Questions-Comments I, II

In even dimensions there are two charge conjugation conjugation matrices C_+, C_-

Supersymmetry selects $t_1 = -1$ Because the supersymmetry is in D=4

$$\{Q_\alpha, Q_\beta\} = -\frac{1}{2}\left(\gamma_\mu C^{-1}\right)_{\alpha\beta} P^\mu$$

the left hand side is symmetric in alpha, beta therefore the right should also be symmetric, since

$$t_1 = -1 \qquad \left(\Gamma^{(r)} C^{-1}\right)^T = -t_r \Gamma^{(r)} C^{-1}$$

Questions-Comments I, II

- Unique irreducible representation of the Clifford algebra
- Traces and the basis of the Clifford algebra

Friendly representations

Recursive construction of generating Clifford algebra for D=2m

We start in $D = 2$ and write

$$\gamma_0 = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} = \mathrm{i}\sigma_2\,, \qquad \gamma_1 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} = \sigma_3$$

Which is really real, hermitian, and friendly representation $\gamma_* = -\gamma_0\gamma_1 = \sigma_1$

is also real. Adding it as gamma2 gives a real representation in D=3.

$$\gamma_\mu = \tilde{\gamma}_\mu \otimes \mathbb{1}\,, \qquad \mu = 0, \ldots, 2m-3\,,$$

$$\gamma_{2m-2} = \tilde{\gamma}_* \otimes \sigma_1\,, \qquad \gamma_{2m-1} = \tilde{\gamma}_* \otimes \sigma_3$$

$\gamma_* = -\tilde{\gamma}_* \otimes \sigma_2$ which can be used as gamma 2m in D=2m+1

Friendly representations

- This construction gives a real representation in 4 dimensions

$$
\begin{aligned}
\gamma_0 &= \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} = \mathrm{i}\sigma_2 \otimes \mathbb{1}\,, \\
\gamma_1 &= \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} = \sigma_3 \otimes \mathbb{1}\,, \\
\gamma_2 &= \begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix} = \sigma_1 \otimes \sigma_1\,, \\
\gamma_3 &= \begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix} = \sigma_1 \otimes \sigma_3\,.
\end{aligned}
$$

This one has an imaginary γ_*

This construction will not give real Representations in higher dimensions

Friendly representations

$$
\begin{aligned}
E_1 &= \sigma_1 \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1}\,, \\
E_2 &= \sigma_3 \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1}\,, \\
E_3 &= \sigma_2 \otimes \sigma_2 \otimes \sigma_1 \otimes \mathbb{1}\,, \\
E_4 &= \sigma_2 \otimes \sigma_2 \otimes \sigma_3 \otimes \mathbb{1}\,, \\
E_5 &= \sigma_2 \otimes \sigma_1 \otimes \mathbb{1} \otimes \sigma_2\,, \\
E_6 &= \sigma_2 \otimes \sigma_3 \otimes \mathbb{1} \otimes \sigma_2\,, \\
E_7 &= \sigma_2 \otimes \mathbb{1} \otimes \sigma_2 \otimes \sigma_1\,, \\
E_8 &= \sigma_2 \otimes \mathbb{1} \otimes \sigma_2 \otimes \sigma_3\,, \\
E_* &= E_1 \dots E_8 = \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2
\end{aligned}
$$

Real representation for Euclidean gamma matrices in D=8

Friendly representations

$$\begin{aligned} \gamma_\mu &= \tilde{\gamma}_\mu \otimes E_* \,, \qquad \mu = 0, \ldots, D-1 \\ \gamma_{D-1+i} &= \mathbb{1} \otimes E_i \,, \qquad i = 1, \ldots, 8 \,. \end{aligned}$$

When the $\tilde{\gamma}_\mu$ are real, the gamma matrices in $D+8$ are also real

real representations in $D = 2, 3, 4 \mod 8$.

$$\gamma_* = \tilde{\gamma}_* \otimes E_*$$

Spinor indexes

The components of the basic spinor λ are indicated as λ_α.

barred spinor are indicated with upper indices: λ^α

raising matrix $\mathcal{C}^{\alpha\beta}$

$$\bar{\lambda}^\alpha = \lambda^\alpha = \mathcal{C}^{\alpha\beta}\lambda_\beta$$

$\mathcal{C}^{\alpha\beta}$ are the components of the matrix C^T Note NW-SE line

$$\lambda_\alpha = \lambda^\beta \mathcal{C}_{\beta\alpha}\,.$$

$$\mathcal{C}^{\alpha\beta}\mathcal{C}_{\gamma\beta} = \delta_\gamma{}^\alpha\,, \qquad \mathcal{C}_{\beta\alpha}\mathcal{C}^{\beta\gamma} = \delta_\alpha{}^\gamma$$

$\mathcal{C}_{\alpha\beta}$ are the components of C^{-1}

Spinor indexes

The gamma matrices have components $(\gamma_\mu)_\alpha{}^\beta$

$$\bar{\chi}\gamma_\mu\lambda = \chi^\alpha(\gamma_\mu)_\alpha{}^\beta\lambda_\beta\,,$$

$$(\gamma_\mu)_{\alpha\beta} = (\gamma_\mu)_\alpha{}^\gamma\mathcal{C}_{\gamma\beta} = (\gamma_\mu C^{-1})_{\alpha\beta}$$

$$(\gamma_{\mu_1\ldots\mu_r})_{\alpha\beta} = -t_r(\gamma_{\mu_1\ldots\mu_r})_{\beta\alpha}$$

$$\lambda^\alpha\chi_\alpha = -t_0\lambda_\alpha\chi^\alpha$$

Fierz rearrangement

- In supergravity we will need changing the pairing of spinors in products of bilinears, which is called Fierz rearrangement

Basic Fierz identity from

$$M = \sum_A m_A \Gamma^A, \qquad m_A = \frac{1}{2^m} \mathrm{Tr}(M \Gamma_A)$$

Expanding any A as

$$A_\alpha{}^\beta = \sum_A (\Gamma^A)_\alpha{}^\beta \, c_A, \quad \to \quad c_A = 2^{-\frac{d}{2}} tr(\Gamma_A \, A)$$

$$A_\alpha{}^\beta = \sum_A (\Gamma^A)_\alpha{}^\beta \, 2^{-\frac{d}{2}} (\Gamma_A)_\gamma{}^\delta A_\delta{}^\gamma$$

Fierz rearrangement

Completeness relation $$\delta_\alpha{}^\beta \delta_\gamma{}^\delta = \frac{1}{2^m} \sum_A (\Gamma_A)_\alpha{}^\delta (\Gamma^A)_\gamma{}^\beta$$

Note that the 'column indices' on the left and right sides have been exchanged

Using

$$\gamma^\nu \gamma^{\mu_1 \ldots \mu_j} = \eta^{\nu\mu_1} \gamma^{\mu_2 \ldots \mu_j} - \eta^{\nu\mu_2} \gamma^{\mu_1 \mu_3 \ldots \mu_j} + \ldots + \gamma^{\nu\mu_1 \ldots \mu_j}$$

We get

$$(\gamma^\mu)_\alpha{}^\beta (\gamma_\mu)_\gamma{}^\delta = \frac{1}{2^m} \sum_A v_A (\Gamma_A)_\alpha{}^\delta (\Gamma^A)_\gamma{}^\beta$$

$v_A = (-)^{r_A}(D - 2r_A)$ Where r_A Is the rank of Γ_A

Fierz rearrangement

Given any set of 4 anti-commuting spinor fields

$$\bar{\lambda}_1 \lambda_2 \bar{\lambda}_3 \lambda_4 = -\frac{1}{2^m} \sum_A \bar{\lambda}_1 \Gamma^A \lambda_4 \bar{\lambda}_3 \Gamma_A \lambda_2$$

chiral Fierz identities for $D = 4$

$$\begin{aligned} P_L \chi \bar{\lambda} P_L &= -\tfrac{1}{2} P_L \left(\bar{\lambda} P_L \chi \right) + \tfrac{1}{8} P_L \gamma^{\mu\nu} \left(\bar{\lambda} \gamma_{\mu\nu} P_L \chi \right) \\ P_L \chi \bar{\lambda} P_R &= -\tfrac{1}{2} P_L \gamma^{\mu} \left(\bar{\lambda} \gamma_{\mu} P_L \chi \right) \; . \end{aligned}$$

Cyclic identities

Multiplying by four commuting spinors $\lambda_\alpha \lambda_\beta \lambda_\gamma \lambda_\delta$

In d=4 dimensions

$$4I_0 - 2I_1 + 2I_3 - 4I_4 = 2^2 \, I_1$$

$(C\gamma_\mu), (C\gamma_{\mu\nu})$ are symmetric and $(C), (C\gamma_{\mu\nu\rho})$ are anti-symmetric

$$I_0 = I_3 = I_4 = 0$$

Which implies the cyclic identity

$$(C\gamma^\mu)_{(\alpha\beta}(C\gamma_\mu)_{\gamma\delta)} = 0, \qquad (\alpha\beta\gamma\delta) \ symmetric \ sum.$$

Analogously one can prove

$$(C\gamma^\mu)_{(\alpha\beta}(C\gamma_{\mu\nu})_{\gamma\delta)} = 0, \qquad (\alpha\beta\gamma\delta) \ symmetric \ sum$$

Cyclic identity useful to study the kappa invariance of M2 brane

Cyclic identities

- Notice the vector $\bar{\psi}\gamma^\mu\psi$ Is light-like

$$(\bar{\psi}\gamma^\mu\psi)(\bar{\psi}\gamma_\mu\psi) = 0$$

Charge conjugate spinor

Complex conjugation is necessary to verify that the lagrangian involving spinor bilinears is hermitian.

In practice complex conjugation is replaced by charge conjugation

Charge conjugate of any spinor

$$\lambda^C \equiv B^{-1}\lambda^* \qquad B = \mathrm{i}t_0 C\gamma^0$$

Barred charge conjugate spinor

$$\overline{\lambda^C} = (-t_0 t_1)\mathrm{i}\lambda^\dagger\gamma^0$$

It coincides withe Dirac conjugate except for the numerical factor $(-t_0 t_1)$

[3] We use the convention that we interchange fermion fields in the process of complex conjugation.

Reality properties

For a matrix M charge conjugate is $M^C \equiv B^{-1} M^* B$

$$(\gamma_\mu)^C \equiv B^{-1}\gamma_\mu^* B \ = \ (-t_0 t_1)\gamma_\mu$$

$$(\gamma_*)^C = (-)^{D/2+1}\gamma_*$$

$$(\bar{\chi} M \lambda)^* \equiv (\bar{\chi} M \lambda)^C = (-t_0 t_1)\overline{\chi^C} M^C \lambda^C$$

Majorana spinors

- Majorana fields are Dirac fields that satisfy and addtional "reality" condition, whic reduces the number degrees of freedom by two. More fundamental like Weyl fields

Particles described by a Majorana field are such that particles and antiparticles are identical

Majorana field

$$\psi = \psi^C = B^{-1}\psi^* \,, \qquad \text{i.e.} \qquad \psi^* = B\psi$$

We have $\psi = B^* B \, \psi$ which implies $B^* B = \mathbb{1}$

Recall

$$B^* B = -t_1 \mathbb{1}$$

which implies $t_1 = -1$

Majorana spinors

Two cases $t_0 = \pm 1$

$t_0 = +1$ holds for spacetime dimension $D = 2, 3, 4, \bmod 8$.

In this case we have Majorana spinors. We have that the barred conjugated spinor and Dirac adjoint spinor coincide

In the Majorana case we can have real representations for the gamma Matrices . For D=4

$\bar{\Psi}$

$$
\begin{aligned}
\gamma_0 &= \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} = \mathrm{i}\sigma_2 \otimes \mathbb{1} \\
\gamma_1 &= \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} = \sigma_3 \otimes \mathbb{1} \, , \\
\gamma_2 &= \begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix} = \sigma_1 \otimes \sigma_1 \\
\gamma_3 &= \begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix} = \sigma_1 \otimes \sigma_3
\end{aligned}
$$

Majorana spinors

Note that the γ_i are symmetric, while γ_0 is antisymmetric

We have B=1, then Implies $\Psi^* = \Psi$ also $C = \mathrm{i}\gamma^0$

Properties

$$(\bar{\chi}\gamma_{\mu_1\ldots\mu_r}\psi)^* = (\bar{\chi}\gamma_{\mu_1\ldots\mu_r}\psi)^C = \bar{\chi}(\gamma_{\mu_1\ldots\mu_r})^C\psi = \bar{\chi}\gamma_{\mu_1\ldots\mu_r}\psi$$

Pseudo-Majorana spinors

- In case $t_0 = -1$ $t_1 = -1$

We have pseudo-Majorana spinors, no real reprsentations of gamma matrices

$$(\gamma^\mu)^* = -\gamma^\mu$$

Mostly relevant for D=8 or 9

Weyl-Majorana spinors

Consider (pseudo) Majorana spinors for D=0,2,4 mod 8

$D = 2 \bmod 8$ we have $(\gamma_* \psi)^C = \gamma_* \psi^C$

The two constraints

$$\text{Majorana:} \qquad \psi^C = \psi \,, \qquad \text{Weyl:} \qquad P_{L,R}\psi = \psi$$

are compatible since

$$(P_L\psi)^C = P_L\psi \,, \qquad (P_R\psi)^C = P_R\psi$$

We have Majorana-Weyl spinor

They have 2^{m-1} independent 'real' components in dimension

D=2 mod 8. Supergravity and string theory in D=10 are based in Majorana-Weyl spinors

Incompatibilty of Majorana and Weyl condition

For $D = 4 \bmod 4$ dimensions we have $(\gamma_* \psi)^C = -\gamma_* \psi^C$.

which implies

$$(P_L \psi)^C = P_R \psi \,, \qquad (P_R \psi)^C = P_L \psi \,.$$

The “left” and “right” components of a Majorana spinor are related by charge by charge conjugation

Symplectic-Majorana spinors

When $t_1 = 1$ we cannot define Majorana spinors

We can define sympletic Majorana spinors

$$\chi^i = \varepsilon^{ij} B^{-1} (\chi^j)^* \quad i = 1, \ldots, 2k$$

$$(\chi^k)^* = -B\epsilon^{ki}\chi^i$$

which implies

$$t_1 = 1$$

For dimensions D=6 mod 8 we can show that the sympletic Majorana constraint is compatible with chirality

Dimensions of minimal spinors

Dim	Spinor	min # components	antisymmetric
2	MW	1	1
3	M	2	1,2
4	M	4	1,2
5	S	8	2,3
6	SW	8	3
7	S	16	0,3
8	M	16	0,1
9	M	16	0,1
10	MW	16	1
11	M	32	1,2

Majorana spinors in physical theories

we consider a prototype action for a Majorana spinor field

for D=2,3, 4 mod 8 . Majorana and Dirac fields transform in the same way under Lorentz transformations, but half degrees of freedom

$$S[\Psi] = -\frac{1}{2}\int \mathrm{d}^D x\, \bar{\Psi}[\gamma^\mu \partial_\mu - m]\Psi(x)$$

For commuting spinors $\Psi^T C \Psi'$ vanishes

$\Psi^T C \gamma^\mu \partial_\mu \Psi$ Is a total derivative, we need anticommuting Majorana spinors

$$\delta S[\Psi] = -\int \mathrm{d}^D x\, \delta\bar{\Psi}[\gamma^\mu \partial_\mu - m]\Psi(x)$$

The Majorana field satisfies the conventional Dirac equation

Majorana spinors in physical theories

Majorna action in terms of “Weyl” fields, D=4

$$
\begin{aligned}
S[\psi] &= -\frac{1}{2}\int \mathrm{d}^4x\ \left[\bar{\Psi}\gamma^\mu\partial_\mu - m\right](P_L + P_R)\Psi \\
&= -\int \mathrm{d}^4x\ \left[\bar{\Psi}\gamma^\mu\partial_\mu P_L\Psi - \frac{1}{2}m\bar{\Psi}P_L\Psi - \frac{1}{2}m\bar{\Psi}P_R\Psi\right]
\end{aligned}
$$

equations of motion

$$
\partial\!\!\!/ P_L\Psi = mP_R\Psi\,, \qquad \partial\!\!\!/ P_R\Psi = mP_L\Psi
$$

D=4 Majorana spinors in terms of Weyl spinors

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}$$

Weyl representation

The Majorana condition $\Psi = B^{-1}\Psi^* = \gamma^0\gamma^1\gamma^3\Psi^*$

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_2^* \\ -\psi_1^* \end{pmatrix} \qquad \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \qquad \tilde{\psi} = \begin{pmatrix} \psi_2^* \\ -\psi_1^* \end{pmatrix}$$

$$\begin{pmatrix} \psi \\ 0 \end{pmatrix} = P_L\Psi\,, \qquad \begin{pmatrix} 0 \\ \tilde{\psi} \end{pmatrix} = (P_L\Psi)^C = P_R\Psi$$

$\mathrm{U}(1)$ *symmetries of a Majorana field*

$(\mathrm{i}\gamma_*)^C = \mathrm{i}\gamma_*$ implies $\Psi \rightarrow \Psi' = \mathrm{e}^{-\mathrm{i}\gamma_*\theta}\Psi$

preserves the Majorana condition

$$\begin{aligned}\delta S[\Psi] &= \mathrm{i}\theta\int \mathrm{d}^4x\, \bar{\Psi}\gamma_*[\gamma^\mu\partial_\mu - m]\Psi \\ &= \mathrm{i}\theta\int \mathrm{d}^4x\, [\tfrac{1}{2}\partial_\mu(\bar{\Psi}\gamma_*\gamma^\mu\Psi) - m\bar{\Psi}\gamma_*\Psi]\end{aligned}$$

- The conventional vector $\mathrm{U}(1)$ symmetry is incompatible with the Majorana condition.
- The axial transformation above is compatible and is a symmetry of the action for a massless Majorana field only.

U(n) symmetry of Majorana fields

The Lagrangian

$$\mathcal{L} = -\frac{1}{2}\bar{\chi}^I \gamma \cdot \partial \chi^I$$

for n Majorana spinors ($I = 1, 2, \ldots, n$) has an obvious $O(n)$ symmetry

there is a larger U(n) chiral symmetry

$$\delta\chi^I = \left(a^{IJ} + s^{IJ}\gamma_5\right)\chi^J$$

where $a^{IJ} = -a^{JI}$ and $s^{IJ} = s^{JI}$ are n^2 real parameters

The symmetry is manifest if we use the chiral projections

$$\chi_+ \to \chi'_+ = e^{-iH}\chi_+ \,, \qquad \chi_- \to \chi'_- = e^{iH^*}\chi_-$$

n,1 (,-1) $\bar{n}$,1

$$H^I{}_J = s^{IJ} + ia^{IJ}$$

U(n) symmetry of Majorana fields

$$\mathcal{L}_{kin} = -\frac{1}{2}\bar{\chi}^I \gamma \cdot \partial \chi^I = -\bar{\chi}_+^I \gamma \cdot \partial \chi_-{}^I + \text{total derivative}$$

This is manifestly U(n) invariant