Clifford algebras and spinors

we need Lorentzian γ's , all we need to do is pick

any single matrix from the Euclidean construction, multiply it by i and label it γ^0 for the time-like direction. This matrix is anti-hermitian and satisfies $(\gamma^0)^2 = -\mathbb{1}$. We then relabel the remaining $D-1$ matrices to obtain the Lorentzian set γ^μ, $0 \le \mu \le D-1$. The hermiticity properties of the Lorentzian γ's are summarized by

$$\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0$$

Clifford algebras and spinors

The full Clifford algebra consists of the identity $\mathbb{1}$, the D generating elements γ^μ, plus all independent matrices formed from products of the generators. Since symmetric products reduce to a product containing fewer γ-matrices by (3.1), the new elements must be antisymmetric products. We thus define

$$\gamma^{\mu_1\ldots\mu_r} = \gamma^{[\mu_1}\ldots\gamma^{\mu_r]}\,, \qquad \text{e.g.} \qquad \gamma^{\mu\nu} = \tfrac{1}{2}\gamma^\mu\gamma^\nu - \tfrac{1}{2}\gamma^\nu\gamma^\mu\,,$$

Non-vanishing tensor components

$$\gamma^{\mu_1\mu_2\ldots\mu_r} = \gamma^{\mu_1}\gamma^{\mu_2}\cdots\gamma^{\mu_r} \qquad \text{for } \mu_1 \neq \mu_2 \neq \cdots \neq \mu_r\,.$$

The antysymmetrization indicated with […] is always with total weight 1

C_r^D distinc indexes choices

Clifford algebras and spinors

properties

$$\begin{aligned}
\gamma^{\mu\nu} &= \frac{1}{2}[\gamma^{\mu}, \gamma^{\nu}]\,, \\
\gamma^{\mu_1\mu_2\mu_3} &= \frac{1}{2}\{\gamma^{\mu_1}, \gamma^{\mu_2\mu_3}\}\,, \\
\gamma^{\mu_1\mu_2\mu_3\mu_4} &= \frac{1}{2}[\gamma^{\mu_1}, \gamma^{\mu_2\mu_3\mu_4}]\,,
\end{aligned}$$

$$\gamma^{[\mu\nu\rho]} = \gamma^{\mu}\gamma^{[\nu\rho]} - \eta^{\nu\mu}\gamma^{\rho} + \eta^{\nu\rho}\gamma^{\mu}$$

Levi-Civita tensor

$$\varepsilon_{012(D-1)} = 1\,, \qquad \varepsilon^{012(D-1)} = -1\,.$$

$$\varepsilon_{\mu_1\ldots\mu_n\nu_1\ldots\nu_p}\varepsilon^{\mu_1\ldots\mu_n\rho_1\ldots\rho_p} = -\,p!\,n!\,\delta^{\rho_1\cdots\rho_p}_{\nu_1\cdots\nu_p}\,, \qquad p = D-n$$

$$\delta^{\nu_1\ldots\nu_q}_{\rho_1\cdots\rho_p} \;\equiv\; \delta^{\nu_1}_{[\rho_1}\delta^{\nu_2}_{\rho_2}\cdots\delta^{\nu_p}_{\rho_p]}\,,$$

Schouten identity

$$0 \;=\; 5\delta_\mu{}^{[\nu}\varepsilon^{\rho\sigma\tau\lambda]} \equiv \delta_\mu{}^\nu\varepsilon^{\rho\sigma\tau\lambda} + \delta_\mu{}^\rho\varepsilon^{\sigma\tau\lambda\nu} + \delta_\mu{}^\sigma\varepsilon^{\tau\lambda\nu\rho} + \delta_\mu{}^\tau\varepsilon^{\lambda\nu\rho\sigma} + \delta_\mu{}^\lambda\varepsilon^{\nu\rho\sigma\tau}$$

Practical gamma matrix manipulation

Consider first products with index contractions such as

$$\gamma^{\mu\nu}\gamma_{\nu} = (D-1)\gamma^{\mu}\,.$$

You can memorize this rule, but it is easier to recall the simple logic behind it: ν runs over all values except μ, so there are $(D-1)$ terms in the sum. Similar logic

$$\gamma^{\mu\nu\rho}\gamma_{\rho} = (D-2)\gamma^{\mu\nu}\,,$$

More generally

$$\gamma^{\mu_1\ldots\mu_r\nu_1\ldots\nu_s}\gamma_{\nu_s\ldots\nu_1} = \frac{(D-r)!}{(D-r-s)!}\gamma^{\mu_1\ldots\mu_r}\,.$$

Practical gamma matrix manipulation

Reverse ordering $$\gamma^{\nu_1\ldots\nu_r} = (-)^{r(r-1)/2}\gamma^{\nu_r\ldots\nu_1}$$

The sign factor $(-)^{r(r-1)/2}$ is negative for $r = 2, 3 \bmod 4$

No index contractions

$$\gamma^\mu\gamma^\nu = \gamma^{\mu\nu} + \eta^{\mu\nu} \, .$$

Useful to prove the susy invariance of the supergravity action

$$\gamma^{\mu\nu\rho}\gamma_{\sigma\tau} = \gamma^{\mu\nu\rho}{}_{\sigma\tau} + 6\gamma^{[\mu\nu}{}_{[\tau}\delta^{\rho]}{}_{\sigma]} + 6\gamma^{[\mu}\delta^{\nu}{}_{[\tau}\delta^{\rho]}{}_{\sigma]}$$

Practical gamma matrix manipulation

- Other useful relations

$$\gamma^{\mu_1\ldots\mu_4}\gamma_{\nu_1\nu_2} = \gamma^{\mu_1\ldots\mu_4}{}_{\nu_1\nu_2} + 8\gamma^{[\mu_1\ldots\mu_3}{}_{[\nu_2}\delta^{\mu_4]}{}_{\nu_1]} + 12\gamma^{[\mu_1\mu_2}\delta^{\mu_3}{}_{[\nu_2}\delta^{\mu_4]}{}_{\nu_1]}$$

In general

$$\begin{aligned}\gamma^{\mu_1\cdots\mu_p}\gamma_{\nu_1\ldots\nu_q} &= \gamma^{\mu_1\cdots\mu_p}{}_{\nu_1\ldots\nu_q} + pq\delta^{[\mu_p}_{[\nu_1}\gamma^{\mu_1\cdots\mu_{p-1}]}{}_{\nu_2\ldots\nu_q]} + \ldots \\ &\ldots + r!\begin{pmatrix} p \\ r \end{pmatrix}\begin{pmatrix} q \\ r \end{pmatrix}\delta^{[\mu_p}_{[\nu_1}\cdots\delta^{\mu_{p-r+1}}_{\nu_r}\gamma^{m_1\cdots\mu_{p-r}]}{}_{\nu_{r+1}\ldots\nu_q]} + .\end{aligned}$$

$$\begin{aligned}\gamma^{\mu_1\cdots\mu_4\rho}\gamma_{\rho\nu_1\nu_2} &= (D-6)\gamma^{\mu_1\cdots\mu_4}{}_{\nu_1\nu_2} + 8(D-5)\gamma^{[\mu_1\cdots\mu_3}{}_{[\nu_2}\delta^{\mu_4]}{}_{\nu_1]} \\ &+12(D-4)\gamma^{[\mu_1\mu_2}\delta^{\mu_3}{}_{[\nu_2}\delta^{\mu_4]}{}_{\nu_1]}\,.\end{aligned}$$

Basis of the algebra for even dimensions

The basis is denoted by the following list $\{\Gamma^A\}$ of matrices

$$\{\Gamma^A = \mathbb{1}, \gamma^\mu, \gamma^{\mu_1\mu_2}, \gamma^{\mu_1\mu_2\mu_3}, \cdots, \gamma^{\mu_1,\cdots\mu_D}\} \; \mu_1 < \mu_2 <, \ldots < \mu_r$$

C_r^D index choices at each rank r and a total of 2^D matrices.

Other possible basis $\{\Gamma_A = \mathbb{1}, \gamma_\mu, \gamma_{\mu_2\mu_1}, \gamma_{\mu_3\mu_2\mu_1}, \cdots, \gamma_{\mu_D,\cdots\mu_1}\}$

$$\Gamma^A\Gamma^B = \pm\Gamma^C,$$

$$\mathrm{Tr}(\Gamma^A\Gamma_B) = 2^m\,\delta^A_B\,.$$

$$M = \sum_A m_A\Gamma^A, \qquad m_A = \frac{1}{2^m}\mathrm{Tr}(M\Gamma_A)$$

The highest rank Clifford algebra element

$$\gamma_* \equiv (-\mathrm{i})^{m+1}\gamma_0\gamma_1\cdots\gamma_{D-1}$$

Provides the link bewteen even and odd dimensions

$\gamma_*^2 = \mathbb{1}$ in every even dimension and is hermitian

$D = 2m$, the matrix γ_* is frequently called γ_{D+1}

Properties

$$\gamma_{\mu_1\mu_2\cdots\mu_D} = \mathrm{i}^{m+1}\varepsilon_{\mu_1\mu_2\cdots\mu_D}\,\gamma_*$$

$$\begin{aligned} \{\gamma_*, \gamma^\mu\} &= 0\,, \\ [\gamma_*, \gamma^{\mu\nu}] &= 0\,. \end{aligned}$$

Since $\gamma_*^2 = \mathbb{1}$ and $\mathrm{Tr}\,\gamma_* = 0$

$$\gamma_* = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}$$

Explicit representations

Assume

$$\gamma^\mu = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

$$\{\gamma_*, \gamma^\mu\} \quad = \quad 0$$

implies

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}$$

$\sigma^\mu \;\; and \; \bar{\sigma}^\mu \;\; are \; 2^{m-1} \times 2^{m-1}$ $\qquad Weyl \; matrices$

$$\sigma_\mu \bar{\sigma}_\nu + \sigma_\nu \bar{\sigma}_\mu \quad = \quad 2\eta_{\mu\nu}\mathbb{1}$$

$$\mathrm{Tr}(\sigma^\mu \bar{\sigma}_\nu) \; = \; 2^{(m-1)}\delta^\mu_\nu$$

Explicit representations

$$[\gamma_*, \gamma^{\mu\nu}] \quad = \quad 0$$ implies

$$\Sigma^{\mu\nu} = \frac{1}{2}\gamma^{\mu\nu} = \frac{1}{4}\begin{pmatrix} \sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu & 0 \\ 0 & \bar{\sigma}^\mu\sigma^\nu - \bar{\sigma}^\mu\sigma^\nu \end{pmatrix}$$

the Dirac representation of $\mathfrak{so}(D-1,1)$ is reducible (for even D)

Weyl spinors

chiral projectors

$$P_L = \tfrac{1}{2}(\mathbb{1} + \gamma_*)\,, \qquad P_R = \tfrac{1}{2}(\mathbb{1} - \gamma_*)$$

$$\begin{pmatrix} \psi \\ 0 \end{pmatrix} \equiv P_L \Psi\,, \qquad \begin{pmatrix} 0 \\ \bar{\chi} \end{pmatrix} \equiv P_R \Psi$$

$$P_L P_L \; = \; P_L, \;\; P_R P_R \; = \; P_R \;\; and \;\; P_L P_R \; = \; 0$$

No explicity Weyl representation will be used in these lectures

Odd space dimension D=2m+1

The Clifford algebra for dimension D=2m+1 can be obtained by reorganazing the matrices in the Clifford algebra for dimension D= 2m

we can define *two sets* of $2m+1$ generating elements

$$\gamma_{\pm}{}^{\mu} = (\gamma^0, \gamma^1, \ldots, \gamma^{(2m-1)}, \gamma^{2m} = \pm\gamma_*)$$

Not all the matrices are independent

The rank r and rank D-r sectors are related by duality relations

$$\gamma_{\pm}^{\mu_1\ldots\mu_r} = \pm \mathrm{i}^{m+1} \frac{1}{(D-r)!} \varepsilon^{\mu_1\ldots\mu_D} \gamma_{\pm\,\mu_D\ldots\mu_{r+1}}$$

Odd space dimension D=2m+1

we already have enough matrices if we consider the matrices up to $\gamma_{\mu_1 \ldots \mu_{(D-1)/2}}$

$$\{\Gamma^A = \mathbb{1}, \gamma^{\mu}, \gamma^{\mu_1\mu_2}, \gamma^{\mu_1\mu_2\mu_3}, \quad \cdots, \gamma_{\mu_1 \ldots \mu_{(D-1)/2}}\}$$

Symmetries of gamma matrices

the $2^m \times 2^m$ matrices, for both $D = 2m$ and $D = 2m + 1$

distinguish between symmetric and the antisymmetric

exists a unitary matrix charge conjugation matrix, such that

$$(C\Gamma^{(r)})^T = -t_r C\Gamma^{(r)}\,, \qquad t_r = \pm 1$$

implies

$$C^T = -t_0 C\,, \qquad \gamma^{\mu T} = t_0 t_1\, C \gamma^\mu C^{-1}$$

Explicit forms conjugation matrix

$$\begin{aligned} C_+ &= \sigma_1 \otimes \sigma_2 \otimes \sigma_1 \otimes \sigma_2 \otimes \ldots \qquad && t_0 t_1 = 1\,, \\ C_- &= \sigma_2 \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_1 \otimes \ldots \qquad && t_0 t_1 = -1 \end{aligned}$$

For odd dimension C is unique (up to phase factor)

The possible sign factors depend on the spacetime dimension D modulo 8
And on r modulo 4

Symmetries of gamma matrices

D (mod 8)	$t_r = -1$	$t_r = +1$
0	0, 3	2, 1
	0, 1	**2, 3**
1	0, 1	2, 3
2	0, 1	2, 3
	1, 2	**0, 3**
3	1, 2	0, 3
4	**1, 2**	**0, 3**
	2, 3	0, 1
5	2, 3	0, 1
6	2, 3	0, 1
	0, 3	**1, 2**
7	0, 3	1, 2

Symmetries of gamma matrices

- Since we use hermitian representations, the symmetry properties of gamma matrices determines also its complex conjugation

$$\gamma^{\mu *} = -t_0 t_1 \, B \gamma^\mu B^{-1} \qquad B = \mathrm{i} t_0 C \gamma^0$$

$$B^* B = -t_1 \mathbb{1}$$

Adjoint spinor

- We have defined the Dirac adjoint, which involves the complex conjugate. Here we define the conjugate of "any" spinor using the transpose and the charge conjugation matrix

$$\bar{\lambda} \equiv \lambda^T C$$

Symmetry properties for bilinears

$$\bar{\lambda}\gamma_{\mu_1\ldots\mu_r}\chi = t_r\bar{\chi}\gamma_{\mu_1\ldots\mu_r}\lambda$$

Majorana flip

More in general

$$\bar{\lambda}\Gamma^{(r_1)}\Gamma^{(r_2)}\cdots\Gamma^{(r_p)}\chi = t_0^{p-1}t_{r_1}t_{r_2}\cdots t_{r_p}\,\bar{\chi}\Gamma^{(r_p)}\cdots\Gamma^{(r_2)}\Gamma^{(r_1)}\lambda$$

Adjoint spinor

We have the rule

$$\chi_{\mu_1\ldots\mu_r} = \gamma_{\mu_1\ldots\mu_r}\lambda \implies \bar{\chi}_{\mu_1\ldots\mu_r} = t_0 t_r \bar{\lambda}\gamma_{\mu_1\ldots\mu_r}$$

$$\chi = \Gamma^{(r_1)}\Gamma^{(r_2)}\cdots\Gamma^{(r_p)}\lambda \implies \bar{\chi} = t_0^p t_{r_1} t_{r_2}\cdots t_{r_p}\,\bar{\lambda}\Gamma^{(r_p)}\cdots\Gamma^{(r_2)}\Gamma^{(r_1)}$$

In even dimensions for chiral spinors

$$\chi = P_L\lambda \;\rightarrow\; \bar{\chi} = \begin{cases} \bar{\lambda}P_L & \text{for } D = 0,4,8,\ldots, \\ \bar{\lambda}P_R & \text{for } D = 2,6,10,\ldots. \end{cases}$$