THE PROPAGATION OF PARTICLES IN SPACETIME FOAM

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Scalar particles would be strongly affected if spacetime had a foamlike structure on the scale of the Planck length. They would acquire an effective $\lambda \phi^4$ interaction with $\lambda \sim -1$ and, possibly, a tachyonic mass of order the Planck value. On the other hand, fermions and vector particles at energies small compared with the Planck mass would be very little affected. There would be, however, small effective interactions which might lead to the gravitational decay of the proton in about $10^{50}$ years.

The scale dependence of the gravitational action implies that there should be very large fluctuations of the metric and of the topology of spacetime on length scales of order the Planck length, $10^{-33}$ cm, or less. One is thus led to a picture of “spacetime foam” [1,2] in which the dominant contribution to the path integral comes from metrics with about one unit of topology per Planck volume. On the other hand, everyday observations indicate that spacetime is nearly flat when viewed on normal scales.

The aim of this paper is to show that common experience is not inconsistent with foamlike structure. We find that low-energy fermions or vector particles propagating through foam are very little affected, though there are some small interactions which might lead to the gravitational decay of the proton in about $10^{50}$ years. On the other hand, scalar particles would acquire an effective four-point vertex of order one and, possibly, a negative mass-squared term of order the Planck value. These effects would not be inconsistent with observation, since, as far as we know, there are no elementary scalar fields. They do, however, suggest that the Higgs mechanism, if it occurs at all, operates with bound states rather than with fundamental scalar fields, at least for electroweak interactions where the symmetry breaking occurs at energies low compared with the Planck energy.

We adopt the euclidean approach [3,4] in which the path integral is evaluated over all positive-definite metrics. We restrict attention to spacetime manifolds that are simply connected [2] because, firstly, it is impossible to classify non-simply-connected four-manifolds and, secondly, topology resulting from multiple connections is in a sense trivial and can be removed by “unwrapping” i.e. by going to the universal covering space. The quadratic form on the second homology group given by the intersection matrix between two-cycles classifies simply connected compact four-manifolds up to homotopy, and up to homeomorphisms if the Poincaré conjecture holds (as it almost certainly does). A non-compact asymptotically euclidean manifold can be represented as a compact manifold in which a point has been sent to infinity. It is therefore sufficient to consider only compact manifolds. The quadratic forms can be divided into two classes according to whether the self-intersection numbers of the fundamental cycles are odd or even.

In the former case the manifolds will not admit spin structures [5] though they can be given generalized spin structures [6]. These manifolds can be represented as the topological sum of $\frac{1}{2}(\chi - \tau - 2)$ copies of $\mathbb{C}P^2$ (complex projective two-space) and $\frac{1}{2}(\chi + \tau - 2)$ copies of $\overline{\mathbb{C}P^2}$, where the bar denotes the opposite orientation, $\chi$ is the Euler number (2 plus the rank of the quadratic form) and $\tau$ is the Hirzebruch signature. (The topological sum $X \# Y$ of two manifolds X and Y is formed by removing a small ball from each and gluing together the resulting boundaries.)
If the self-intersection numbers are even, the manifold admits a spin structure. Providing the intersection matrix is indefinite, i.e. $|\tau| \neq \chi - 2$, the manifold can be represented as the topological sum of $\frac{1}{8} \chi - \frac{1}{2} |\tau| - 1$ copies of $S^2 \times S^2$ and $\frac{1}{8} \tau$ copies of $K3$, Kummer's quartic surface (if $\tau < 0$, then it is $-\frac{1}{8} \tau$ copies of $\overline{K3}$ with the opposite orientation).

One can therefore regard spacetime foam as being made up of three basic kinds of "gravitational bubbles", $\mathbb{CP}^2$ in the odd case, and $S^2 \times S^2$ and $K3$ in the even case. We do not yet know how to treat the $K3$ bubbles because they necessarily have zero or negative eigenvalues for the conformally invariant scalar operator [7]. This means that if one performs the conformal transformation to send a point to infinity and obtain a metric with $R = 0$ (which one needs to evaluate the path integral [8]), the conformal factor passes through zero so the metric is singular [3,4]. However, one would expect that the number of $K3$ bubbles would be very small compared with the number of $S^2 \times S^2$ bubbles. Their contribution would be damped by the $|\tau|$ fermion zero modes that they must contain by the index theorem [9]. The zero modes give helicity-changing amplitudes for the fermions as in the Yang-Mills case but these would not give the fermions a mass unless there was only one species of Majorana spinors. We have, however, been able to compute the effects of certain $\mathbb{CP}^2$ and $S^2 \times S^2$ bubbles on the propagation of massless particles. The method is similar to that developed for the propagation of particles in a gas of Yang-Mills instantons [10].

One wants to compute the $S$-matrix elements between asymptotic plane wave in and out states in asymptotically euclidean spaces which have the topologies of $\mathbb{CP}^2$ or $S^2 \times S^2$ with a point removed. The first step is to compactify the asymptotically euclidean space by replacing the point $I$ at infinity and performing a conformal transformation to make the metric regular there. The conformal factor can be fixed on the light cone $\mathcal{O}$ of the point $I$ by the conformal gauge condition

$$R_{\mu \nu}u^\mu u^\nu = 0,$$

where $u^\mu$ is the null vector tangent to the light cone. This completely determines the conformal factor on $\mathcal{O}$ in terms of its value and gradient at $I$. Changing the value at $I$ corresponds to dilations of the bubble and changing the gradient corresponds to translations of the bubble. The asymptotic in and out states are then defined to have the same Cauchy data on $\mathcal{O}$ as they would have in compactified flat space [10]. One then calculates the amplitude in this metric by propagating the in states using the conformally invariant Green's function in the compactified space. This amplitude is weighted by the amplitude of the bubble, which is $(\det \Delta)^{-1/2} \exp(-\int \! [g^*])$, where $\Delta$ is the conformally invariant scalar operator and $g^*$ is the conformally related asymptotically euclidean metric with $R = 0$ [8]. By the positive action result [8,11–14], $\int \! [g^*]$ is positive if $g^*$ is nonsingular. One then averages over all bubble metrics and divides by the vacuum-to-vacuum amplitude obtained by averaging the amplitude for the bubbles.

If one averaged over all bubble metrics, one would obtain the full $S$-matrix amplitude. Unfortunately, we do not know the Green's function in a general bubble metric. However, we do know the Green's function in certain special metrics. These metrics have zero or self-dual conformal tensor and thus satisfy the Huijgens principle [15]. They might be expected to give the dominant contribution to the $S$ matrix. Restriction to these special metrics reduces the averaging over all metrics to a finite-dimensional integral. We would expect that averaging over more general metrics would change the $S$-matrix elements by numerical factors but would not alter their low-energy power-law dependence on the external momenta, which is fixed by dimensional considerations.

In the case of $\mathbb{CP}^2$ bubbles we take the standard Fubini–Study metric on $\mathbb{CP}^2$ [16–18],

$$ds^2 = (\partial^2 K/\partial \xi_i \partial \xi_j) d\xi_i d\xi_j, \quad i, j = 1, 2,$$  

where

$$K = \log(1 + \xi_1 \xi_1 + \xi_2 \xi_2).$$

The four quantities $\xi_1, \xi_2, \xi_2, \xi_2$ are regarded as complex coordinates. The euclidean section is given by $\xi_i = \xi_i$. The metric has self-dual conformal tensor and satisfies the Einstein equations with $\Lambda = 6$. It therefore obeys the conformal gauge condition (1). The conformally invariant scalar Green's function is

$$G(\xi, \xi') = [4\pi^2(1 - L)]^{-1},$$

$$L = \frac{(1 + \xi_1 \xi_1 + \xi_2 \xi_2)(1 + \xi_1 \xi_1 + \xi_2 \xi_2)}{(1 + \xi_1 \xi_1 + \xi_2 \xi_2)(1 + \xi_1 \xi_1 + \xi_2 \xi_2)}.$$
On the light cone of the point I, which can be taken to be at the origin, the denominator of the expression for $L$ becomes unity.

If one uses this Green's function to calculate the amplitude to propagate between two states of momenta $k_1$ and $k_2$ on $\mathcal{D}$, one obtains the flat-space contribution plus

$$0$$

or

$$(1/8\pi)\rho^2 J_0(\rho[-1/2 \cdot k_1 \cdot k_2 (1 + \cos^2 \theta)]^{1/2})$$

$$\times (1 + \cos^2 \theta) e^{i(k_1+k_2) \cdot \nu},$$

(6b)

where $\rho$, $\theta$ and $\nu$ describe the scale, orientation and position of the bubble. Which of the two values (6) one obtains is determined by the contours of integration on $\mathcal{D}$. The action $I [g^*] = \frac{1}{4} \pi \rho^2$ in units in which $G = c = \hbar = 1$. It is not clear what measure one should use on the integration over scale sizes $\rho$, but, provided it is of the form $\rho^a$ where $a > -1$, the dominant contribution will come from $\rho$ of order one, i.e. the Planck length. For low-energy particles, $k_1 \cdot k_2$ will be very much less than one, so the Bessel function will be nearly one. The averaging over orientations $\theta$ will ensure angular momentum conservation and the averaging over positions $\nu$ of the bubble will ensure momentum conservation. In the case of two particles propagating in a bubble, there seems to be an effective four-point vertex corresponding to an interaction $\lambda \rho^4$ with $\lambda \sim -1$. In the case of a single particle propagating in a bubble it is not clear which contour of integration to choose, but the one that gives a non-zero amplitude would correspond to an effective tachyonic mass of order the Planck mass.

One can choose a conformal frame in which these bubbles are flat space with $n$ points identified. The conformally invariant Green’s functions are thus just the ordinary flat space Green’s functions but with $n - 1$ extra image charges. The amplitude for a particle to propagate between states of momenta $k_1$ and $k_2$ in the $n = 2$ case will be the flat space value plus

$$0$$

or

$$(1/8\pi)q^2 J_{2\nu}((k_1 \cdot q)(k_2 \cdot q) - \frac{1}{2} q^2 k_1 \cdot k_2)^{1/2})$$

$$\times e^{i(k_1+k_2) \cdot \nu},$$

(7b)

where $s$ is the spin of the field, $q$ is a four-vector describing the scale and orientation of the bubble and $\nu$ is the position. The action $I [g^*] = \frac{3}{8} \pi q^2$. The results for the higher $n$ cases are similar.

Scalar particles will again acquire an effective four-point vertex with $\lambda \sim -1$ and, possibly, a tachyonic effective mass of the Planck value. On the other hand, the Bessel functions $J_1$ and $J_2$ are small for small arguments. This means that fermions and vectors will not acquire a mass. There will be effective four-point interactions of the form $(\bar{\psi} \psi)^2/m_p^2$ and $(F_{\mu\nu})^4/m_p^4$, where $m_p$ is the Planck mass. The vector interaction would be completely negligible compared with vacuum polarization effects which are of order $(F_{\mu\nu})^4 \times e^4/m_e^4$. The fermion interactions would be small compared to weak interactions which are of order $(\bar{\psi} \psi)^2/m_w^2$ but they might be distinguished by certain acausal properties which would be the analogue of black hole processes and which might lead to the gravitational decay of the proton in about $10^{50}$ years (e.g., by two quarks turning into an antiquark and a neutrino). This lifetime is longer than the $10^{31}$ years predicted by some of the grand unified theories [22] on the basis of a grand unified mass of about $10^{14}$ GeV. If, however, the unification does not occur until one gets
to the Planck energy, the predicted lifetime would be of the same order as that of gravitational decay.

Further details of these results will be published elsewhere.

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References