The Maxwell and Yang-Mills Gauge Fields

Gauge invariances

a) Relativistic covariance is maintained
b) The field equations do not determine certain longitudinal components
c) We have constraints, that restrict the initial data

The classical degrees of freedom are the independent functions required as Initial data for the Cauchy problem of hyperbolic equations.

An elliptic equation $\nabla^2 \phi = 0.$ Does not contain degrees of freedom

Abelian Gauge Field

- Principle of minimal coupling

For a complex Dirac spinor of charge q

$$\Psi(x) \rightarrow \Psi'(x) \equiv \mathrm{e}^{\mathrm{i}q\theta(x)}\Psi(x)$$

Gauge field

$$A_\mu(x) \rightarrow A'_\mu(x) \equiv A_\mu(x) + \partial_\mu\theta(x)$$

Covariant derivative

$$D_\mu\Psi(x) \equiv (\partial_\mu - \mathrm{i}qA_\mu(x))\Psi(x)$$

Abelian Gauge Field

- Free gauge field

$$[D_\mu, D_\nu]\Psi \;=\; -\mathrm{i}eqF_{\mu\nu}\Psi$$

$$F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)$$

$E_i \;=\; F_{0i}$ $\qquad\qquad$ $B_i = \frac{1}{2}\varepsilon_{ijk}F_{jk}$

Free equation of motion $\qquad \partial^\mu F_{\mu\nu} \;=\; 0$

Noether identity $\qquad \partial^\nu\partial^\mu F_{\mu\nu} = \;0$ signal of the gauge symmetry

$$A_\mu \to A_\mu + \partial_\mu\theta$$

Bianchi identity $\qquad \partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} + \partial_\rho F_{\mu\nu} \;=\; 0$

Maxwell algebra

- For constant electromagnetic field there is a generalization of the Poincare group

$$[P_\mu, P_\nu] = iZ_{\mu\nu}$$

together with the generators of Lorentz transformations

Abelian Gauge Field

- Degrees of freedom

Gauge fixing, eg Coulomb gauge $\partial_i A_i(\vec{x}, t) = 0$

This condition does eliminate the gauge freedom

$\partial_i A'_i = \partial^i (A_i + \partial_i \theta) = 0$ If $\nabla^2 \theta = 0$

which implies $\theta(x) \equiv 0$

Maxwell equation

$$\begin{aligned} \nabla^2 A_0 - \partial_0(\partial_i A_i) &= 0 \\ \Box A_i - \partial_i \partial_0 A_0 - \partial_i(\partial_j A_j) &= 0 \end{aligned}$$

Abelian Gauge Field

- Degrees of freedom

In the Coulomb gauge

$\nabla^2 A_0 = 0$ implies no degrees of freedom

$\Box A_i = 0$ massless particle

Initial data $A_i(\vec{x}, 0),\ \dot{A}_i(\vec{x}, 0)$

$2(D-2)$ independent degrees of freedom

Off-shell degrees of freedom

$(D-2)$ on-shell degrees of freedom helicity states

Abelian Gauge Field

- The field strength verifies

$$\Box F_{\mu\nu} = 0$$

Gauge invriant description that electromagnetic field describes massless particles

Abelian Gauge Field

- Hamiltonian counting of degrees of freedom

Primary constraints $\pi^0 = 0$

Secondary constraint $\nabla\vec{\pi} = 0$

First class constraints $\pi^0 = 0, \quad \nabla\vec{\pi} = 0$

Gauge fixing constraints $A^0 = 0 \quad \nabla\vec{A} = 0$

Dirac bracket and number of degrees of freedom , 8-4=4=2+2

Noether identities

- Combination of equation of motion that vanish identically

Action $S = S[\phi]$ equations of motion $\frac{\delta S}{\delta \phi^i}$ =0

Gauge transformations $\delta\phi = R^i_\alpha \epsilon^\alpha$

Variation of the action $\delta S[\phi] = \frac{\delta S}{\delta \phi^i} R^i_\alpha \epsilon^\alpha$

Noether identities $\frac{\delta S}{\delta \phi^i} R^i_\alpha = 0$

Abelian Gauge Field

- QED

$$S[A_\mu, \bar{\Psi}, \Psi] = \int \mathrm{d}^D x \left[-\tfrac{1}{4} F^{\mu\nu} F_{\mu\nu} + \bar{\Psi}(\gamma^\mu D_\mu - m)\Psi \right]$$

$$\partial^\mu F_{\mu\nu} - \mathrm{i} e q \bar{\Psi} \gamma_\nu \Psi = 0 \,.$$

Abelian Gauge Field

- Dual tensors

In D=4

$$\tilde{H}^{\mu\nu} \equiv -\tfrac{1}{2}\mathrm{i}\varepsilon^{\mu\nu\rho\sigma}H_{\rho\sigma}$$

Selfdual-antiselfdual

$$H^{\pm}_{\mu\nu} = \tfrac{1}{2}\left(H_{\mu\nu} \pm \tilde{H}_{\mu\nu}\right)\,, \qquad H^{\pm}_{\mu\nu} = \left(H^{\mp}_{\mu\nu}\right)^{*}$$

$$-\tfrac{1}{2}\mathrm{i}\varepsilon_{\mu\nu}{}^{\rho\sigma}H^{\pm}_{\rho\sigma} = \pm H^{\pm}_{\mu\nu}$$

properties

$$G^{+\mu\nu}H^{-}{}_{\mu\nu} = 0\,, \qquad G^{\pm\rho(\mu}H^{\pm\nu)}{}_{\rho} = -\tfrac{1}{4}\eta^{\mu\nu}G^{\pm\rho\sigma}H^{\pm}{}_{\rho\sigma}\,, \qquad G^{+}{}_{\rho[\mu}H^{-}{}_{\nu]}{}^{\rho} = 0\,.$$

Duality for the electromagnetic field

- Free case

The Maxwell and Bianchi equations

$$\partial_\mu F^{\mu\nu} = 0\,, \qquad \partial_\mu \tilde{F}^{\mu\nu} = 0$$

are invariant under the transformation

$$F^{\mu\nu} \rightarrow F'^{\mu\nu} = \mathrm{i}\tilde{F}^{\mu\nu}$$

property

$$F_{\mu\nu}\tilde{F}^{\mu\nu} = -\mathrm{i}\partial_\mu(\varepsilon^{\mu\nu\rho\sigma}A_\nu F_{\rho\sigma})$$

Chern-Simmons action L =A \wedge dA topological action

Equations of motion F=dA=0 flat connections