Duality for the electromagnetic field

Duality for gauge field and complex scalar

Interacting theory with one ableian gauge field and a complex scalar

\[ \mathcal{L} = -\frac{1}{4} (\text{Re } Z) F_{\mu\nu} F^{\mu\nu} + \frac{1}{8} (\text{Im } Z) \varepsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} \]

Bianchi identity and equation of motion

\[ \partial_\mu \tilde{F}^{\mu\nu} = 0, \quad \partial_\mu \left[ (\text{Re } Z) F^{\mu\nu} - i(\text{Im } Z) \tilde{F}^{\mu\nu} \right] = 0 \]

Define the tensor

\[ G^{\mu\nu} \equiv \varepsilon^{\mu\nu\rho\sigma} \frac{\partial S}{\partial F^{\rho\sigma}} = -i(\text{Re } Z) \tilde{F}^{\mu\nu} - (\text{Im } Z) F^{\mu\nu} \]

\[ G^{\mu\nu} = iZF^{\mu\nu}, \quad G^{\mu\nu}+ = -i\tilde{Z}F^{\mu\nu}+ \]
Duality for the electromagnetic field

\[ \partial_\mu \text{Im} \, F^{\mu \nu} = 0, \quad \partial_\mu \text{Im} \, G^{\mu \nu} = 0 \]

These equations are invariant under the transformation

\[ \begin{pmatrix} F'^- \\ G'^- \end{pmatrix} = S \begin{pmatrix} F^- \\ G^- \end{pmatrix} \]

Where

\[ S \equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad ad - bc = 1 \]

is an SL(2,R) transformation

\[ G'^{\mu \nu} = iZ' F'^{\mu \nu} \]

If

\[ iZ' = \frac{c + idZ}{a + ibZ} \]

Which is the transformation of the scalar
Duality for the electromagnetic field

\[ \mathcal{L}(F, Z) = -\frac{1}{2} \text{Re}(ZF_{\mu\nu}^- F^{\mu\nu}^-) \]

Magnetic and electric charges appear as sources for the Bianchi identity and generalized Maxwell equation

The vector of charges \( \left( \begin{array}{c} p \\ q \end{array} \right) \), transforms like \( \left( \begin{array}{c} F^- \\ G^- \end{array} \right) \)

Schwinger-Zwanziger quantization condition for dyons \( p_1 q_2 - p_2 q_1 = 2\pi n \)

We have \( \text{SL}(2, \mathbb{Z}) \), often called the modular group

\[
\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}
\]

\( Z' = Z - i \), \( Z' = \frac{1}{Z} \). S-duality
Duality for the electromagnetic field

Gauge fields $A^A_\mu(x)$, indexed by $A = 1, 2, \ldots, m$

Scalar fields $\phi^i$

$$S = \int d^4x \mathcal{L}, \quad \mathcal{L} = -\frac{1}{4} \text{Re} f_{AB} F^A_\mu F^{\mu\nu} B + \frac{1}{4} i \text{Im} f_{AB} F^A_\mu \tilde{F}^{\mu\nu} B$$

$f_{AB}(\phi) = f_{BA}(\phi)$

$$\mathcal{L}(F^+, F^-) = -\frac{1}{2} \text{Re} \left( f_{AB} F^{-A}_{\mu\nu} F^{\mu\nu} - B \right)$$

$$= -\frac{1}{4} \left( f_{AB} F^{-A}_{\mu\nu} F^{\mu\nu} - B + f_{AB}^{*} F^{+A}_{\mu\nu} F^{\mu\nu} + B \right)$$

$$G^\mu_\nu^- = -2i \frac{\partial S(F^+, F^-)}{\partial F^{-A}_{\mu\nu}} = i f_{AB} F^{\mu\nu} - B$$

$$G^\mu_\nu^+ = 2i \frac{\partial S(F^+, F^-)}{\partial F^{+A}_{\mu\nu}} = -i f_{AB}^{*} F^{\mu\nu} + B$$
Duality for the electromagnetic field

\[ \partial^\mu \text{Im} \, F_{\mu\nu}^A = 0 \quad : \quad \text{Bianchi identities,} \]
\[ \partial_\mu \text{Im} \, G_{\mu\nu}^A = 0 \quad : \quad \text{Equations of motion} \]
Duality for the electromagnetic field

\[
\begin{pmatrix}
F' \ \\
G'
\end{pmatrix} = S \begin{pmatrix}
F \ \\
G
\end{pmatrix} = \begin{pmatrix}
A & B \\
C & D
\end{pmatrix} \begin{pmatrix}
F \ \\
G
\end{pmatrix}
\]

\[
if' = (C + iDf)(A + iBf)^{-1}
\]

\[
S = \begin{pmatrix}
A & B \\
C & D
\end{pmatrix} \in \text{Sp}(2m, \mathbb{R})
\]

\[
S^T \Omega S = \Omega \quad \text{where} \quad \Omega = \begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}
\]

Dimension of the symplectic group $m(2m+1)$
Duality for the electromagnetic field

• Duality transformations
  - symmetries of one theory (S-duality)
  - transformations from theory to another theory (M-theory applications)
Non-abelian Gauge Field

An element of the gauge group in the fundamental representation

\[ U(x) = e^{-\Theta(x)}, \quad \text{with} \quad \Theta(x) = \theta^A(x)t_A \]

a spinor field \( \Psi \) in the fundamental representation

\[ \Psi(x) \rightarrow U(x)\Psi(x) \]

given any field in the adjoint representation, such as \( \phi^A(x) \)

\[ U(x)t_A U(x)^{-1} = t_B R(x)^B_A \quad \quad \quad R^B_A = \delta^B_A + \theta^C f_{AC}^B \]

\[ \Phi(x) = t_A \phi^A(x) \quad \quad \quad \Phi(x) \rightarrow U(x)\Phi(x)U(x)^{-1} \]

Gauge potential

\[ A_\mu(x) = t_A A^A_\mu(x) \]

\[ A_\mu(x) \rightarrow A'_\mu(x) \equiv \frac{1}{g} U(x)\partial_\mu U(x)^{-1} + U(x)A_\mu(x)U(x)^{-1} \]
Non-abelian Gauge Field

infinitesimal transformations

\[ \delta A_\mu(x) = \frac{1}{a} \partial_\mu \Theta(x) + [A_\mu(x), \Theta(x)] \]

Covariant derivatives

\[ D_\mu \Psi \equiv (\partial_\mu + gA_\mu) \Psi \]
\[ D_\mu \bar{\Psi} \equiv \partial_\mu \bar{\Psi} - g\bar{\Psi} A_\mu \]

\[ D_\mu \Phi = \partial_\mu \Phi + g[A_\mu, \Phi] \]

\[ D_\mu \Psi \rightarrow U(x) D_\mu \Psi, \quad D_\mu \bar{\Psi} \rightarrow D_\mu \bar{\Psi} U(x)^{-1}, \quad D_\mu \Phi \rightarrow U(x) D_\mu \Phi U(x)^{-1} \]
Non-abelian Gauge Field

non-abelian field strength

\[ F_{\mu\nu} = t_A F^A_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + g[A_\mu, A_\nu] \]

Bianchi identity

\[ D_\mu F_{\nu\rho} + D_\nu F_{\rho\mu} + D_\rho F_{\mu\nu} = 0 \]

\[ D_\mu F^A_{\nu\rho} + D_\nu F^A_{\rho\mu} + D_\rho F^A_{\mu\nu} = 0 \]

Where

\[ D_\mu F^A_{\nu\rho} = \partial_\mu F^A_{\nu\rho} + g f^{A'B'}_{B'} A^A_{\mu} F_{\nu\rho}^B \]

Action

\[ S[A_\mu, \bar{\Psi}, \Psi] = \int d^D x \left[ \frac{1}{2} \text{Tr}(F^{\mu\nu} F_{\mu\nu}) + \bar{\Psi} (\gamma^\mu D_\mu - m) \Psi \right] \]
Non-abelian Chern Simons

\[ S = \frac{k}{4\pi} \int_M tr(A \wedge dA + \frac{2}{3} A \wedge A \wedge A) \]

Equation of motion

\[ F = dA + A \wedge A \quad F = 0 \]
Internal Symmetry for Majorana Spinors

Majorana spinors play a central role in supersymmetric field theories. Spinor fields of super-Yang-Mills theory are denoted as $\lambda^A$ and transform in the adjoint representation

$$\lambda^A \rightarrow \lambda'^A = R^A_B \lambda^B$$

$R^A_B$ is real and therefore compatible with the Majorana condition.
Internal Symmetry for Majorana Spinors

- **D=4**

  Complex representation, we have the highest rank element

  \[ \chi^\alpha \rightarrow \chi'^\alpha \equiv \left( e^{-\theta^A(t_A P_L + t^*_A P_R)} \right)^\alpha_\beta \chi^\beta \]

  The chiral projections transform

  \[ P_L \chi^\alpha \rightarrow P_L \chi'^\alpha \equiv \left( e^{-\theta^A t_A} \right)^\alpha_\beta P_L \chi^\beta \]

  \[ P_R \chi^\alpha \rightarrow P_R \chi'^\alpha \equiv \left( e^{-\theta^A t^*_A} \right)^\alpha_\beta P_R \chi^\beta \]

  Variation of the mass term

  \[ \delta(\bar{\chi}\chi) = -\theta^A \bar{\chi}(t_A + t^T_A) \gamma^*_\chi \]

  If G=SU(n), the mass term is preserved by SO(n)
The free Rarita-Schwinger field

Consider now a free spinor abelian gauge field
\[ \Psi_{\mu}(x) \]  
we omit the spinor indexes

Gauge transformation
\[ \Psi_{\mu}(x) \rightarrow \Psi_{\mu}(x) + \partial_{\mu} \epsilon(x) \]

\(\Psi_{\mu}\) and \(\epsilon\) are complex spinors with \(2[D/2]\) spinor components

This is fine for a free theory, but interacting supergravity theories are more restrictive. We will need to use Majorana and/or Weyl spinors

Field strength
\[ \partial_{\mu} \Psi_{\nu} - \partial_{\nu} \Psi_{\mu} \]  
gauge invariant
The free Rarita-Schwinger field

• Action

Properties: a) Lorentz invariant, b) first order in space-time derivatives c) gauge invariant, d) hermitean

\[ S = - \int d^D x \bar{\Psi}_\mu \gamma^{\mu \nu \rho} \partial_\nu \Psi_\rho \]

contains the third rank Clifford algebra element \( \gamma^{\mu \nu \rho} \)

\( \bar{\Psi}_\mu \) is the Dirac conjugate

\[ \bar{\Psi}_\mu = \Psi^{\dagger} i \gamma^0 \]

The lagrangian is invariant up to a total derivative

\[ \delta \mathcal{L} = -\partial_\mu (\bar{\epsilon} \gamma^{\mu \nu \rho} \partial_\nu \bar{\Psi}_\rho) \]
The free Rarita-Schwinger field

• Equation of motion

\[ \gamma^\mu{}^{\nu\rho} \partial_\nu \Psi_\rho = 0 \]

Noether identities

\[ \gamma^\mu{}^{\nu\rho} \partial_\mu \partial_\nu \Psi_\rho = 0 \]

Using

\[ \gamma_\mu \gamma^{\mu\nu\rho} = (D - 2) \gamma_\nu^{\nu\rho} \]

\[ \gamma^\mu{}^{\nu\rho} = \gamma^\mu \gamma_\nu^{\nu\rho} - \eta_\mu{}^{\nu\rho} \gamma^\rho + \eta^{\mu\rho} \gamma_\mu \]

We can write the equations of motion as

\[ \gamma^\mu (\partial_\mu \Psi_\nu - \partial_\nu \Psi_\mu) = 0 \]
The free Rarita-Schwinger field

• Massless particles

\[ \phi \left( \partial_\rho \Psi_\nu - \partial_\nu \Psi_\rho \right) = 0 \]
The free Rarita-Schwinger field

• Initial value problem

The gauge \( \gamma^i \Psi_i = 0 \) fixes completely the gauge

\[ \gamma^i \Psi_i' = \gamma^i (\Psi_i + \partial_i \epsilon) = \gamma^i \partial_i \epsilon \longrightarrow \nabla^2 \epsilon = 0 \]

the equations of motion in components

\[
\begin{align*}
\gamma^i \partial_i \Psi_0 - \partial_0 \gamma^i \Psi_i &= 0 \\
\gamma \cdot \partial \Psi_i - \partial_i \gamma \cdot \Psi &= 0
\end{align*}
\]

Using the gauge condition one can see that \( \nabla^2 \Psi_0 = 0 \), so \( \Psi_0 = 0 \)
The free Rarita-Schwinger field

• Initial value problem

The spatial components $\Psi_i$ then satisfy the Dirac equation

$$\gamma \cdot \partial \Psi_i = 0$$

We have also

$$\partial^i \Psi_i = 0$$

The restrictions on the initial conditions are

$$\gamma^i \Psi_i(\vec{x}, 0) = 0$$
$$\Psi_0(\vec{x}, 0) = 0$$
$$\partial^i \Psi_i(\vec{x}, 0) = 0$$

there are only $2^{[\frac{D}{2}]}(D - 3)$ initial components
The free Rarita-Schwinger field

• Initial value problem

The on-shell degrees freedom are half of

\[ 2^{\left[ \frac{D}{2} \right]} (D - 3) \]

In D=4, with Majorana conditions, we find two states expected for a massless particle for any s>0. The helicities are +3/2 and -3/2.
Degrees of freedom

D=11 Supergravity

128 bosons degrees of freedom
graviton $g_{\mu\nu}$, 44 of SO(9):

3-form $C_{\mu\nu\rho}$, $C_{(3)}$, 84 of SO(9):

128 fermions degrees of freedom

SO(9) little group of massless particles of ISO(10,1)

$L = \sqrt{-g} \{ R - \frac{1}{2} (F_{(4)})^2 + \cdots \} + C_{(3)} \wedge dC_{(3)} \wedge dC_{(3)} + \cdots$ fermions

$F_{(4)} = \partial C_{(3)}$, 4-form field strength

$d(\ast F_{(4)} + C_{(3)} \wedge dC_{(3)}) = 0$