau_k \tau_a \tau_k = \frac{1}{4}\tau_a. \quad (40)$$

From the definition (14) we obtain

$$\hat{w}_{iI\Delta} = \text{Tr}((\tau_i(\sqrt{V} - h_{s_I}\sqrt{V}h_{s_I}^{-1})). \quad (41)$$

Up to second order in s, we have

$$\hat{w}_{iI\Delta} = -\text{Tr}\left(\tau_i\left([A_I, \sqrt{V}] + \frac{1}{2}[A_I, [A_I, \sqrt{V}]] + \cdots\right)\right),$$

$$A_I = s_I{}^a A_{ia}\tau^i = s_I{}^a \frac{1}{2}[A_{ia}, \sqrt{V}]$$

$$+ s_I{}^a s_I{}^b \frac{1}{8}\epsilon_{ikl}[A_{ka}, [A_{lb}, \sqrt{V}]] + \cdots, \quad (42)$$

which finally allows us to identify

$$w_{ia} = \frac{1}{2}[A_{ia}, \sqrt{V}], \quad w_{iab} = \frac{1}{8}\epsilon_{ikl}[A_{ka}, [A_{lb}, \sqrt{V}]]. \quad (43)$$

The scaling properties under the semiclassical expectation value of the above gravitational operators is

$$\langle W\xi| \ldots w_{i a_1 \ldots a_n} \cdots |W\xi\rangle \to \frac{\ell_P^{3/2}}{\mathcal{L}^n}\left(\frac{\ell_P}{\mathcal{L}}\right)^{n\Upsilon}. \quad (44)$$

V. THE CALCULATION

The detailed calculation of the correction terms is performed according to the following prescription: first we set $\Upsilon = 0$, then we consider an expansion to order ℓ_P^2 and finally we reintroduce the non-zero value for Υ in the corresponding terms.

Let us recall the general expressions from the last section:

$$H_{nl}^F = i\binom{n}{l}\int d^3\vec{x}\,\pi(\vec{x})\tau_k$$

$$\times T^{a_1 a_2 \cdots a_l m_{l+1} m_{l+2} \cdots m_n k}$$

$$\times \tau_{m_{l+1}}\tau_{m_{l+2}} \cdots \tau_{m_n}\partial_{a_1}\partial_{a_1} \cdots \partial_{a_l}\xi(\vec{x}) \quad (45)$$

$$T^{a_1 a_2 \cdots a_l m_{l+1} m_{l+2} \cdots m_n k}$$

$$= \frac{\Theta}{4\ell_P^7}\langle (s_K^{a_1}s_K^{a_2} \cdots s_K^{a_l})(s_K^{a_{l+1}}s_K^{a_{l+2}} \cdots s_K^{a_n})$$

$$\times X_{a_{l+1}a_{l+2} \cdots a_n}^{m_{l+1}m_{l+2} \cdots m_n kK}(v)\rangle \quad (46)$$

$$X_{a_{l+1}a_{l+2} \cdots a_n}^{m_{l+1}m_{l+2} \cdots m_n kK}(v)$$

$$= \epsilon^{ijk}\epsilon^{IJK}\langle W,\xi|\hat{A}_{a_{l+1}}^{m_{l+1}}(v)\hat{A}_{a_{l+2}}^{m_{l+2}}(v)$$

$$\times \ldots \hat{A}_{a_n}^{m_n}(v)\hat{w}_{iI\Delta}(v)$$

$$\times \hat{w}_{jJ\Delta}(v)|W,\xi\rangle. \quad (47)$$

The partition is made of the number n such that we have l derivatives and $(n-l)$ τ-matrices, all arising from the presence of n covariant derivatives in Eq. (24). We recall that

$$\hat{w}_{iI\Delta}(v)\hat{w}_{jJ\Delta}(v) = (s_I{}^a s_J{}^b w_{ia}w_{jb}) + (s_I{}^a s_I{}^b s_J{}^c w_{ia}w_{jbc}$$

$$+ s_I{}^a s_I{}^b s_J{}^c w_{iab}w_{jc}) + O(s^4),$$

$$= (\hat{w}_{iI\Delta}(v)\hat{w}_{jJ\Delta}(v))_2$$

$$+ (\hat{w}_{iI\Delta}(v)\hat{w}_{jJ\Delta}(v))_3 \cdots$$

$$+ (\hat{w}_{iI\Delta}(v)\hat{w}_{jJ\Delta}(v))_N + \cdots, \tag{48}$$

where N counts the powers of s in the term $\hat{w}_{iI\Delta}(v)\hat{w}_{jJ\Delta}(v)$. Under the semiclassical state we can estimate the contribution of each term in Eq. (48) as

$$\langle W,\xi| \ldots (\hat{w}_{iI\Delta}(v)\hat{w}_{jJ\Delta}(v))_N \ldots |W,\xi\rangle = \ell_P^3 \left(\frac{\ell_P}{\mathcal{L}}\right)^{N(1+\Upsilon)}, \tag{49}$$

where we have used Eqs. (37) and (44).

A. The case $n=1$

Here we have two possibilities which we consider separately: $(n=1, l=1)$ and $(n=1, l=0)$.

1. Case $n=1$, $l=1$

$$H_{11}^F = i \int d^3\vec{x}\, \pi(\vec{x})\tau_k T_{11}^{a\,k}\partial_a \xi(\vec{x}) \tag{50}$$

$$T_{11}^{a\,k} = \frac{\Theta}{4\ell_P^7}\langle s_K^a X^{kK}(v)\rangle \tag{51}$$

$$X^{kK}(v) = \epsilon^{ijk}\epsilon^{IJK}\langle W,\xi|\hat{w}_{iI\Delta}(v)\hat{w}_{jJ\Delta}(v)|W,\xi\rangle. \tag{52}$$

We present this first calculation in some detail since it sets the stage for all the remaining estimates

$$T_{11}^{a\,k} = \frac{\Theta}{4\ell_P^7}\langle s_K^a \epsilon^{ijk}\epsilon^{IJK}\langle W,\xi|\hat{w}_{iI\Delta}(v)\hat{w}_{jJ\Delta}(v)|W,\xi\rangle\rangle$$

$$= \delta^{ak}\frac{\Theta}{4\ell_P^7}\ell_P \sum_{N=2,3,}\vartheta_{11N}\ell_P^3\left(\frac{\ell_P}{\mathcal{L}}\right)^{N(1+\Upsilon)}$$

$$= \delta^{ak}\frac{\Theta}{4\ell_P^3}\left(\vartheta_{112}\left(\frac{\ell_P}{\mathcal{L}}\right)^{2(1+\Upsilon)} + \vartheta_{113}\left(\frac{\ell_P}{\mathcal{L}}\right)^{3(1+\Upsilon)} + \cdots\right)$$

$$T_{11}^{a\,k} = \delta^{ak}\frac{\Theta}{4\ell_P^3}\left(\frac{\ell_P}{\mathcal{L}}\right)^{2(1+\Upsilon)}\left(\vartheta_{112} + \vartheta_{113}\left(\frac{\ell_P}{\mathcal{L}}\right)^{(1+\Upsilon)}\cdots\right). \tag{53}$$

The notation is ϑ_{nlN} and these numbers are assumed to be of order one. Choosing $\vartheta_{112}=1$ we are left with

$$T_{11}^{a\,k} = \delta^{ak}\frac{\Theta}{4\ell_P^3}\left(\frac{\ell_P}{\mathcal{L}}\right)^{2(1+\Upsilon)}\mathcal{F}_{11} \tag{54}$$

where

$$\mathcal{F}_{11} = \left(1 + \vartheta_{113}\left(\frac{\ell_P}{\mathcal{L}}\right)^{(1+\Upsilon)}\cdots\right). \tag{55}$$

Our convention in the sequel is to write $\mathcal{F}_{nl}$ so that its first term is ϑ_{nl2}, which is a pure number not depending either on ℓ_P or on $\mathcal{L}$. In this way

$$H_{11}^F = \frac{\Theta}{4\ell_P^3}\left(\frac{\ell_P}{\mathcal{L}}\right)^{2(1+\Upsilon)}\mathcal{F}_{11}\int d^3\vec{x}\, i\,\pi(\vec{x})\tau_a\partial_a \xi(\vec{x}). \tag{56}$$

In order to recover the standard kinetic term we have to choose

$$\Theta = 4\ell_P^3\left(\frac{\mathcal{L}}{\ell_P}\right)^{2(1+\Upsilon)}, \tag{57}$$

which reduces to our choice in Ref. [4], for $\Upsilon=0$. We obtain

$$H_{11}^F = \mathcal{F}_{11}\int d^3\vec{x}\, N(\vec{x})i\,\pi(\vec{x})\tau_a\partial_a \xi(\vec{x}) \tag{58}$$

as the final result for the kinetic term piece of the Hamiltonian.

2. Case $n=1$, $l=0$

The basic quantities are

$$H_{10}^F = i\int d^3\vec{x}\, \pi(\vec{x})\tau_k\tau_m T_{10}^{m\,k}\xi(\vec{x}) \tag{59}$$

$$T_{10}^{m\,k} = \frac{\Theta}{4\ell_P^7}\langle s_K^a \epsilon^{ijk}\epsilon^{IJK}\langle W,\xi|\hat{A}_a^m(v)\hat{w}_{iI\Delta}(v)\hat{w}_{jJ\Delta}(v)|W,\xi\rangle\rangle. \tag{60}$$

Under the scaling properties we obtain

$$T_{10}^{m\,k} = \delta^{mk}\frac{1}{\mathcal{L}}\left(\frac{\ell_P}{\mathcal{L}}\right)^{\Upsilon}\mathcal{F}_{10} \tag{61}$$

with

$$\mathcal{F}_{10} = \left(\vartheta_{102} + \vartheta_{103}\left(\frac{\ell_P}{\mathcal{L}}\right)^{(1+\Upsilon)} + \cdots\right) \tag{62}$$

$$H_{10}^F = -\frac{3}{4}\mathcal{F}_{10}\int d^3\vec{x}\, i\,\pi(\vec{x})\xi(\vec{x})\frac{1}{\mathcal{L}}\left(\frac{\ell_P}{\mathcal{L}}\right)^{\Upsilon}. \tag{63}$$

B. The case $n=2$

Here we have three possibilities: $(n=2, l=0), (n=2, l=1)$ and $(n=2, l=2)$

1. Case $(n=2, l=2)$

$$H_{22}^F = i \int d^3\vec{x}\, \pi(\vec{x}) \tau_k T_{22}^{ab\,k} \partial_a \partial_b \xi(\vec{x}) \qquad (64)$$

$$T_{22}^{ab\,k} = \frac{\Theta}{4\ell_P^7} \langle s_K^a s_K^b \epsilon^{ijk} \epsilon^{IJK} \langle W,\xi | \hat{w}_{iI\Delta}(v) \hat{w}_{jJ\Delta}(v) | W,\xi \rangle \rangle. \qquad (65)$$

From the above equation we see that T_{22}^{abk} must be symmetric in the indices a,b. Nevertheless, the only three index tensor at our disposal in flat space is ϵ^{abk} so we must conclude that this contribution is zero, that is to say $H_{22}^F = 0$.

2. Case $(n=2, l=1)$

$$H_{21}^F = 2i \int d^3\vec{x}\, \pi(\vec{x}) \tau_k \tau_m T_{21}^{am\,k} \partial_a \xi(\vec{x}) \qquad (66)$$

$$T_{21}^{am\,k} = \frac{\Theta}{4\ell_P^7} \langle s_K^a s_K^b \epsilon^{ijk} \epsilon^{IJK} \langle W,\xi | \hat{A}_b^m(v)$$

$$\times \hat{w}_{iI\Delta}(v) \hat{w}_{jJ\Delta}(v) | W,\xi \rangle \rangle. \qquad (67)$$

Here we have no symmetry requirement, so the antisymmetric tensor is allowed and we have

$$T_{21}^{am\,k} = \epsilon^{amk} \left(\frac{\ell_P}{\mathcal{L}} \right)^{1+\Upsilon} \mathcal{F}_{21}, \qquad$$

$$\mathcal{F}_{21} = \left(\vartheta_{212} + \vartheta_{213} \left(\frac{\ell_P}{\mathcal{L}} \right)^{1+\Upsilon} + \cdots \right). \qquad (68)$$

The final contribution is

$$H_{21}^F = -2\mathcal{F}_{21} \int d^3\vec{x}\, i\, \pi(\vec{x}) \tau_a \partial_a \xi(\vec{x}) \left(\frac{\ell_P}{\mathcal{L}} \right)^{1+\Upsilon}. \qquad (69)$$

This is a correction of order $(\ell_P/\mathcal{L})^{1+\Upsilon}$ to the standard kinetic term.

3. Case $n=2, l=0$

$$H_{20}^F = i \int d^3\vec{x}\, \pi(\vec{x}) \tau_k T_{20}^{nmk} \tau_n \tau_m \xi(\vec{x}) \qquad (70)$$

$$T_{20}^{nmk} = \frac{\Theta}{4\ell_P^7} \langle s_K^a s_K^b \epsilon^{ijk} \epsilon^{IJK} \langle W,\xi | \hat{A}_a^n(v) \hat{A}_b^m(v)$$

$$\times \hat{w}_{iI\Delta}(v) \hat{w}_{jJ\Delta}(v) | W,\xi \rangle \rangle. \qquad (71)$$

Here we do have symmetry requirements. Since the operators $\hat{A}_a^n(v)$ and $\hat{A}_b^m(v)$ commute, the above tensor must be symmetrical in the indices n,m. Again, in flat space the only three index tensor at our disposal is the ϵ^{nmk}, which leads to $H_{20}^F = 0$.

C. The case $n=3$

Here we have four possibilities: $(n=3, l=0), (n=3, l=1), (n=3, l=2)$ and $(n=3, l=3)$.

1. Case $(n=3, l=3)$

$$H_{33}^F = i \int d^3\vec{x}\, \pi(\vec{x}) \tau_k T_{33}^{abck} \partial_a \partial_b \partial_c \xi(\vec{x}) \qquad (72)$$

$$T_{33}^{abc\,k} = \frac{\Theta}{4\ell_P^7} \langle s_K^a s_K^b s_K^c \epsilon^{ijk} \epsilon^{IJK} \langle W,\xi | \hat{w}_{iI\Delta}(v) \hat{w}_{jJ\Delta}(v) | W,\xi \rangle \rangle. \qquad (73)$$

Here we have to impose the symmetry in the indices a,b,c. In flat space the required tensor is

$$t_{33}^{abck} = \frac{1}{3} (\delta^{ab} \delta^{ck} + \delta^{bc} \delta^{ak} + \delta^{ca} \delta^{bk}) \qquad (74)$$

which turns out to be symmetric in all four indices. Then

$$T_{33}^{abc\,k} = t_{33}^{abck} \ell_P^2 \mathcal{F}_{33},$$

$$\mathcal{F}_{33} = \left(\vartheta_{332} + \vartheta_{333} \left(\frac{\ell_P}{\mathcal{L}} \right)^{1+\Upsilon} + \cdots \right). \qquad (75)$$

Combining the above results we obtain

$$H_{33}^F = \mathcal{F}_{33} \int d^3\vec{x}\, i\, \pi(\vec{x}) \tau_a \partial_a \partial^2 \xi(\vec{x}) \ell_P^2, \qquad (76)$$

the leading term of which is independent of the scale $\mathcal{L}$.

2. Case $n=3, l=2$

$$H_{32}^F = 3 \int d^3\vec{x}\, i\, \pi(\vec{x}) \tau_k T_{32}^{abmk} \tau_m \partial_a \partial_b \xi(\vec{x}) \qquad (77)$$

$$T_{32}^{abm\,k} = \frac{\Theta}{4\ell_P^7} \langle s_K^a s_K^b s_K^c \epsilon^{ijk} \epsilon^{IJK} \langle W,\xi | \hat{A}_c^m(v)$$

$$\times \hat{w}_{iI\Delta}(v) \hat{w}_{jJ\Delta}(v) | W,\xi \rangle \rangle. \qquad (78)$$

The above tensor must be symmetrical in the indices a,b only. In flat space we can construct

$$t_{32}^{abmk} = \alpha_{32} \delta^{ab} \delta^{mk} + \beta_{32} (\delta^{am} \delta^{bk} + \delta^{bm} \delta^{ak}). \qquad (79)$$

Then we have

$$T_{32}^{abm\,k} = t_{32}^{abmk} \ell_P^2 \frac{1}{\mathcal{L}} \left(\frac{\ell_P}{\mathcal{L}} \right)^{\Upsilon} \mathcal{F}_{32},$$

$$\mathcal{F}_{32} = \left(\vartheta_{322} + \vartheta_{323} \left(\frac{\ell_P}{\mathcal{L}} \right)^{(1+\Upsilon)} + \cdots \right) \qquad (80)$$

which leads to

$$H_{32}^F = -3\left(\frac{3}{4}\alpha_{32} + \frac{1}{2}\beta_{32}\right)\mathcal{F}_{32}\int d^3\vec{x}\, i\,\pi(\vec{x})\ell_P^2\partial^2\xi(\vec{x})$$

$$\times \frac{1}{\mathcal{L}}\left(\frac{\ell_P}{\mathcal{L}}\right)^Y. \tag{81}$$

3. Case $n=3$, $l=1$

$$H_{31}^F = 3\int d^3\vec{x}\, i\,\pi(\vec{x})\tau_k T_{31}^{amnk}\tau_m\tau_n\partial_a\xi(\vec{x}) \tag{82}$$

$$T_{31}^{amn\,k} = \frac{\Theta}{4\ell_P^7}\langle s_K^a s_K^b s_K^c \epsilon^{ijk}\epsilon^{IJK}\langle W,\xi|\hat{A}_b^m(v)\hat{A}_c^n(v)$$

$$\times\hat{w}_{iI\Delta}(v)\hat{w}_{jJ\Delta}(v)|W,\xi\rangle. \tag{83}$$

The above tensor must be symmetrical in m,n. We have

$$t_{31}^{amnk} = \alpha_{31}\delta^{mn}\delta^{ak} + \beta_{31}(\delta^{ma}\delta^{nk} + \delta^{na}\delta^{mk}). \tag{84}$$

Then

$$T_{31}^{amnk} = t_{31}^{amnk}\left(\frac{\ell_P}{\mathcal{L}}\right)^{2(1+Y)}\mathcal{F}_{31},$$

$$\mathcal{F}_{31} = \left(\vartheta_{312} + \vartheta_{313}\left(\frac{\ell_P}{\mathcal{L}}\right)^{(1+Y)} + \cdots\right) \tag{85}$$

which produces

$$H_{31}^F = -\frac{1}{4}\mathcal{F}_{31}(9\alpha_{31} + 6\beta_{31})\int d^3\vec{x}\, i\,\pi(\vec{x})\tau_a\partial_a\xi(\vec{x})$$

$$\times\left(\frac{\ell_P}{\mathcal{L}}\right)^{2(1+Y)}. \tag{86}$$

4. Case $n=3$, $l=0$

$$H_{30}^F = \int d^3\vec{x}\, i\,\pi(\vec{x})T_{30}^{mnpk}\tau_k\tau_m\tau_n\tau_p\xi(\vec{x}) \tag{87}$$

$$T_{30}^{mnp\,k} = \frac{\Theta}{4\ell_P^7}\langle s_K^a s_K^b s_K^c \epsilon^{ijk}\epsilon^{IJK}\langle W,\xi|\hat{A}_a^m(v)\hat{A}_b^n(v)$$

$$\times\hat{A}_c^p(v)\hat{w}_{iI\Delta}(v)\hat{w}_{jJ\Delta}(v)|W,\xi\rangle. \tag{88}$$

The above tensor is symmetrical with respect to the indices m,n,p. This means that we need

$$t_{30}^{mnpk} = \frac{1}{3}(\delta^{mn}\delta^{pk} + \delta^{np}\delta^{mk} + \delta^{pm}\delta^{nk}) \tag{89}$$

so that

$$T_{30}^{mnp\,k} = t_{30}^{mnpk}\frac{\ell_P^2}{\mathcal{L}^3}\left(\frac{\ell_P}{\mathcal{L}}\right)^{3Y}\mathcal{F}_{30},$$

$$\mathcal{F}_{30} = \left(\vartheta_{302} + \vartheta_{303}\left(\frac{\ell_P}{\mathcal{L}}\right)^{(1+Y)} + \cdots\right), \tag{90}$$

which produces

$$H_{30}^F = \frac{15}{48}\mathcal{F}_{30}\int d^3\vec{x}\, i\,\pi(\vec{x})\xi(\vec{x})\frac{\ell_P^2}{\mathcal{L}^3}\left(\frac{\ell_P}{\mathcal{L}}\right)^{3Y}. \tag{91}$$

D. The full contribution H^F

It is given by

$$H^F = \sum_{n,l} H_{nl}^F, \tag{92}$$

where the different pieces were calculated in the previous subsection. Next we make a clear separation among terms containing either ℓ_P or $\mathcal{L}$ and those purely numerical factors not including these quantities, through the following series of redefinitions. First we change $\mathcal{F}_{nl}$ into $\mathcal{G}_{nl}$, where

$$\mathcal{G}_{nl} = \left(\varkappa_{nl2} + \varkappa_{nl3}\left(\frac{\ell_P}{\mathcal{L}}\right)^{(1+Y)} + \cdots\right), \tag{93}$$

by absorbing all numerical factors in the latter coefficients, in such a way that the Hamiltonian is written as

$$H^F = \int d^3\vec{x}\, N(\vec{x})i\,\pi(\vec{x})\left(\mathcal{G}_{11} + \mathcal{G}_{21}\left(\frac{\ell_P}{\mathcal{L}}\right)^{1+Y}\right.$$

$$\left.+\mathcal{G}_{31}\left(\frac{\ell_P}{\mathcal{L}}\right)^{2(1+Y)} + \mathcal{G}_{33}\ell_P^2\nabla^2 + \cdots\right)\tau_a\partial_a\xi(\vec{x})$$

$$+\int d^3\vec{x}\frac{i}{4}\pi(\vec{x})\frac{1}{\mathcal{L}}\left(\frac{\ell_P}{\mathcal{L}}\right)^Y\left(\mathcal{G}_{10} + \mathcal{G}_{30}\left(\frac{\ell_P}{\mathcal{L}}\right)^{2(1+Y)}\right.$$

$$\left.+\mathcal{G}_{32}\ell_P^2\nabla^2 + \cdots\right)\xi(\vec{x}).$$

In particular we have $\mathcal{G}_{11} \equiv \mathcal{F}_{11}$ and what we have done amounts effectively to a numerical redefinition of each ϑ_{nlN} into $\varkappa_{nlN}$. Finally we factor out the contributions arising from different powers of $(\ell_P/\mathcal{L})$ and ℓ_P. To the order considered we have

$$\mathcal{G}_{11} + \mathcal{G}_{21}\left(\frac{\ell_P}{\mathcal{L}}\right)^{1+Y} + \mathcal{G}_{31}\left(\frac{\ell_P}{\mathcal{L}}\right)^{2(1+Y)} + \cdots$$

$$= 1 + \kappa_1\left(\frac{\ell_P}{\mathcal{L}}\right)^{1+Y} + \kappa_2\left(\frac{\ell_P}{\mathcal{L}}\right)^{2(1+Y)} + \cdots$$

$$\mathcal{G}_{33} = \frac{\kappa_3}{2} \tag{94}$$

where κ_P are numbers not containing either ℓ_P or $\mathcal{L}$. The factor one in the right-hand side (RHS) of the first equation arises from the condition that in the limit $\ell_P \to 0$ we recover the standard fermionic Hamiltonian. Analogously we redefine

$$\mathcal{G}_{10}+\mathcal{G}_{30}\left(\frac{\ell_P}{\mathcal{L}}\right)^{2(1+\Upsilon)}=\kappa_4+\kappa_5\left(\frac{\ell_P}{\mathcal{L}}\right)^{1+\Upsilon}+\kappa_6\left(\frac{\ell_P}{\mathcal{L}}\right)^{2(1+\Upsilon)}$$

$$+\cdots$$

$$\mathcal{G}_{32}=\frac{\kappa_7}{2}\qquad(95)$$

where the remaining κ's satisfy the same condition as the previous ones. The above leads to

$$H^F=\int d^3\vec{x}N(\vec{x})i\,\pi(\vec{x})\left(1+\kappa_1\left(\frac{\ell_P}{\mathcal{L}}\right)^{1+\Upsilon}+\kappa_2\left(\frac{\ell_P}{\mathcal{L}}\right)^{2(1+\Upsilon)}\right.$$

$$+\frac{\kappa_3}{2}\ell_P^2\nabla^2+\cdots\bigg)\tau_a\partial_a\xi(\vec{x})$$

$$+\int d^3\vec{x}\frac{i}{4}\,\pi(\vec{x})\frac{1}{\mathcal{L}}\left(\frac{\ell_P}{\mathcal{L}}\right)^{\Upsilon}\left(\kappa_4+\kappa_5\left(\frac{\ell_P}{\mathcal{L}}\right)^{1+\Upsilon}\right.$$

$$+\kappa_6\left(\frac{\ell_P}{\mathcal{L}}\right)^{2(1+\Upsilon)}+\frac{\kappa_7}{2}\ell_P^2\nabla^2+\cdots\bigg)\xi(\vec{x})\qquad(96)$$

which is our final expression for this piece of the effective Hamiltonian.

E. The mass term

Since the spinors ξ,π are half-densities regularization of the mass term requires a treatment along the lines of [55] that effectively dedensitizes them. For our purposes, however, it is enough to determine the leading contribution and this can be done consistently with our strategy above. Namely, using the triangulation again,

$$H_m=\frac{m}{2\hbar}\int d^3xN\xi^T(i\sigma^2)\xi+\text{c.c.}$$

$$=\sum_\Delta N(v(\Delta))\frac{8}{3!E(v)}\epsilon^{IJK}\epsilon^{abc}s_I^a(\Delta)s_J^b(\Delta)$$

$$\times s_K^c(\Delta)\xi^T(s(\Delta))(i\sigma^2)h_{s(\Delta)}\xi(v(\Delta))+\text{c.c.}\quad(97)$$

and in the quantum theory, by adapting the triangulation to the flat WBSC state with fermions, we get

$$H_m=\langle W,\xi|\hat{H}_m|W,\xi\rangle=\sum_\Delta N(v(\Delta))\frac{8}{3!E(v)}\epsilon^{IJK}\epsilon^{abc}s_I^a(\Delta)s_J^b(\Delta)s_K^c(\Delta)\langle W,\xi|\theta^T(s(\Delta))(i\sigma^2)h_{s(\Delta)}\theta(v(\Delta))|W,\xi\rangle+\text{c.c.}$$

$$=\sum_{v\in V(\gamma)}N(v(\Delta))\frac{8}{3!E(v)}\epsilon^{IJK}\epsilon^{abc}s_I^a(\Delta)s_J^b(\Delta)s_K^c(\Delta)(\langle W,\xi|\theta^T(v)(i\sigma^2)\theta(v)|W,\xi\rangle+\langle W,\xi|s^a\partial_a\theta^T(v)(i\sigma^2)\theta(v)|W,\xi\rangle$$

$$+\langle W,\xi|\theta^T(v)(i\sigma^2)s^aA_a(v)\theta(v)|W,\xi\rangle+\langle W,\xi|s^a\partial_a\theta^T(v)(i\sigma^2)s^aA_a(v)\theta(v)|W,\xi\rangle+\cdots)+\text{c.c.}\qquad(98)$$

Hence, to leading order, the modifications coming from Planck scale to standard flat space mass term for fermions are just

$$H_m=\frac{m}{2\hbar}\int d^3x\left(\xi^T(i\sigma^2)\xi+\kappa_9\ell_P\xi^T(i\sigma^2)\tau^a\partial_a\xi\right.$$

$$+\kappa_8\left(\frac{\ell_P}{\mathcal{L}}\right)^{\Upsilon+1}\xi^T(i\sigma^2)\xi+\kappa_{11}\ell_P\left(\frac{\ell_P}{\mathcal{L}}\right)^{\Upsilon+1}$$

$$\times\xi^T(i\sigma^2)\tau^a\partial_a\xi+\text{c.c.}\bigg),\qquad(99)$$

where we have set $N=1$.

F. The extra contributions

Here we study the second term of Eq. (3)

$$H^{(2)}_{\text{spin}-1/2}:=\int_\Sigma d^3x\frac{E_i^a}{2\sqrt{q}}[\mathcal{D}_a(\pi^T\tau_i\xi)+\text{c.c.}].\qquad(100)$$

Repeating the procedure applied to $H^{(1)}_{\text{spin}-1/2}$ step by step with the pertinent modification in the fermionic term leads to

$$\hat{H}^{(2)}_{\text{spin}-1/2}[N]=-\frac{m_P}{2\ell_P^3}\sum_{v\in V(\gamma)}N_v\sum_{v(\Delta)=v}\epsilon^{ijk}\epsilon^{IJK}$$

$$\times\text{tr}(\tau_ih_{s_I(\Delta)}[h_{s_I(\Delta)}^{-1},\sqrt{\hat{V}_v}])$$

$$\times\text{tr}(\tau_jh_{s_J(\Delta)}[h_{s_J(\Delta)}^{-1},\sqrt{\hat{V}_v}])$$

$$\times\bigg[[(h_{s_K(\Delta)}^{-1}\tau_kh_{s_K(\Delta)}\theta)(s_K(\Delta)(\delta))]_A$$

$$\times\frac{\partial}{\partial\theta_A(s_K(\Delta)(\delta))}-[\tau_k\theta(v)]_A$$

$$\times\frac{\partial}{\partial\theta_A(v)}+\text{H.c.}\bigg].\qquad(101)$$

The corresponding expectation value behaves as

$$\langle\hat{H}^{(2)}_{\text{spin}-1/2}\rangle=\int_\Sigma d^3x[Z_1\partial_aB^a+Z_2\partial_a\partial_bB^{ab}+\cdots]\qquad(102)$$

where the Z_i's represent the estimate of the nonfermionic factors in Eq. (101) and the B's represent the fermionic bilinears. Clearly every B term, being a boundary term, will not affect the effective dynamics of the spin-$\frac{1}{2}$ particle.

We proceed to describe the third term in Eq. (3)

$$H^{(3)}_{\text{spin-1/2}} := \int_\Sigma d^3x \frac{E_i^a}{2\sqrt{q}} \frac{1}{2} [iK_a^i \pi^T \xi + \text{c.c.}]. \qquad (103)$$

To regularize it we can treat the $E_i^a/2\sqrt{q}$ factor similarly as with $H^{(1)}_{\text{spin-1/2}}$ term. As for the K_a^i factor we recall [55]

$$K_a^j = -\frac{1}{\kappa}\{A_a^j, \{V, H_E\}\} \qquad (104)$$

$$H^E = \frac{2}{\kappa}\int_\Sigma d^3x \, \epsilon^{abc}\text{tr}(F_{ab}\{A_c, V\}). \qquad (105)$$

Upon regularization of the quantum H^E one gets

$$H^E = \sum_\Delta H_\Delta^E, \qquad (106)$$

$$H_\Delta^E = -\frac{2N_v}{3i\kappa\ell_P^2}\epsilon^{ijk}\text{tr}(h_{\alpha_{ij(\Delta)}}h_{s_k(\Delta)}[h_{s_k(\Delta)}^{-1}, V_v]). \qquad (107)$$

Following the standard procedure to incorporate holonomies and fermions one arrives at the quantum version of Eq. (103) as

$$\left\{ \sum_{v \in V(\gamma)} \frac{\hbar}{\ell_P^{10}} \sum_{v(\Delta)=v} \sum_{v(\Delta')=v} \epsilon^{IJK}\epsilon^{LMN}\epsilon_{ijk}w_{iI\Delta}w_{jJ\Delta} \right.$$
$$\times \text{tr}(\tau_k h_{s_K}[h_{s_K}^{-1}, \{V_v, \text{tr}(h_{\alpha_{LM(\Delta')}}h_{s_N(\Delta')}$$
$$\left. \times [h_{s_N(\Delta')}^{-1}, V_v])\}])\xi_A(v(\Delta))\frac{\partial}{\partial \xi_A(v(\Delta))} + \text{H.c.} \right\} f_\gamma$$
$$(108)$$

where the prefactor $\hbar/\ell_P^{10}$ was obtained from $(1/\kappa^2)(\hbar)(1/\hbar^2)(1/\kappa^2\hbar^2\ell_P^2)$ with the κ factors coming from the different Poisson brackets identities, the $\hbar$ factors from the quantization procedure and the ℓ_P^2 factor arises similarly as in previous steps. We want to estimate the leading order contribution of Eq. (103) to the effective dynamics of the spin-$\frac{1}{2}$ particle. For simplicity in the discussion we set $Y=0$, but an extension can be given for $Y>0$. The leading order is estimated by noticing that under $\langle\langle \cdots \rangle\rangle$ the contribution of an holonomy h is 1 if it is not an entry of a commutator, otherwise it scales like $\ell_P/\mathcal{L}$. Such a leading order contribution behaves then like $(\hbar/\ell_P^{10})(\ell_P^{3/2}(\ell_P/\mathcal{L}))^2(\ell_P/\mathcal{L})^2\ell_P^6 = (\hbar/\ell_P) \times (\ell_P/\mathcal{L})^4 = m_P(\ell_P/\mathcal{L})^4$, which is highly suppressed given the quartic order and $\ell_P \ll \mathcal{L}$.

VI. EFFECTIVE DYNAMICS FOR SPIN-$\frac{1}{2}$ PARTICLES

The total effective Hamiltonian H_T for our spin-$\frac{1}{2}$ particle is then

$$H_T = H^F + H_m$$
$$= \int d^3x \left[i\pi(\vec{x})\tau^d\partial_d \ \hat{A}\xi(\vec{x}) + \text{c.c.} + \frac{i}{4\hbar}\frac{1}{\mathcal{L}}\pi(\vec{x})\hat{C}\xi(\vec{x}) \right.$$
$$+ \frac{m}{2\hbar}\xi^T(\vec{x})(i\sigma^2)(\alpha + 2\hbar\beta\tau^a\partial_a)\xi(\vec{x})$$
$$\left. + \frac{m}{2\hbar}\pi^T(\vec{x})(\alpha + 2\hbar\beta\tau^a\partial_a)(i\sigma^2)\pi(\vec{x}) \right], \qquad (109)$$

where

$$\hat{A} = \left(1 + \kappa_1\left(\frac{\ell_P}{\mathcal{L}}\right)^{Y+1} + \kappa_2\left(\frac{\ell_P}{\mathcal{L}}\right)^{2Y+2} + \frac{\kappa_3}{2}\ell_P^2\nabla^2 \right),$$

$$\hat{C} = \hbar\left(\kappa_4\left(\frac{\ell_P}{\mathcal{L}}\right)^{Y} + \kappa_5\left(\frac{\ell_P}{\mathcal{L}}\right)^{2Y+1} + \kappa_6\left(\frac{\ell_P}{\mathcal{L}}\right)^{3Y+2} \right.$$
$$\left. + \frac{\kappa_7}{2}\left(\frac{\ell_P}{\mathcal{L}}\right)^{Y}\ell_P^2\nabla^2 \right),$$

$$\alpha = \left(1 + \kappa_8\left(\frac{\ell_P}{\mathcal{L}}\right)^{Y+1} \right),$$

$$\beta = \frac{\kappa_9}{2\hbar}\ell_P + \frac{\kappa_{11}}{2\hbar}\ell_P\left(\frac{\ell_P}{\mathcal{L}}\right)^{Y+1}. \qquad (110)$$

It can be seen that the terms proportional to κ_4, κ_5 and κ_6 correspond to a renormalization of the fermion mass. They will not be considered in the sequel, although they would give rise to standard neutrino oscillations even if the neutrino bare mass is zero.

The wave equation becomes

$$\left[i\hbar\frac{\partial}{\partial t} - i\hbar\hat{A}\vec{\sigma}\cdot\nabla + \frac{\hat{C}}{2\mathcal{L}} \right]\xi(t,\vec{x}) + m(\alpha - \beta i\hbar\vec{\sigma}\cdot\nabla)\chi(t,\vec{x})$$
$$= 0, \qquad (111)$$

$$\left[i\hbar\frac{\partial}{\partial t} + i\hbar\hat{A}\vec{\sigma}\cdot\nabla - \frac{\hat{C}}{2\mathcal{L}} \right]\chi(t,\vec{x}) + m(\alpha - \beta i\hbar\vec{\sigma}\cdot\nabla)\xi(t,\vec{x})$$
$$= 0, \qquad (112)$$

with $\chi(t,\vec{x}) = i\sigma_2\xi^*(t,\vec{x})$. Following the standard steps one can verify the consistency of the above equations.

Eliminating χ from Eq. (111) we obtain

$$\chi = -\frac{1}{m}\frac{1}{\alpha - \beta i\hbar\vec{\sigma}\cdot\nabla}\left[i\hbar\frac{\partial}{\partial t} - i\hbar\hat{A}\vec{\sigma}\cdot\nabla + \frac{\hat{C}}{2\mathcal{L}} \right]\xi. \qquad (113)$$

Substituting in Eq. (112) we obtain the second order equation

$$\left(\left[i\hbar\frac{\partial}{\partial t}+i\hbar\hat{A}\vec{\sigma}\cdot\nabla-\frac{\hat{C}}{2\mathcal{L}}\right]\left[i\hbar\frac{\partial}{\partial t}-i\hbar\hat{A}\vec{\sigma}\cdot\nabla+\frac{\hat{C}}{2\mathcal{L}}\right]\right.$$
$$\left.-m^2(\alpha-\beta i\hbar\vec{\sigma}\cdot\nabla)^2\right)\xi(t,\vec{x})=0. \tag{114}$$

The above equation has positive and negative energy solutions

$$W(\vec{p},h)e^{-(i/\hbar)Et+(i/\hbar)\vec{p}\cdot\vec{x}}, \quad W(\vec{p},h)e^{(i/\hbar)Et-(i/\hbar)\vec{p}\cdot\vec{x}}, \tag{115}$$

where it is convenient to take the spinorial part $W(\vec{p},h)$ as helicity $(\vec{\sigma}\cdot\hat{p})$ eigenstates, with $h=\pm 1$, so that

$$W(\vec{p},1)=\begin{pmatrix}\cos\left(\dfrac{\theta}{2}\right)\\[2mm]e^{i\phi}\sin\left(\dfrac{\theta}{2}\right)\end{pmatrix},$$

$$W(\vec{p},-1)=\begin{pmatrix}-e^{-i\phi}\sin\left(\dfrac{\theta}{2}\right)\\[2mm]\cos\left(\dfrac{\theta}{2}\right)\end{pmatrix}. \tag{116}$$

The dispersion relation has the following form:

$$E_\pm=\sqrt{(A^2+m^2\beta^2)|\vec{p}|^2+m^2\alpha^2+\left(\frac{C}{2\mathcal{L}}\right)^2}\pm B|\vec{p}|, \tag{117}$$

with

$$A=\left(1+\kappa_1\left(\frac{\ell_P}{\mathcal{L}}\right)^{\Upsilon+1}+\kappa_2\left(\frac{\ell_P}{\mathcal{L}}\right)^{2\Upsilon+2}-\frac{\kappa_3}{2}\ell_P^2 p^2\right),$$

$$B=A\left(\frac{C}{\mathcal{L}}+2\alpha\beta m^2\right),$$

$$C=-\frac{\hbar\kappa_7}{2}\left(\frac{\ell_P}{\mathcal{L}}\right)^{\Upsilon}\ell_P^2 p^2,$$

$$\alpha=\left(1+\kappa_8\left(\frac{\ell_P}{\mathcal{L}}\right)^{\Upsilon+1}\right),$$

$$\beta=\frac{\kappa_9}{2\hbar}\ell_P+\frac{\kappa_{11}}{2\hbar}\ell_P\left(\frac{\ell_P}{\mathcal{L}}\right)^{\Upsilon+1}. \tag{118}$$

The $\pm$ in Eq. (117) refers to the dispersion relation of the helicities $\pm$ respectively. Let us emphasize that the solution $\xi(t,\vec{x})$ to either Eq. (111) or Eq. (112) is given by an appropriate linear combination of plane waves and helicity eigenstates. This is not unexpected since we are dealing with massive particles.

In the sequel we write down the dispersion relation in terms of an expansion up to second order in $(\ell_P/\mathcal{L})^{\Upsilon}$. The coefficient of each power is subsequently expanded in powers of m in the combinations either (m/p) or $(m\ell_P)$. In this way we obtain

$$E_\pm(p,\mathcal{L})=\left[p+\frac{m^2}{2p}\pm\ell_P\left(\frac{1}{2}m^2\kappa_9\right)+\ell_P^2\left(-\frac{1}{2}\kappa_3 p^3+\frac{1}{8}\right.\right.$$
$$\left.\times(2\kappa_3+\kappa_9^2)m^2 p\right)\left]+\left(\frac{\ell_P}{\mathcal{L}}\right)^{\Upsilon+1}\left[\left(\kappa_1 p-\frac{\Theta_{11}m^2}{4p}\right)\right.\right.$$
$$\left.\pm\ell_P\left(-\kappa_7\frac{p^2}{4}+\Theta_{12}\frac{m^2}{16}\right)\right]+\left(\frac{\ell_P}{\mathcal{L}}\right)^{2\Upsilon+2}$$
$$\times\left(\kappa_2 p-\frac{m^2}{64p}\Theta_{22}\right) \tag{119}$$

where the previous coefficients denoted by κ appear in the following combinations:

$$\Theta_{11}=(2\kappa_1-4\kappa_8), \quad \Theta_{12}=(8\kappa_{11}+2\kappa_7+8\kappa_8\kappa_9), \tag{120}$$

$$\Theta_{22}=-32\kappa_1^2+32\kappa_2+64\kappa_1\kappa_8. \tag{121}$$

The velocity is defined as

$$v_\pm(p,\mathcal{L})=\frac{\partial E_\pm(p,\mathcal{L})}{\partial p} \tag{122}$$

and is

$$v_\pm(p,\mathcal{L})=\left[\left(1-\frac{m^2}{2p^2}\right)+\ell_P^2\left(-\frac{3}{2}\kappa_3 p^2+\frac{1}{8}(2\kappa_3+\kappa_9^2)m^2\right)\right]$$
$$+\left(\frac{\ell_P}{\mathcal{L}}\right)^{\Upsilon+1}\left[\left(\kappa_1+\frac{\Theta_{11}m^2}{4p^2}\right)\mp\frac{\kappa_7}{2}(\ell_P p)\right]$$
$$+\left(\frac{\ell_P}{\mathcal{L}}\right)^{2\Upsilon+2}\left(\kappa_2+\frac{m^2}{64p^2}\Theta_{22}\right), \tag{123}$$

within the same approximation.

We will be mainly interested in the case of ultrarelativistic cosmological neutrinos, whose mass we take to be $m=10^{-9}$ GeV, in the range of momenta $10^5\leq p\leq 10^{10}$ GeV, where

$$p\gg m \Rightarrow (p\ell_P)\gg(m\ell_P)\frac{m}{p}, \quad (p\ell_P)^2\gg(m\ell_P)^2. \tag{124}$$

Such a regime allows us to introduce some simplifications in the coefficients of $(\ell_P/\mathcal{L})^{\Upsilon}$. The results are

$$\ell_P E_\pm(p,\mathcal{L}) = (p\ell_P) + (\ell_P m)\frac{m}{2p} \pm \left(\frac{1}{2}(\ell_P m)^2 \kappa_9\right)$$
$$- \frac{1}{2}\kappa_3(\ell_P p)^3 + \left(\frac{\ell_P}{\mathcal{L}}\right)^{\Upsilon+1}$$
$$\times \left[\kappa_1(p\ell_P) \mp \kappa_7 \frac{(\ell_P p)^2}{4}\right]$$
$$+ \left(\frac{\ell_P}{\mathcal{L}}\right)^{2\Upsilon+2} \kappa_2(p\ell_P), \qquad (125)$$

together with

$$v_\pm(p,\mathcal{L}) = 1 - \frac{m^2}{2p^2} - \frac{3}{2}\kappa_3(\ell_P p)^2 + \left(\frac{\ell_P}{\mathcal{L}}\right)^{\Upsilon+1}$$
$$\times \left[\kappa_1 \mp \frac{\kappa_7}{2}(\ell_P p)\right] + \left(\frac{\ell_P}{\mathcal{L}}\right)^{2\Upsilon+2}\kappa_2. \quad (126)$$

VII. PARAMETER ESTIMATES

A. The scale $\mathcal{L}$

In order to estimate some numerical values for the modifications to the velocity of propagation we must further fix the value of the scale $\mathcal{L}$. Two distinguished cases arise.

1. The mobile scale $\mathcal{L} = \lambda = \frac{1}{p}$

Recall that $\mathcal{L}$ is a macroscopic length scale, being defined by the given geometry which indicates where the nonperturbative states of the spin network can be approximated by the classical flat metric. The neutrino is characterized by energies which probe to distances of order λ. To be described by a classical continuous flat geometry equation, as the found Dirac equation, it is necessary that one remains in the correct range with respect to $\mathcal{L}$, i.e. $\mathcal{L} < \lambda$. We take the marginal choice of the equality in order to be able to make some further estimates. Previous expressions reduce to

$$\ell_P E_\pm(p,\mathcal{L})|_{\mathcal{L}=1/p}$$
$$= (p\ell_P) + (\ell_P m)\frac{m}{2p} \pm \frac{1}{2}(\ell_P m)^2 \kappa_9$$
$$- \frac{1}{2}\kappa_3(\ell_P p)^3 + (p\ell_P)^{\Upsilon+1}$$
$$\times \left[\kappa_1(p\ell_P) \mp \kappa_7 \frac{(\ell_P p)^2}{4}\right] + (p\ell_P)^{2\Upsilon+2}\kappa_2(p\ell_P)$$
$$\qquad (127)$$

and

$$v_\pm(p,\mathcal{L})|_{\mathcal{L}=1/p} = 1 - \frac{m^2}{2p^2} - \frac{3}{2}\kappa_3(\ell_P p)^2 + (\ell_P p)^{\Upsilon+1}$$
$$\times \left[\kappa_1 \mp \frac{\kappa_7}{2}(\ell_P p)\right] + (\ell_P p)^{2\Upsilon+2}\kappa_2. \qquad (128)$$

In the case $\Upsilon = 0$ we recover the expressions (9) and (10) of Ref. [4].

2. A universal scale for $\mathcal{L}$

Recently, in [13] a universal value for $\mathcal{L}$ was considered in the context of the GZK anomaly. The study of the different reactions involved produces a preferred bound on $\mathcal{L}$:4.6×10^{-8} GeV$^{-1} \geq \mathcal{L} \geq 8.3 \times 10^{-9}$ GeV^{-1}. Actually, in this case the general expressions (125) and (126) are valid.

B. The exponent Υ

In this section we follow the conclusion derived fom the Super-Kamiokande atmospheric neutrino experiment that neutrino oscillations are well described by the mass differences contribution to the energy ($n = -0.9 \pm 0.4$ at 90% C.L., in the standard notation [62]) [9]. That is to say, any additional contribution to the oscillation length must be highly suppressed. This condition will set a lower bound to Υ which we will use in further estimates.

The oscillation length $L = 2\pi/|\Delta E|$ is

$$\frac{1}{L} = \frac{1}{L_m} + \frac{1}{L_{QG}}, \quad L = L_m \frac{1}{1 + \frac{L_m}{L_{QG}}}, \quad L_m = \frac{4\pi E}{(\Delta m)^2},$$
$$\qquad (129)$$

where we have used $p \approx E$, $\ell_P = 1/M_P$ and L_{QG} is calculated according to each specific additional term in $|\Delta E|$. Since the dominant contribution to the oscillation arises form the mass term, i.e. $L \approx L_m$, we must have the condition

$$L_{QG} > L_m \approx X, \qquad (130)$$

where X is the distance travelled by the neutrinos in each experiment. Let us observe that the energy range for current neutrino observatories lies between 10^{-2} GeV and 10^{+2} GeV, which amply satisfies the condition $(p\mathcal{L}) \lesssim 1$. In particular we will consider the case of the SNO experiment characterized by the following parameters [63]:

$$E = 10^{-2} \text{ GeV}, \quad X = 10^8 \text{ km} = 10^{27} \quad \text{GeV}^{-1}. \quad (131)$$

The estimates proceed by considering first the separate contribution of each arbitrary parameter $\Delta\kappa$ and subsequently the case where we have more than one non-zero contribution to the oscillation.

1. The case $\mathcal{L} = \lambda = \frac{1}{p}$

(i) $\Delta\kappa_3 \neq 0$, $\Delta\kappa_1 = \Delta\kappa_7 = 0$.

The quantum gravity contribution to the oscillation length is

$$L_{QG} = \frac{4\pi}{\Delta\kappa_3}\left(\frac{M}{E}\right)^2 \frac{1}{E}, \qquad (132)$$

from which we get that

$$\Delta\kappa_3 < 6.28 \times 10^{17},$$

which is a very weak bound on the variable. The reason is that this term corresponds to a second order correction in l_P.

(ii) $\Delta\kappa_1 \neq 0$, $\Delta\kappa_3 = \Delta\kappa_7 = 0$.

Here we have

$$L_{QG} = \frac{2\pi}{\Delta\kappa_1}\left(\frac{M}{E}\right)^{\Upsilon+1} \frac{1}{E}. \qquad (133)$$

Using $\Delta\kappa_1 \sim 1$, we obtain

$$\Upsilon > 0.152. \qquad (134)$$

(iii) $\Delta\kappa_7 \neq 0$, $\Delta\kappa_1 = \Delta\kappa_3 = 0$.

Here we have

$$L_{QG} = \frac{8\pi}{\Delta\kappa_7}\left(\frac{M}{E}\right)^{\Upsilon+2} \frac{1}{E}. \qquad (135)$$

Using $\Delta\kappa_7 \sim 1$, we obtain

$$\Upsilon > -0.87. \qquad (136)$$

(iv) When two of the $\Delta\kappa_i$ do not simultaneously vanish, we get the following situations:

(iv-1) $\Delta\kappa_1 \neq 0$.

In this case the term proportional to $\Delta\kappa_7$ is suppressed with respect to the one proportional to $\Delta\kappa_1$. The comparison among the terms proportional to $\Delta\kappa_1$ and $\Delta\kappa_3$ leads to the threshold $\Upsilon = 1$. When $\Upsilon > 1$ the term proportional to $\Delta\kappa_3$ dominates, while the term proportional to $\Delta\kappa_1$ dominates in the other situation.

(iv-2) $\Delta\kappa_1 = 0$.

In this case the competition is among the terms proportional to $\Delta\kappa_3$ and $\Delta\kappa_7$. The threshold here is $\Upsilon = 0$, so that we consider only the case $\Upsilon > 0$, where the term proportional to $\Delta\kappa_3$ dominates.

2. A universal scale for $\mathcal{L}$

In this estimate we will assume that $\mathcal{L} = 10^{-8}$ GeV^{-1} [13].

(i) $\Delta\kappa_3 \neq 0$, $\Delta\kappa_1 = \Delta\kappa_7 = 0$.

Since this contribution does not depend upon $\mathcal{L}$, we obtain the same result as the corresponding case in the mobile scale.

(ii) $\Delta\kappa_1 \neq 0$, $\Delta\kappa_3 = \Delta\kappa_7 = 0$.

Here we have

$$L_{QG} = \frac{2\pi}{\Delta\kappa_1}\left(\frac{\mathcal{L}}{\ell_P}\right)^{\Upsilon+1} \frac{1}{E}. \qquad (137)$$

Using $\Delta\kappa_1 \sim 1$, we obtain

$$\Upsilon > 1.2. \qquad (138)$$

(iii) $\Delta\kappa_7 \neq 0$, $\Delta\kappa_1 = \Delta\kappa_3 = 0$.

Here we have

$$L_{QG} = \frac{8\pi}{\Delta\kappa_7}\left(\frac{\mathcal{L}}{\ell_P}\right)^{\Upsilon+1} \frac{M_P}{E} \frac{1}{E}. \qquad (139)$$

Assuming $\Delta\kappa_7 \sim 1$, we get a bound on Υ:

$$\Upsilon > -0.764, \qquad (140)$$

which is also a weak bound on the variable.

(iv) When two of the $\Delta\kappa_i$ do not simultaneously vanish, we get the following situations:

(iv-1) $\Delta\kappa_1 \neq 0$, which leads to the threshold $\Upsilon > \frac{17}{11}$. When $\Upsilon > 17/11$ the term $\Delta\kappa_3$ dominates and we are back to the case (i). On the other hand, when $\Upsilon < \frac{17}{11}$ we obtain the interval $\frac{17}{11} > \Upsilon > 1.2$. It should be stressed that the latter bound on Υ is also compatible with the bound on the maximum speed of Ref. [64], which in our case reads

$$\Delta v_{QG} = \left| v_{\pm}(p,\mathcal{L}) - \left(1 - \frac{m^2}{2p^2}\right)\right| = |\kappa_1|\left(\frac{\ell_P}{\mathcal{L}}\right)^{\Upsilon+1} < 10^{-22}. \qquad (141)$$

(iv-2) $\Delta\kappa_1 = 0$.

The threshold here is $\Upsilon = 3/11$. When $\Upsilon > 3/11$ the term proportional to $\Delta\kappa_3$ dominates over the one proportional to $\Delta\kappa_7$.

So, the present data on neutrino oscillations do not prescribe the theory considered in this work with a universal scale $\mathcal{L} \sim 10^{-8}$ GeV^{-1}.

VIII. SUMMARY AND DISCUSSION

In this work we have derived an effective Hamiltonian exhibiting Planck scale corrections with respect to standard propagation for the theory describing spin-1/2 fermions, using an heuristical approach based upon Thiemann's regularization within the framework of loop quantum gravity. Corrections arise as a consequence of the discrete nature of space which are manifest at Planck scale. The effective spin-1/2 particle Hamiltonian given in Eqs. (109),(110) was obtained by taking the expectation value of the regularized version of the quantum operator corresponding to Eq. (3) with respect to a *would be semiclassical state* $|W,\xi\rangle$ describing a large scale flat metric together with a slowly varying classical spinor field. Only the basic properties of such a state were used: (i) peakedness both in a flat space metric together with a flat connection for large distances $d \gg \mathcal{L} \gg \ell_P$, where $\mathcal{L}$ can be thought of as the scale that settles the transition between a discrete and a continuous description of space, (ii) well defined expectation values, (iii) existence of a coarse-grained expansion involving ratios of the relevant scales of the problem: the Planck length ℓ_P, the characteristic length $\mathcal{L}$ of the state and the de Broglie wavelength λ of the spin-1/2 particle and (iv) invariance under rotations at scales larger than $\mathcal{L}$,

which amounts to expressing the box-averaged values introduced after Eq. (18) only in terms of flat space tensors.

The effective theory violates Lorentz invariance and, in analogy with the photon case [5], we assume that the effective Hamiltonian so found corresponds to that in the particular frame of reference where the cosmic radiation background looks isotropic.

Some improvements with respect to our original presentation [4] are the following: (i) in Sec. V F we have elucidated the contribution of the derivative term and the extrinsic curvature dependent term in Eq. (3) by showing that they are highly suppressed in powers of ℓ_P. (ii) We have extended the corrections to the scaling of the connection by including the new parameter Υ, already considered in our discussion of photons [5], in the form

$$\langle W, \xi | A_{ia} | W, \xi \rangle = 0 + \frac{1}{\mathcal{L}} \left(\frac{\ell_P}{\mathcal{L}} \right)^{\Upsilon}, \tag{142}$$

where $\Upsilon > 0$ can be any real number.

In Sec. VIII we have estimated some bounds for Υ, based on the observation that atmospheric neutrino oscillations at average energies of the order $10^{-2} - 10^2$ GeV are dominated by the corresponding mass differences via the oscillation length L_m in Eq. (129). This means that additional contributions to the oscillation length, in particular the quantum gravity correction L_{QG}, should satisfy $L_{QG} > L_m$, which is used to set a lower bound upon Υ. Within the proposed two different ways of estimating the scale $\mathcal{L}$ of the process we obtain: (i) $\Upsilon > 0.15$ when $\mathcal{L}$ is considered as a mobile scale and it is estimated by $1/E$ and (ii) $1.2 < \Upsilon$ when the scale $\mathcal{L}$ takes the universal value $\mathcal{L} = 10^{-8}$ GeV^{-1}, according to Ref. [13]. In the above estimates we have satisfied the condition $(p\mathcal{L}) \lesssim 1$.

Let us observe that according to Eq. (127) the mass-difference contribution to the neutrino oscillation process will be highly suppressed at high energies while other mechanisms, like those arising in the quantum gravity framework, could be the dominant ones [31]. It is an experimental issue to settle this question. Since the bound upon Υ strongly depends on the dominant mechanism there is the possibility that the exponent Υ be energy-dependent. This situation is not considered in the present approach. Nevertheless, in order to make some numerical estimates related to cosmological neutrinos we will make the extrapolation of our bounds in Υ, which have been obtained in the range of a few GeV, to energies of the order of 10^5 GeV, corresponding to typical neutrinos arising from gamma ray bursts. Let us consider specifically ultrarelativistic neutrinos of mass $m = 10^{-9}$ GeV, energy $p \approx E = 10^5$ GeV, traveling a cosmological distance $L = 10^{10}$ light years $= 0.5 \times 10^{42}$ GeV^{-1}, where $p\mathcal{L} \lesssim 1$. Next we give an estimate of two types of delay times (for a more realistic calculation including the effect of the expanding universe see Refs. [5,13,31]).

(i) $\Delta t_\nu \approx (L/c^2)(v - v_0)$, which measures the time delay of the arrival of the neutrino coming from a distance L, flying with velocity $v = \partial E / \partial p$, with respect to the time of flight corresponding to a particle of mass m with velocity $v_0 = p/\sqrt{p^2 + m^2} \approx 1$.

(ii) $\Delta t_\pm \approx (L/c^2)(v_- - v_+)$, which is a measure of the birefringence effects and it is defined by the difference in the arrival time of two neutrinos with opposite polarizations.

In the range $0.6 < \Upsilon < 2.0$ and within the two scenarios presented here for dealing with the scale $\mathcal{L}$, we obtain the estimates

$$10^{-10} \text{ s} < \Delta t_\nu < 10^{-4} \text{ s}, \tag{143}$$

$$\Delta t_\pm < 10^{-17} \text{ s}. \tag{144}$$

To conclude we point out that an effective dynamics and dispersion relations for gravity plus matter were thoroughly studied recently by Thiemann and Sahlmann [21], where the semiclassical states were taken as coherent states. They included photons and scalar particles and their results for the dispersion relations essentially agree with what we obtain here for fermions, even the possibility of having nonintegral powers in ℓ_P for the correcting terms, which is encoded in our parameter Υ. Besides they provide a detailed classification of the quantum geometry aspects required in defining the semiclassical regime yielding the effective dynamics. Yet another avenue to tackle the problem of defining semiclassical states in quantum geometry is currently under investigation that establishes a relation between Fock space and the kinematical Hilbert space for diffeomorphism covariant theories of connections such as quantum geometry (loop quantum gravity) [65,66] (see also [67]). It will be interesting to compare all of the above proposed semiclassical states in the context of the quantum gravity phenomenology.

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