Duality for the electromagnetic field

Duality for gauge field and complex scalar

Interacting theory with one ableian gauge field and a complex scalar

$$\mathcal{L} = -\tfrac{1}{4}(\mathrm{Re}\, Z)F_{\mu\nu}F^{\mu\nu} + \tfrac{1}{8}(\mathrm{Im}\, Z)\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}$$

Bianchi identity and equation of motion

$$\partial_\mu \tilde{F}^{\mu\nu} = 0\,, \qquad \partial_\mu \left[(\mathrm{Re}\, Z)\, F^{\mu\nu} - \mathrm{i}(\mathrm{Im}\, Z)\, \tilde{F}^{\mu\nu}\right] = 0$$

Define the tensor

$$G^{\mu\nu} \equiv \varepsilon^{\mu\nu\rho\sigma}\frac{\partial S}{\partial F^{\rho\sigma}} = -\mathrm{i}(\mathrm{Re}\, Z)\tilde{F}^{\mu\nu} - (\mathrm{Im}\, Z)F^{\mu\nu}$$

$$G^{\mu\nu -} = \mathrm{i}ZF^{\mu\nu -}\,, \qquad G^{\mu\nu +} = -\mathrm{i}\bar{Z}F^{\mu\nu +}$$

Duality for the electromagnetic field

$$\partial_\mu \, \mathrm{Im}\, F^{\mu\nu\,-} \;=\; 0\,, \qquad \partial_\mu \, \mathrm{Im}\, G^{\mu\nu\,-} \;=\; 0$$

These equations are invariant under the transformation

$$\begin{pmatrix} F'^{-} \\ G'^{-} \end{pmatrix} = \mathcal{S} \begin{pmatrix} F^{-} \\ G^{-} \end{pmatrix}$$

Where

$$\mathcal{S} \equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad ad - bc = 1$$

is an SL(2,R) transformation

$$G'^{\mu\nu\,-} \;=\; \mathrm{i} Z' F'^{\mu\nu\,-}$$

If $$\mathrm{i}Z' = \frac{c + \mathrm{i}dZ}{a + \mathrm{i}bZ}$$ Which is the transformation of the scalar

Duality for the electromagnetic field

$$\mathcal{L}(F,Z) \;=\; -\tfrac{1}{2}\,\mathrm{Re}(ZF^{-}_{\mu\nu}F^{\mu\nu\,-})$$

Magnetic and electric charges appear as sources for the Bianchi identity and generalized Maxwell equation

$\text{vector of charges} \begin{pmatrix} p \\ q \end{pmatrix},$ Transforms like $\begin{pmatrix} F^- \\ G^- \end{pmatrix}$

Schwinger-Zwanziger quantization condition for dyons $p_1 q_2 - p_2 q_1 = 2\pi n$

We have $\mathrm{SL}(2,\mathbb{Z})$, often called the modular group

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \quad , \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$Z' = Z - \mathrm{i} \quad , \quad Z' = \frac{1}{Z}\,.$$ S-duality

Duality for the electromagnetic field

gauge fields $A_\mu^A(x)$, indexed by $A = 1, 2, \ldots, m$

scalar fields ϕ^i

$$S = \int \mathrm{d}^4 x \, \mathcal{L}\,, \qquad \mathcal{L} = -\tfrac{1}{4}(\mathrm{Re}\, f_{AB}) F_{\mu\nu}^A F^{\mu\nu\, B} + \tfrac{1}{4}\mathrm{i}(\mathrm{Im}\, f_{AB}) F_{\mu\nu}^A \tilde{F}^{\mu\nu\, B}$$

$$f_{AB}(\phi) = f_{BA}(\phi)$$

$$\begin{aligned} \mathcal{L}(F^+, F^-) &= -\tfrac{1}{2}\, \mathrm{Re}\left(f_{AB} F_{\mu\nu}^{-\,A} F^{\mu\nu\,-\,B}\right) \\ &= -\tfrac{1}{4}\left(f_{AB} F_{\mu\nu}^{-\,A} F^{\mu\nu\,-\,B} + f_{AB}^* F_{\mu\nu}^{+\,A} F^{\mu\nu\,+\,B}\right) \end{aligned}$$

$$G_A^{\mu\nu\,-} = -2\mathrm{i} \frac{\partial S(F^+, F^-)}{\partial F_{\mu\nu}^{-\,A}} = \mathrm{i} f_{AB} F^{\mu\nu\,-\,B}$$

$$G_A^{\mu\nu\,+} = 2\mathrm{i} \frac{\partial S(F^+, F^-)}{\partial F_{\mu\nu}^{+\,A}} = -\mathrm{i} f_{AB}^* F^{\mu\nu\,+\,B}$$

Duality for the electromagnetic field

$$
\begin{aligned}
\partial^\mu \,\mathrm{Im}\, F^{A\,-}_{\mu\nu} &= 0 \quad &: \quad &\text{Bianchi identities,}\\
\partial_\mu \,\mathrm{Im}\, G^{\mu\nu\,-}_{A} &= 0 \quad &: \quad &\text{Equations of motion}
\end{aligned}
$$

Duality for the electromagnetic field

$$\begin{pmatrix} F'^- \\ G'^- \end{pmatrix} = \mathcal{S} \begin{pmatrix} F^- \\ G^- \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} F^- \\ G^- \end{pmatrix}$$

$$\boxed{\mathrm{i}f' = (C + \mathrm{i}Df)(A + \mathrm{i}Bf)^{-1}}$$

$$\mathcal{S} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(2m, \mathbb{R})$$

$$\mathcal{S}^T \Omega \mathcal{S} = \Omega \qquad \text{where} \qquad \Omega = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}$$

Dimension of the symplectic group m(2m+1)

Duality for the electromagnetic field

- Duality transformations

-symmetries of one theory (S-duality)

-transformations from theory to another theory (M-theory applications)

Non-abelian Gauge Field

An element of the gauge group in the fundamental representation

$$U(x) \doteq \mathrm{e}^{-\Theta(x)}, \text{ with } \Theta(x) = \theta^A(x) t_A$$

a spinor field Ψ in the fundamental representation

$$\Psi(x) \to U(x)\Psi(x)$$

given any field in the adjoint representation, such as $\phi^A(x)$

$$U(x) t_A U(x)^{-1} = t_B R(x)^B{}_A \qquad R^B{}_A = \delta^B_A + \theta^C f_{AC}{}^B$$

$$\mathbf{\Phi}(x) = t_A \phi^A(x) \qquad \mathbf{\Phi}(x) \to U(x)\mathbf{\Phi}(x)U(x)^{-1}$$

Gauge potential $\mathbf{A}_\mu(x) = t_A A^A_\mu(x)$

$$\mathbf{A}_\mu(x) \to \mathbf{A}'_\mu(x) \equiv \frac{1}{g} U(x)\partial_\mu U(x)^{-1} + U(x)\mathbf{A}_\mu(x)U(x)^{-1}$$

Non-abelian Gauge Field

infinitesimal transformations

$$\delta A_\mu(x) = \frac{1}{g}\partial_\mu \Theta(x) + [A_\mu(x), \Theta(x)]$$

Covarint derivatives

$$\begin{aligned} D_\mu \Psi &\equiv (\partial_\mu + g\mathbf{A}_\mu)\Psi \\ D_\mu \bar{\Psi} &\equiv \partial_\mu \bar{\Psi} - g\bar{\Psi}\mathbf{A}_\mu \end{aligned}$$

$$D_\mu \mathbf{\Phi} = \partial_\mu \mathbf{\Phi} + g[\mathbf{A}_\mu, \mathbf{\Phi}]$$

$$D_\mu \Psi \to U(x) D_\mu \Psi, \quad D_\mu \bar{\Psi} \to D_\mu \bar{\Psi} U(x)^{-1}, \quad D_\mu \mathbf{\Phi} \to U(x) D_\mu \mathbf{\Phi} U(x)^{-1}$$

Non-abelian Gauge Field

non-abelian field strength

$$\mathbf{F}_{\mu\nu} = t_A F^A_{\mu\nu} = \partial_\mu \mathbf{A}_\nu - \partial_\nu \mathbf{A}_\mu + g[\mathbf{A}_\mu, \mathbf{A}_\nu]$$

Bianchi identity $D_\mu F_{\nu\rho} + D_\nu F_{\rho\mu} + D_\rho F_{\mu\nu} = 0$

$$D_\mu F^A_{\nu\rho} + D_\nu F^A_{\rho\mu} + D_\rho F^A_{\mu\nu} = 0$$

Where

$$D_\mu F^A_{\nu\rho} = \partial_\mu F^A_{\nu\rho} + g f_{BC}{}^A A^B_\mu F^A_{\nu\rho}$$

Action

$$S[\mathbf{A}_\mu, \bar{\Psi}, \Psi] = \int \mathrm{d}^D x \left[\frac{1}{2} \mathrm{Tr}(\mathbf{F}^{\mu\nu}\mathbf{F}_{\mu\nu}) + \bar{\Psi}(\gamma^\mu \mathrm{D}_\mu - \mathrm{m})\Psi \right]$$

Non-abelian Chern Simons

$$S = \frac{k}{4\pi}\int_M tr(A \wedge dA + \frac{2}{3}A \wedge A \wedge A)$$

Equation of motion

$$F = dA + A \wedge A \quad F = 0$$

Internal Symmetry for Majorana Spinors

Majorana spinors play a central role in supersymmetric field theories

spinor fields of super-Yang-Mills theory are denoted as λ^A

transform in the adjoint representation

$$\lambda^A \to \lambda'^A = R^A{}_B \lambda^B$$

$R^A{}_B$. real and therefore compatible with the Majorana condition

Internal Symmetry for Majorana Spinors

- D=4 t_A Complex representation, we have the highest rank element

$$\chi^\alpha \to \chi'^\alpha \equiv (\mathrm{e}^{-\theta^A(t_A P_L + t_A^* P_R)})^\alpha{}_\beta \chi^\beta$$

The chiral projectiosn transform

$$\begin{aligned} P_L\chi^\alpha &\to P_L\chi'^\alpha \equiv (\mathrm{e}^{-\theta^A t_A})^\alpha{}_\beta P_L\chi^\beta \\ P_R\chi^\alpha &\to P_R\chi'^\alpha \equiv (\mathrm{e}^{-\theta^A t_A^*})^\alpha{}_\beta P_R\chi^\beta \end{aligned}$$

Variation of the mass term

$$\delta(\bar\chi\chi) = -\theta^A \bar\chi(t_A + t_A^T)\gamma_*\chi$$

If G=SU(n), the mass term is preserved by SO(n)

The free Rarita-Schwinger field

Consider now a free spinor abelian gauge field

$\Psi_\mu(x)$ we omit the spinor indexes

Gauge transformation $\Psi_\mu(x) \rightarrow \Psi_\mu(x) + \partial_\mu \epsilon(x)$

Ψ_μ and ϵ are complex spinors with $2^{[D/2]}$ spinor components

This is fine for a free theory, but interacting supergravity theories are more restrictive .
We will need to use Majorana and/or Weyl spinors

Field strenght $\partial_\mu \Psi_\nu - \partial_\nu \Psi_\mu$ gauge invariant

The free Rarita-Schwinger field

- Action

Properties: a) Lorentz invariant, b) first order in space-time derivatives
c) gauge invariant, d) hermitean

$$S = -\int \mathrm{d}^D x\, \bar{\Psi}_\mu \gamma^{\mu\nu\rho} \partial_\nu \Psi_\rho$$

contains the third rank Clifford algebra element $\gamma^{\mu\nu\rho}$

$\bar{\Psi}_\mu$ is the Dirac conjugate $\qquad \bar{\Psi}_\mu = \Psi^\dagger \mathrm{i}\gamma^0$

The lagrangian is invariant up to a total derivative

$$\delta\mathcal{L} = -\partial_\mu (\bar{\epsilon}\gamma^{\mu\nu\rho}\partial_\nu \bar{\Psi}_\rho)$$

The free Rarita-Schwinger field

- Equation of motion

$$\gamma^{\mu\nu\rho}\partial_\nu\Psi_\rho = 0$$

Noether identities $\quad \gamma^{\mu\nu\rho}\partial_\mu\partial_\nu\Psi_\rho = 0$

Using $\quad \gamma_\mu\gamma^{\mu\nu\rho} = (D-2)\gamma^{\nu\rho}$

$$\gamma^{\mu\nu\rho} = \gamma^\mu\gamma^{\nu\rho} - \eta^{\mu\nu}\gamma^\rho + \eta^{\mu\rho}\gamma^\mu$$

We can write the equations of motion as

$$\gamma^\mu(\partial_\mu\Psi_\nu - \partial_\nu\Psi_\mu) = 0$$

The free Rarita-Schwinger field

- Massless particles

$$\partial\!\!\!/(\partial_\rho \Psi_\nu - \partial_\nu \Psi_\rho) = 0$$

The free Rarita-Schwinger field

- Initial value problem

The gauge $\gamma^i \Psi_i = 0$ fixes completely the gauge

$$\gamma^i \Psi_i' = \gamma^i (\Psi_i + \partial_i \epsilon) = \gamma^i \partial_i \epsilon \longrightarrow \nabla^2 \epsilon = 0$$

the equations of motion in components

$$\begin{aligned} \gamma^i \partial_i \Psi_0 - \partial_0 \gamma^i \Psi_i &= 0 \\ \gamma \cdot \partial \Psi_i - \partial_i \gamma \cdot \Psi &= 0 \end{aligned}$$

Using the gauge condition one can see that $\nabla^2 \Psi_0 = 0$, so $\Psi_0 = 0$

The free Rarita-Schwinger field

- Initial value problem

The spatial components Ψ_i then satisfy the Dirac equation

$$\gamma \cdot \partial \Psi_i = 0$$

We have also $\partial^i \Psi_i = 0$

The restrictions on the initial conditions are

$$\begin{aligned} \gamma^i \Psi_i(\vec{x}, 0) &= 0 \\ \Psi_0(\vec{x}, 0) &= 0 \\ \partial^i \Psi_i(\vec{x}, 0) &= 0 \end{aligned}$$

there are only $2^{[\frac{D}{2}]}(D-3)$ initial components

The free Rarita-Schwinger field

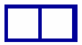
- Initial value problem

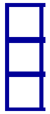
The on-sheel degrees freedom are half of $2^{[\frac{D}{2}]}(D-3)$

In D=4, with Majorana conditions, we frind two states expected for a
a masslees particle for any s>0. The helicities are +3/2 and -3/2

Degrees of freedom

D=11 Supergravity

128 bosons degrees of freedom
graviton $g_{\mu\nu}$, **44** of SO(9): 

3-form $C_{\mu\nu\rho}$, $C_{(3)}$, **84** of SO(9): 

128 fermions degrees of freedom

SO(9) little group of massless particles of ISO(10,1)

$$L = \sqrt{-g}\{R - \frac{1}{2}(F_{(4)})^2 + \cdots\} + C_{(3)} \wedge dC_{(3)} \wedge dC_{(3)} + \cdots \textit{fermions}$$

$F_{(4)} = \partial C_{(3)}$, 4-form field strength

$$d(*F_{(4)} + C_{(3)} \wedge dC_{(3)}) = 0$$