Dimensional Reduction on $\text{Minkowski}_D \times S^1$.

- Scalar fields

We consider a massive complex scalar field $\phi(x^\mu, y)$

$$[\Box_{D+1} - m^2]\phi = \left[\Box_D + \left(\frac{\partial}{\partial y}\right)^2 - m^2\right]\phi = 0$$

Acceptable solutions must be single-valued on S^1

$$\phi(x^\mu, y) = \sum_{k=-\infty}^{\infty} \mathrm{e}^{\frac{iky}{L}} \phi_k(x^\mu)$$

We have

$$\left[\Box_D - \left(\frac{k}{L}\right)^2 - m^2\right]\phi_k = 0 \qquad {m_k}^2 = \left(\tfrac{k}{L}\right)^2 + m^2$$

Dimensional reduction

- Spinor fields

Consider D=2m, the spinors in D+1 have the same number of components

$$\text{if } \Psi(x) \text{ satisfies} \qquad [\not{\partial}_D - m]\Psi(x) = 0$$

$$\tilde{\Psi} \equiv \mathrm{e}^{-\mathrm{i}\gamma_* \beta}\Psi \qquad [\not{\partial}_D - m(\cos 2\beta + \mathrm{i}\gamma_* \sin 2\beta)]\tilde{\Psi}$$

The sign of of m has no physical significance, since it can be changed By a field redefinition with $\beta = \pi/2$

Dimensional reduction

- Periodic and antiperiodic boundary conditions

$$\Psi(x^\mu, y) = \pm\Psi(x^\mu, y+2\pi L))$$

Fourier expansion

$$\Psi(x^\mu, y) = \sum \mathrm{e}^{\frac{\mathrm{i}ky}{L}}\Psi_k(x^\mu),$$

mode number k is integer or half-integer

The D+1 Dirac equation $[\not{\partial}_{D+1} - m]\Psi((x^\mu, y)$

$D = 2m+1, \quad \gamma^D = \pm\gamma_*$ where γ_*

highest rank Clifford element in $D = 2m$ dimensions

implies

$$\left[\not{\partial}_D - \left(m + \mathrm{i}\gamma_* \frac{k}{L}\right)\right]\Psi_k(x^\mu) = 0\,.$$

Dimensional reduction

- Periodic and antiperiodic boundary conditions

With a chiral transformation with phase $\tan 2\beta = k/(mL)$

We see $\Psi_k(x^\mu)$ describes particles of mass ${m_k}^2 = (\frac{k}{L})^2 + m^2$

Dimensional reduction

- Maxwell Field

$$\partial^N F_{NM} = \partial^2 A_M - \partial_M(\partial^N A_N)$$

$$A_\mu(x,y) = \sum_k \mathrm{e}^{\frac{\mathrm{i}ky}{L}} A_{\mu k}(x)\,, \qquad A_D(x,y) = \sum_k \mathrm{e}^{\frac{\mathrm{i}ky}{L}} A_{Dk}(x)$$

Gauge fixing condition $\partial_Y A_D = 0$ implies $A_{Dk}(x) = 0 \text{ for } k \neq 0$

$\mu \to D \text{ component}$

$$\begin{aligned} k = 0 \quad &: \quad \Box_{D+1} A_{D0} = \Box_D A_{D0} \;=\; 0 \\ k \neq 0 \quad &: \quad \partial^\mu A_{\mu k} \;=\; 0\,, \end{aligned}$$

Dimensional reduction

$A_{D0}(x)$ simply describes a massless scalar in D dimensions

Dimensional reduction

- Maxwell Field

$$\left[\Box_D - \frac{k^2}{L^2}\right] A_{\mu k} - \partial_\mu(\partial^\nu A_{\nu k}) = 0\,.$$

For mode number $k = 0$ this is just the Maxwell equation

For mode number $k \neq 0$ massive vector field with mass $m_k^2 = k^2/L^2$

Degrees of freedom (initial conditions) $k \neq 0$ 2(D-1), the on-shell degrees of freedom are D-1

k=0, 2(D-2) corresponding to the vector $A_{\mu 0}$ and 2 associated to an scalar A_{D0}

It coincides with the counting of a massless gauge vector in D+1

Dimensional reduction

- Action of the massive gauge field

$$S = \int \mathrm{d}^D x \, \left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} m^2 A_\mu A^\mu \right]$$

Dimensional reduction

- Rarita Schwinger Field

Consider a massless Rarita-Schwinger field in D+1 with D=2m.
We assume $\Psi_\mu(x,y)$ is antiperiodic, so the Fourier series modes involve only half-integer k

We choose the gauge $\Psi_D = 0$

The reduced equations are

$$\mu = D \qquad \gamma^{\nu\rho}\partial_\nu \Psi_{\rho k} \;=\; 0\,,$$

$$\mu \leq D-1 \qquad \left[\gamma^{\mu\nu\rho}\partial_\nu - \mathrm{i}\frac{k}{L}\gamma_*\gamma^{\mu\rho}\right]\Psi_{\rho k} \;=\; 0$$

$$chiral\ \ transformation\ \ \Psi_{\rho k} \;=\; \mathrm{e}^{(-\mathrm{i}\pi\gamma_*/4)}\Psi'_{\rho k}$$

Dimensional reduction

- Rarita Schwinger Field

Gives the equation of motion of a massive RS field

$$(\gamma^{\mu\nu\rho}\partial_\nu - m\gamma^{\mu\rho})\,\Psi_\rho \;=\; 0$$

There are two constraints

$$\gamma^\mu \Psi_\mu = 0$$
$$\partial^\mu \psi_\mu = 0$$

Th equation of motion becomes $$\left[\partial\!\!\!/ + m\right]\Psi_\mu = 0$$

$$\frac{1}{2}(D-2) \times 2^{[D/2]} \text{ on-shell physical states}$$

Dimensional reduction

Ex. 5.11 *Study the Kaluza-Klein reduction for the Rarita-Schwinger field assuming periodicity $\Psi_\mu(x, y + 2\pi) = \Psi_\mu(x, y)$ in y. Show that the spectrum seen in Minkowski$_D$ consists of a massive gravitino for each Fourier mode $k \neq 0$ plus a massless gravitino and massless Dirac particle for the zero mode.*

Differential geometry

- The metric and the frame field

Line element $\mathrm{d}s^2 = g_{\mu\nu}(x)\mathrm{d}x^\mu \mathrm{d}x^\nu$ Non-degenerate metric

$$g^{\mu\rho}g_{\rho\nu} = g_{\nu\rho}g^{\rho\mu} = \delta^\mu_\nu$$

Frame field $g_{\mu\nu}(x) = e^a_\mu(x)\eta_{ab}e^b_\nu(x)$ $\eta_{ab} = \mathrm{diag}(-1, 1, \ldots, 1)$

Inverse frame field $e^\mu_a(x)$ $e^a_\mu e^\mu_b = \delta^a_b$ and $e^\mu_a e^a_\nu = \delta^\mu_\nu$

Given the metric $g_{\mu\nu}(x)$, the frame field $e^a_\mu(x)$ is not uniquely determined. Any local Lorentz transformation $\Lambda^a{}_b(x)$, which leaves η_{ab} invariant, produces an equally good frame field

$$e'^a_\mu(x) = \Lambda^{-1\,a}{}_b(x)e^b_\mu(x)\,. \tag{6.27}$$

Differential geometry

- Frame field

$$e'^{a}_{\mu}(x') = \frac{\partial x^{\rho}}{\partial x'^{\mu}} e^{a}_{\rho}(x) , \qquad e'^{\mu}_{a}(x') = \frac{\partial x'^{\mu}}{\partial x^{\rho}} e^{\rho}_{a}(x) .$$

$$V^{\mu}(x) = V^{a}(x) e^{\mu}_{a}(x) \text{ with } V^{a}(x) = V^{\mu}(x) e^{a}_{\mu}(x).$$

Vector under Lorentz transformations

$$V'^{a}(x) = \Lambda^{-1\,a}{}_{b}(x) V^{b}(x).$$

Vector field $\qquad E_{a} \equiv e^{\mu}_{a}(x) \dfrac{\partial}{\partial x^{\mu}} .$

Dual form $\qquad e^{a} \equiv e^{a}_{\mu}(x) \mathrm{d}x^{\mu} . \qquad \langle e^{a} | E_{b} \rangle = \delta^{a}_{b}$

$$[E_{a}, E_{b}] = \Omega^{c}_{ab} E_{c} \qquad \Omega^{c}_{ab} = e^{\mu}_{a} e^{\nu}_{b} (\partial_{\mu} e^{c}_{\nu} - \partial_{\nu} e^{c}_{\mu})$$

Volume forms and integration

any top degree D-form $\omega^{(D)}$ can be integrated

$$
\begin{aligned}
I &= \int \omega^{(D)} \\
&= \frac{1}{D!}\int \omega_{\mu_1\cdots\mu_D}(x)\mathrm{d}x^{\mu_1}\wedge\ldots\wedge\mathrm{d}x^{\mu_D} \\
&= \int \omega_{01\cdots D-1}\mathrm{d}x^0\mathrm{d}x^1\ldots\mathrm{d}x^{D-1}\,.
\end{aligned}
$$

Canonical volume form depends of the metric or frame field

$$
\begin{aligned}
\mathrm{d}V &\equiv e^0\wedge e^1\wedge\ldots\wedge e^{D-1} \\
&= \frac{1}{D!}\varepsilon_{a_1\cdots a_D}e^{a_1}\wedge\cdots\wedge e^{a_D} \\
&= \frac{1}{D!}e\,\varepsilon_{\mu_1\cdots\mu_D}\mathrm{d}x^{\mu_1}\wedge\ldots\wedge\mathrm{d}x^{\mu_D}
\end{aligned}
$$

Volume forms and integration

$$\begin{aligned} \mathrm{dV} &= e\,\mathrm{d}x^0 \ldots \mathrm{d}x^{D-1} \\ &= \mathrm{d}^D x\,\sqrt{-\det g}\,. \end{aligned}$$

$$\varepsilon_{a_1 a_2 \cdots a_D} = \begin{cases} +1 & a_1 a_2 \cdots a_D \text{ an even permutation of } 01\ldots(D-1) \\ -1 & a_1 a_2 \cdots a_D \text{ an odd permutation of } 01\ldots(D-1) \\ 0 & \text{otherwise.} \end{cases}$$

$$\begin{aligned} \varepsilon_{\mu_1 \mu_2 \cdots \mu_D} &= e^{-1} \varepsilon_{a_1 a_2 \cdots a_D} e^{a_1}_{\mu_1} e^{a_2}_{\mu_2} \cdots e^{a_D}_{\mu_D} \\ \varepsilon^{\mu_1 \mu_2 \cdots \mu_D} &= e\,\varepsilon^{a_1 a_2 \cdots a_D} e^{\mu_1}_{a_1} e^{\mu_2}_{a_2} \cdots e^{\mu_D}_{a_D}\,. \end{aligned}$$

$$e = \det e^a_\mu$$

Action for fields

$$S = \int \mathrm{d}V\,\mathcal{L} = \int \mathrm{d}^D x \sqrt{-\det g}\,\mathcal{L}$$

Hodge duality of forms

$$^{*}e^{a_1} \wedge \ldots e^{a_p} = \frac{1}{q!} e^{b_1} \wedge \ldots e^{b_q} \varepsilon_{b_1 \cdots b_q}{}^{a_1 \cdots a_p}$$

$$\begin{aligned} \Omega^{(q)} =^{*} \omega^{(p)} &= {}^{*}\left(\frac{1}{p!} \omega_{a_1 \cdots a_p} e^{a_1} \wedge \ldots e^{a_p}\right) \\ &= \frac{1}{p!} \omega_{a_1 \cdots a_p} {}^{*}e^{a_1} \wedge \ldots e^{a_p} . \end{aligned}$$

Lorentzian signature $\quad {}^{*}({}^{*}\omega^{(p)}) = -(-)^{pq} \omega^{(p)}$

Euclidean signature $\quad {}^{*}({}^{*}\omega^{(p)}) = (-)^{pq} \omega^{(p)}$

For D=2m it is possible the constraint of self-duality or antiself duality

$$\Omega^{(m)} = \pm {}^{*}\Omega^{(m)}$$

Hodge duality of forms

Lorentzian siganture $\quad -(-)^m = +1$

self-dual $F^{(5)}$ is possible in $D = 10$

Euclidean signature $\quad (-)^m = +1$

self-dual Yang-Mills instantons in 4 Euclidean

${}^*\omega^{(p)} \wedge \omega^{(p)}$ Is a top form and can be integrated

$$\int {}^*\omega^{(p)} \wedge \omega^{(p)} = \frac{1}{p!} \int \mathrm{d}^D x \sqrt{-g}\, \omega^{\mu_1 \cdots \mu_p} \omega_{\mu_1 \cdots \mu_p}$$

p-forms gauge fields

$$S_0 = -\tfrac{1}{2}\int {}^*F^{(1)} \wedge F^{(1)}\,, \qquad F^{(1)} = \mathrm{d}\phi\,,$$

$$S_1 = -\tfrac{1}{2}\int {}^*F^{(2)} \wedge F^{(2)}\,, \qquad F^{(2)} = \mathrm{d}A^{(1)}$$

Bianchi identity $\mathrm{d}F^{(1)} = 0 \text{ and } \mathrm{d}F^{(2)} = 0.$

$$S_p = -\tfrac{1}{2}\int {}^*F^{(p+1)} \wedge F^{(p+1)}\,, \qquad F^{(p+1)} = \mathrm{d}A^{(p)}$$

$$S_p = -\frac{1}{2\,(p+1)!}\int \mathrm{d}^D x\,\sqrt{-g}\,F^{\mu_1\cdots\mu_{p+1}}F_{\mu_1\cdots\mu_{p+1}}$$

$$F_{\mu_1\cdots\mu_{p+1}} = (p{+}1)\partial_{[\mu_1}A_{\mu_2\ldots\mu_{p+1}]}$$

p-forms gauge fields

$$\mathrm{d}^* F^{(p+1)} = 0$$ equations of motion

$$\mathrm{d} F^{(p+1)} = 0$$ Bianchi identity

A p-form and D-p-2 form are dual

$$S_p = -\int \left[\frac{1}{2} {}^* F^{(p+1)} \wedge F^{(p+1)} + b^{(D-p-2)} \wedge \mathrm{d} F^{(p+1)}\right]$$

can consider $F^{(p+1)}$ and $b^{(D-p-2)}$ as the independent fields.

Algebraic equation of motion $${}^* F^{(p+1)} = (-)^{D-p}\, \mathrm{d} b^{(D-p-2)}$$

$b^{(D-p-2)}$ takes the role of $A^{(p)}$

p-forms gauge fields

$$\binom{D}{p} - \binom{D}{p-1} + \binom{D}{p-2} - \ldots = \binom{D-1}{p}$$

Off-shell degrees of freedom, number of compoents of a p-form in D-1 Dimensions.

On-shell degrees of freedom $\binom{D-2}{p}$