Lectures on Supergravity

Joaquim Gomis

Based on the SUGRA book of Dan Freedman and Antoine Van Proeyen to appear in Cambridge University Press

Public Material

Lectures on supergravity, Amsterdam-Brussels-Paris doctoral school, Paris 2009, October-November 2009: PDF-file.

http://itf.fys.kuleuven.be/~toine/SUGRA_DoctSchool.pdf

A. Van Proeyen, Tools for supersymmetry, hep-th 9910030

History and overview of Supergravity

60’ and 70’s. Yang Mills theories, Spontaneous symmetry breaking.
Standard model

Internal symmetry group G	→	spontaneous breakdown	→	Goldstone particles, s=0	→	Goldstone particles
	→	localization	→	Y–M gauge fields, s=1		Gauge fields Higgs effect

Supersymmetry

Yu. Gol’fand , E. Lichtman (1971)
J.L. Gervais and B. Sakita (1971)
A,Neveu, J. Schwarz, P.Ramond (1971)
D. Volkov, V. Akulov (1972)
J. Wess, B. Zumino (1974)

History and overview of Supergravity

- Yu. Gol'fand , E. Lichtman-→Parity violation in QFT, 4d
- J.L. Gervais and B. Sakita & A,Neveu, J. Schwarz→ String theory-Dual models. Worls sheet supersymmetry 2d
- D. Volkov, V. Akulov-→ Goldstone particles of spin ½? 4d
- J. Wess, B. Zumino → Supersymmetric field theory in 4d

$$\boxed{\begin{array}{l}\text{May the Goldstone fermion}\\ \text{with } \mathrm{s} = \mathbf{1/2} \text{ exist ?}\end{array}} \Longrightarrow \boxed{\begin{array}{l}\text{Super–Poincaré}\\ \text{group,}\\ \text{Supergravity}\end{array}} \longleftarrow \boxed{\begin{array}{l}\text{A group}\\ \text{of which kind?}\end{array}}$$

Supergroup, superalgebra

History and overview of Supergravity

- Super Poincare

Translations P_μ Lorentz transformations $J_{[\nu\sigma]}$

Spinor supercharge Q_α

Massless multiplets contains spins (s, s-1/2), for s=1/2, 1, 2,

R symmetry

History and overview of Supergravity

Supergravity

Gauged supersymmetry was expected to be an extension of general Relativity with a superpartner of the gravito call gravition

$e^a_\mu(x)$ $\psi_{\mu\alpha}(x)$ Multiplet (2,3/2)

S. Ferrara, D. Freedman, P. Van Nieuwenhuizen (1976)
S. Deser, B. Zumino (1976)
D. Volkov, V. Soroka (1973), massive gravitinos,..

Extensions with more supersymmetries and extension has been considered, N=2 supergravity, special geometry. N=1 Supergravity in 11d

Motivation for Supergravity

Supergravity (SUGRA) is an extension of Einstein's general relativity to include supersymmetry (SUSY). General relativity demands extensions since it has shortcomings including at least the following:

Motivation for Supergravity

- Space time singularities. The singularity theorems of Penrose, Hawking and Geroch shows that general relativity is incomplete.
- Failure to unify gravity with the strong and electro weak forces.
- Einstein gravity is not power counting renormalizable. It is renormalizable as an effective theory. It is not a fundamental theory
- If we include supersymmetry in a theory of gravity. The simple example of divergences: zero point energy of the vacuum, can potentially be cancelled by super partners of ordinary particles

The current status of supergravity

- A reliable approximation to M-theory.
- An essential ingrediente for supersymmetric phenomenology (minimal supersymmetric estándar model coupled to N=1 supergravity).
- Applications in cosmology
- An crucial part for the AdS/CFT correspondence

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- Scalar field and its symmetries
- The Dirac Field
- Clifford algebras ans spinors
- The Maxwell and Yang-Mills Gauge fields
- Free Rarita-Schwinger field
- Differential geometry
- First and second order formulation of gravity
- N=1 Global Supersymmetry in D=4

Index

- N=1 pure supergravity in 4 dimensions
- D=11 supergravity

Scalar field

Metric (-,+,+,+…+)

our fields satisfy the Klein-Gordon equation

$$\boxed{\Box\phi^i(x) = m^2\phi^i(x)\,,}$$

$\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu$ is the Lorentz invariant d'Alembertian wave operator.

Symmetry transformations

$$\phi^i(x) \;\to\; \phi'^i(x)$$

map solutions into solutions

Noether symmetry leaves the action invariant

$$S[\phi^i] = S[\phi'^i]\,.$$

General internal symmetry

We will be interested in an n-dimensional representation of G in which the generators of its Lie algebra are a set of $n \times n$ matrices $(t_A)^i{}_j$, $A = 1, 2, \ldots, \dim G$. Their commutation relations are[5]

$$[t_A, t_B] = f_{AB}{}^C t_C \, ,$$

and the $f_{AB}{}^C$ are structure constants of the Lie algebra. The representative of a general element of the Lie algebra is a matrix Θ which is a superposition of the generators with real parameters θ^A, i.e.

$$\Theta = \theta^A t_A .$$

$$U(\Theta) = \mathrm{e}^{-\Theta} = \mathrm{e}^{-\theta^A t_A} \, .$$

$$\phi^i(x) \rightarrow \phi'^i(x) \equiv U(\Theta)^i{}_j \phi^j(x) \, .$$

Infinitesimal transformations

$$\delta\phi = -\Theta\phi \, ,$$

General internal symmetry

Commutator of infinitesimal transformations

$$\begin{aligned}
[\delta_1, \delta_2]\phi &= -[\Theta_1, \Theta_2]\phi \equiv \delta_3\phi \, , \\
\Theta_3 &= [\Theta_1, \Theta_2] = f_{AB}{}^{C}\theta_1^A\theta_2^B t_C \, .
\end{aligned}$$

Spacetime symmetries

Lorentz group

$$x^\mu \;=\; \Lambda^\mu{}_\nu x'^\nu \qquad \text{or} \qquad x'^\mu \;=\; \Lambda^{-1\mu}{}_\nu x^\nu\,,$$

Lorentz condition

$$\boxed{\Lambda^\mu{}_\rho \eta_{\mu\nu} \Lambda^\nu{}_\sigma = \eta_{\rho\sigma}\,.}$$

Poincaré group $\quad x'^\mu \;=\; \Lambda^{-1\mu}{}_\nu \left(x^\nu \;-\; a^\nu\right)\,.$

Relations among Lorentz transformations

$$\begin{aligned} \Lambda_{\mu\nu} \;&=\; (\Lambda^{-1})_{\nu\mu}\,, \qquad \Lambda^\mu{}_\nu = (\Lambda^{-1})_\nu{}^\mu\,, \\ x'_\mu \;&=\; (\Lambda^{-1})_\mu{}^\nu x_\nu \;=\; x_\nu \Lambda^\nu{}_\mu\,. \end{aligned}$$

infinitesimal transformation

$$\Lambda^\mu{}_\nu = \delta^\mu_\nu + \epsilon m^\mu{}_\nu + \cdots\,. \qquad m_{\mu\nu} \equiv \eta_{\mu\rho} m^\rho{}_\nu = -m_{\nu\mu}\,.$$

Vector representation

$$m_{[\rho\sigma]}{}^\mu{}_\nu \equiv \delta^\mu_\rho \eta_{\nu\sigma} - \delta^\mu_\sigma \eta_{\rho\nu} \;=\; -m_{[\sigma\rho]}{}^\mu{}_\nu\,. \qquad \Lambda = \mathrm{e}^{\frac{1}{2}\lambda^{\rho\sigma} m_{[\rho\sigma]}}\,.$$

Spacetime symmetries

Lorentz algebra

$$\boxed{[m_{[\mu\nu]}, m_{[\rho\sigma]}] = \eta_{\nu\rho} m_{[\mu\sigma]} - \eta_{\mu\rho} m_{[\nu\sigma]} - \eta_{\nu\sigma} m_{[\mu\rho]} + \eta_{\mu\sigma} m_{[\nu\rho]} \,.}$$

Let $\psi^i(x)$
denote a set of fields with i an index of the components of a general representation. There is a corresponding Lie algebra representation with matrices $m_{[\rho\sigma]}$ which act on the indices and differential/matrix generators

$$J_{[\rho\sigma]} = L_{[\rho\sigma]} \mathbb{1} + m_{[\rho\sigma]} \,,$$

$$\psi(x) \to \psi'(x) = U(\Lambda)\psi(x) = \mathrm{e}^{-\frac{1}{2}\lambda^{\rho\sigma} m_{[\rho\sigma]}} \psi(\Lambda x) \,.$$

Orbital part $$L_{[\rho\sigma]} \equiv x_\rho \partial_\sigma - x_\sigma \partial_\rho \,.$$

$$\begin{aligned}
[J_{[\mu\nu]}, J_{[\rho\sigma]}] &= \eta_{\nu\rho} J_{[\mu\sigma]} - \eta_{\mu\rho} J_{[\nu\sigma]} - \eta_{\nu\sigma} J_{[\mu\rho]} + \eta_{\mu\sigma} J_{[\nu\rho]} \,, \\
[J_{[\rho\sigma]}, P_\mu] &= P_\rho \eta_{\sigma\mu} - P_\sigma \eta_{\rho\mu} \,, \\
[P_\mu, P_\nu] &= 0 \,.
\end{aligned}$$

Noether charges

$\Delta_A \phi^i$ Infinitesimal Noether symmetry

$$\delta\mathcal{L} \equiv \epsilon^A \left[\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi^i}\partial_\mu\Delta_A\phi^i + \frac{\delta\mathcal{L}}{\delta\phi^i}\Delta_A\phi^i \right] = \epsilon^A\partial_\mu K^\mu_A$$

Noether current

$$J^\mu{}_A = -\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi^i}\Delta_A\phi^i + K^\mu_A$$

Noether trick. Consider $\epsilon^A(x)$

$$\begin{aligned}
\delta S &= \int \mathrm{d}^D x \left[\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi^i}\partial_\mu(\epsilon^A\Delta_A\phi^i) + \frac{\delta\mathcal{L}}{\delta\phi^i}\epsilon^A\Delta_A\phi^i \right] \\
&= \int \mathrm{d}^D x \left[\epsilon^A\partial_\mu K^\mu_A + (\partial_\mu\epsilon^A)\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi^i}\Delta_A\phi^i \right] \\
&= -\int \mathrm{d}^D x \, (\partial_\mu\epsilon^A) J^\mu_A \, .
\end{aligned}$$

Noether charges

$$Q_A = \int \mathrm{d}^{D-1}\vec{x}\, J^0{}_A(\vec{x},t)$$

For internal symmetries

$$Q_A = -\int \mathrm{d}^{D-1}\vec{x}\, \pi_i \Delta_A \phi^i$$

Hamiltonian formalism

$$\Delta_A \phi^i(x) = \{Q_A, \phi^i(x)\} = -\int \mathrm{d}^{D-1}\vec{y}\{\pi_j(\vec{y})\Delta_A\phi^j(\vec{y}), \phi^i(x)\}$$

$$\{Q_A\,,\,Q_B\} = f_{AB}{}^C Q_C$$

Noether charges

At quantum level

$$\begin{aligned} \Delta_A \mathbf{\Phi}^i &= -\mathrm{i}\left[\mathbf{Q}_A, \mathbf{\Phi}^i\right]_{\mathrm{qu}} \\ \left[\mathbf{Q}_A\,,\,\mathbf{Q}_B\right]_{\mathrm{qu}} &= \mathrm{i}\, f_{AB}{}^C \mathbf{Q}_C\,. \end{aligned}$$

The homomorphism of $\mathrm{SL}(2,\mathbb{C}) \to \mathrm{SO}(3,1)$

Spinor representations exist for all spacetime dimensions D.

$D = 4$ a general spinor representation is labeled $(j\,,\,j')$

The fundamental spinor representations

$(\frac{1}{2},0)$ and its complex conjugate $(0,\frac{1}{2})$

$$\sigma_\mu = (\,-\mathbb{1},\sigma_i\,)\;, \qquad \bar{\sigma}_\mu = \sigma^\mu = (\,\mathbb{1},\sigma_i\,)$$

Properties

$$\begin{aligned} \sigma_\mu\bar{\sigma}_\nu + \sigma_\nu\bar{\sigma}_\mu &= 2\eta_{\mu\nu}\mathbb{1}\,, \\ \mathrm{tr}(\sigma^\mu\,\bar{\sigma}_\nu) &= 2\delta^\mu{}_\nu\,. \end{aligned}$$

Hermitean matrix $\quad \mathbf{x} = \bar{\sigma}_\mu x^\mu\,, \qquad x^\mu = \frac{1}{2}\,\mathrm{tr}(\sigma^\mu\mathbf{x})$

The transformation $\quad \mathbf{x} \to \mathbf{x}' \equiv A\mathbf{x}A^\dagger$ induces a Lorentz transformation

The homomorphism of $\mathrm{SL}(2,\mathbb{C}) \to \mathrm{SO}(3,1)$

Lie algebra $\mathfrak{so}(3,1)$ in the $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$ representations

$$\sigma_{\mu\nu} = \tfrac{1}{4}(\sigma_\mu\bar{\sigma}_\nu - \sigma_\nu\bar{\sigma}_\mu)$$
$$\bar{\sigma}_{\mu\nu} = \tfrac{1}{4}(\bar{\sigma}_\mu\sigma_\nu - \bar{\sigma}_\nu\sigma_\mu)$$

The finite Lorentz transformation

$$\begin{aligned} L(\lambda) &= \mathrm{e}^{\frac{1}{2}\lambda^{\mu\nu}\sigma_{\mu\nu}} \\ \bar{L}(\lambda) &= \mathrm{e}^{\frac{1}{2}\lambda^{\mu\nu}\bar{\sigma}_{\mu\nu}} \end{aligned}$$

The Dirac Field

Dirac postulated that the electron is described by a complex valued multi-component field $\Psi(x)$ called a spinor field, which satisfies the first order wave equation

$$\boxed{\partial\!\!\!/\Psi(x) \equiv \gamma^\mu \partial_\mu \Psi(x) = m\Psi(x)\,.}$$

Applying the Dirac operator

$$\begin{aligned} \partial\!\!\!/^2 \Psi &= m^2 \Psi\,, \\ \tfrac{1}{2}\{\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu\}\partial_\mu \partial_\nu \Psi &= m^2 \Psi\,. \end{aligned}$$

Clifford algebra

$$\boxed{\{\gamma^\mu, \gamma^\nu\} \equiv \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\,\eta^{\mu\nu}\,\mathbb{1}\,,}$$

The Dirac Field

Explicit representation for D=4 in terms of 2×2 Weyl matrices

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} .$$

Lorentz covariance property of the Dirac equation;

$$\Psi'(x) = L(\lambda)^{-1}\Psi(\Lambda(\lambda)x)$$

$$L(\lambda)\gamma^\rho L(\lambda)^{-1} = \gamma^\sigma \Lambda(\lambda)_\sigma{}^\rho .$$

Finite Lorentz transformations $\quad L(\lambda) = \mathrm{e}^{\frac{1}{2}\lambda^{\mu\nu}\Sigma_{\mu\nu}} ,$

$$\Sigma^{\mu\nu} \equiv \frac{1}{4}\left[\gamma^\mu, \gamma^\nu\right] ,$$

The Dirac Field

We then define the Dirac adjoint, a row vector, by

$$(AB)^\dagger \equiv B^\dagger A^\dagger \qquad \bar{\Psi} = \Psi^\dagger \beta = \Psi^\dagger \mathrm{i}\gamma^0 \, ,$$

Dirac action

$$S[\bar{\Psi}, \Psi] = -\int \mathrm{d}^D x \bar{\Psi}[\gamma^\mu \partial_\mu - m]\Psi(x)$$

Equation of motion for adjoint spinor

$$\bar{\Psi}[\gamma^\mu \overleftarrow{\partial}_\mu + m] = 0$$

Weyl spinors

even dimension $D = 2m$ Weyl field $\psi(x)$ with $2^{(m-1)}$ components.

$$\Psi'(x) = L(\lambda)^{-1}\Psi(\Lambda(\lambda)x)$$

Undotted components

$$\Psi'_\alpha(x) = (L(\lambda)^{-1})_\alpha{}^\beta \Psi_\alpha(\Lambda x)$$

field $\bar{\chi}(x)$ that transforms in the conjugate representation

$$\bar{\chi}(x) \to \bar{\chi}'(x) = \bar{L}(\lambda)^{-1}\bar{\chi}(\Lambda(\lambda)x)$$

Dotted components

$$\bar{\chi}^{\dot{\alpha}}(x) = (\bar{L}(\lambda)^{-1})^{\dot{\alpha}}{}_{\dot{\beta}}\bar{\chi}^{\dot{\beta}}(\Lambda x))$$

Weyl spinors

Write the Dirac field as

$$\Psi(x) = \begin{pmatrix} \psi(x) \\ \bar{\chi}(x) \end{pmatrix}$$

Dirac Lagrangian

$$\mathcal{L} = \mathrm{i}\left[-\psi^\dagger\,\bar{\sigma}\cdot\partial\,\psi + \bar{\chi}^\dagger\,\sigma\cdot\partial\,\bar{\chi} - m\,\bar{\chi}^\dagger\,\psi + m\,\psi^\dagger\,\bar{\chi}\right]$$

Energy momentum tensor

symmetric energy-momentum tensor

$$\Theta_{\mu\nu} = \frac{1}{4}\bar{\Psi}(\gamma_\mu \overleftrightarrow{\partial}_\nu + \gamma_\nu \overleftrightarrow{\partial}_\mu)\Psi + \eta_{\mu\nu}\mathcal{L}'$$

where $$\mathcal{L}' \quad = \quad \frac{1}{2}\bar{\Psi}\gamma^\mu \overleftrightarrow{\partial}_\mu \Psi - m\bar{\Psi}\Psi$$

$$A\overleftrightarrow{\partial}_\mu B \equiv A(\partial_\mu B) - (\partial_\mu A)B$$

Clifford algebras and spinors

- Clifford algebras in general dimensions

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\eta^{\mu\nu}\,\mathbb{1}\,.$$

Euclidean Clifford algebras

$$\begin{aligned}
\gamma^1 &= \sigma_1 \otimes \mathbb{1} \otimes \mathbb{1} \otimes \ldots \\
\gamma^2 &= \sigma_2 \otimes \mathbb{1} \otimes \mathbb{1} \otimes \ldots \\
\gamma^3 &= \sigma_3 \otimes \sigma_1 \otimes \mathbb{1} \otimes \ldots \\
\gamma^4 &= \sigma_3 \otimes \sigma_2 \otimes \mathbb{1} \otimes \ldots \\
\gamma^5 &= \sigma_3 \otimes \sigma_3 \otimes \sigma_1 \otimes \ldots \\
\ldots &= \ldots\,.
\end{aligned}$$

Clifford algebras and spinors

These matrices are all hermitian with squares equal to $\mathbb{1}$, and they mutually anticommute. Suppose that $D = 2m$ is even. Then we need m factors in the construction (3.2) to obtain γ^μ, $1 \leq \mu \leq D = 2m$. Thus we obtain a representation of dimension $2^{D/2}$. For odd $D = 2m + 1$ we need one additional matrix, and we take γ^{2m+1} from the list above, but we keep only the first m factors, i.e. deleting a σ_1. Thus there is no increase in the dimension of the representation in going from $D = 2m$ to $D = 2m + 1$, and we can say in general that the construction (3.2) gives a representation of dimension $2^{[D/2]}$, where $[D/2]$ means the integer part of $D/2$.