# First structure equation

- Spin connection

Let us consider the differential of the vielbvein

$$\mathrm{d}e^a = \frac{1}{2}(\partial_\mu e^a_\nu - \partial_\nu e^a_\mu)\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$$

it is not a Lorentz vector. Introduce the spin connection connection one form

$$\omega'^a{}_b = \Lambda^{-1\,a}{}_c \mathrm{d}\Lambda^c{}_b + \Lambda^{-1\,a}{}_c\,\omega^c{}_d\,\Lambda^d{}_b$$

The quantity $\mathrm{d}e^a + \omega^a{}_b \wedge e^b \equiv T^a$ .

transforms as a vector $T'^a \, = \, \Lambda^{-1\,a}{}_b T^b.$

$$\text{torsion tensor } T^a_{\mu\nu} = -T^a_{\nu\mu}$$

$$\omega'_\mu{}^a{}_b = \Lambda^{-1\,a}{}_c\,\partial_\mu \Lambda^c{}_b + \Lambda^{-1\,a}{}_c\,\omega_\mu{}^c{}_d\,\Lambda^d{}_b\,.$$

same transformation properties that YM potential for the group O(D-1,1)

# First structure equation

- Lorentz Covariant derivatives

$$
\begin{aligned}
V'^a(x) &= \Lambda^{-1\,a}{}_b(x) V^b(x)\,, \\
U'_a(x) &= U_b(x) \Lambda^b{}_a(x)\,, \\
T'_{ab}(x) &= T_{cd}(x) \Lambda^c{}_a(x) \Lambda^d{}_b(x)\,,
\end{aligned}
$$

$$
\begin{aligned}
D_\mu V^a &= \partial_\mu V^a + \omega_\mu{}^a{}_b V^b\,, \\
D_\mu U_a &= \partial_\mu U_a - U_b \omega_\mu{}^b{}_a = \partial_\mu U_a + \omega_{\mu a}{}^b U_b\,, \\
D_\mu T_{ab} &= \partial_\mu T_{ab} - T_{cb} \omega_\mu{}^c{}_a - T_{ac} \omega_\mu{}^c{}_b\,.
\end{aligned}
$$

$$
D_\mu \eta_{ab} = -\eta_{cb} \omega_\mu{}^c{}_a - \eta_{ac} \omega_\mu{}^c{}_b = -\omega_{\mu ba} - \omega_{\mu ab} = 0\,.
$$

The metric has vanishing covarint derivative.

# First structure equation

scalar products $V^a U_a$ are preserved under parallel transport

$$\tilde{\dot{V}}^a(x) = V^a(x) + \omega^a_{\mu b} V^a(\dot{x}) \Delta x^\mu$$

The geometrical effect of torsion is seen in the properties of an infinitesimal parallelogram constructed by the parallel transport of two vector fields.

For the Levi-Civita connection the torsion vanishes

$$\mathrm{d}e^a + \omega^a{}_b \wedge e^b = 0$$

Non-vanishing torsion appears in supergravity

# First structure equation

- Covariant derivatives

$$\omega_{\mu[\rho\nu]} = \omega_{\mu ab} e_\rho^a e_\nu^b \text{ in terms of}$$

$$\Omega_{[\mu\nu]\rho} = \left(\partial_\mu e_\nu^a - \partial_\nu e_\mu^a\right) \, e_{a\rho} \, , \qquad T_{[\mu\nu]\rho} = T_{\mu\nu}{}^a e_{a\rho}$$

The structure equation $T_{[\mu\nu]\rho} = \Omega_{[\mu\nu]\rho} + \omega_{\mu[\rho\nu]} - \omega_{\nu[\rho\mu]} \, .$

implies

$$\begin{aligned}
\omega_{\mu[\nu\rho]} &= \omega_{\mu[\nu\rho]}(e) + K_{\mu[\nu\rho]} \, , \\
\omega_{\mu[\nu\rho]}(e) &= \tfrac{1}{2}(\Omega_{[\mu\nu]\rho} - \Omega_{[\nu\rho]\mu} + \Omega_{[\rho\mu]\nu}) = \omega_{\mu ab}(e) e_\nu^a e_\rho^b \, , \\
\omega_\mu{}^{ab}(e) &= 2e^{\nu[a}\partial_{[\mu} e_{\nu]}{}^{b]} - e^{\nu[a} e^{b]\sigma} e_{\mu c} \partial_\nu e_\sigma{}^c \, , \\
K_{\mu[\nu\rho]} &= -\tfrac{1}{2}(T_{[\mu\nu]\rho} - T_{[\nu\rho]\mu} + T_{[\rho\mu]\nu}) \, .
\end{aligned}$$

$K_{\mu[\nu\rho]}$ is called contorsion

# First structure equation

- Covariant derivatives

Matrix representation of Lorentz transformation

$$\Lambda(x) = \exp\left(\tfrac{1}{2}\lambda^{ab}(x) m_{\{ab\}}\right)$$

Spinor representation

$$\begin{aligned} \Psi'(x) &= \exp\left(-\tfrac{1}{4}\lambda^{ab}(x)\gamma_{ab}\right)\Psi(x)\,, \\ D_\mu\Psi(x) &= \left(\partial_\mu + \tfrac{1}{4}\omega_\mu{}^{ab}(x)\gamma_{ab}\right)\Psi(x)\,. \end{aligned}$$

# First structure equation

- The affine connection

Our next task is to transform Lorentz covariant derivatives to covariant derivatives with respect to general conformal transformations

$$\begin{aligned} \nabla_\mu V^\rho &\equiv e_a^\rho D_\mu V^a \\ &= e_a^\rho D_\mu (e_\nu^a V^\nu) \\ &= \partial_\mu V^\rho + e_a^\rho (\partial_\mu e_\nu^a + \omega_\mu{}^a{}_b e_\nu^b) V^\nu \,. \end{aligned}$$

Affine connection $\quad \Gamma^\rho_{\mu\nu} = e_a^\rho (\partial_\mu e_\nu^a + \omega_\mu{}^a{}_b e_\nu^b) \,.$

relates affine connection with spin connection

$$\nabla_\mu V^\rho = \partial_\mu V^\rho + \Gamma^\rho_{\mu\nu} V^\nu$$

$\partial_\mu e_\nu^a + \omega_\mu{}^a{}_b e_\nu^b - \Gamma^\sigma_{\mu\nu} e_\sigma^a = 0$ $\qquad$ vielbein postulate

# First structure equation

$$\nabla_\mu V_\nu \equiv e^a_\nu D_\mu V_a \qquad \nabla_\mu V_\nu \;=\; \partial_\mu V_\nu - e^\rho_a(\partial_\mu e^a_\nu + \omega_\mu{}^a{}_b e^b_\nu)V_\rho \,.$$

$$\nabla_\mu V_\nu \;\;=\;\; \partial_\mu V_\nu - \Gamma^\rho_{\mu\nu} V_\rho$$

For tensors in general $$\nabla_\mu T^{\rho_1\cdots\rho_p}_{\nu_1\cdots\nu_q} \equiv e^{\rho_1}_{a_1} \ldots e^{\rho_p}_{a_p}\, e^{b_1}_{\nu_1} \ldots e^{b_q}_{\nu_q}\, D_\mu T^{a_1\cdots a_p}_{b_1\cdots b_q}$$

$$\nabla_\mu g_{\nu\rho} \equiv \partial_\mu g_{\nu\rho} - \Gamma^\sigma_{\mu\nu} g_{\sigma\rho} - \Gamma^\sigma_{\mu\rho} g_{\nu\sigma} = 0$$

Covariant differentation commutes with index raising

$$\nabla_\mu V^\rho = g^{\rho\nu} \nabla_\mu V_\nu$$

# First structure equation

- The affine connection

$$\Gamma^{\rho}_{\mu\nu} = \Gamma^{\rho}_{\mu\nu}(g) - K_{\mu\nu}{}^{\rho}\,,$$

$$\Gamma^{\rho}_{\mu\nu}(g) = \frac{1}{2}g^{\rho\sigma}(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu})\,.$$

$$\Gamma^{\rho}_{\mu\nu} - \Gamma^{\rho}_{\nu\mu} = -K_{\mu\nu}{}^{\rho} + K_{\nu\mu}{}^{\rho} = T_{\mu\nu}{}^{\rho}\,.$$

For mixed quantities with both coordinate ans frame indexes, it is useful to distinguish among local Lorentz and coordinate covariant derivatives

$$D_\mu \Psi_\nu \equiv \left(\partial_\mu + \tfrac{1}{4}\omega_{\mu ab}\gamma^{ab}\right)\Psi_\nu\,,$$

$$\nabla_\mu \Psi_\nu = D_\mu \Psi_\nu - \Gamma^{\rho}_{\mu\nu}\Psi_\rho\,.$$

Vielbein postulate euivalent to $\nabla_\mu e^a_\nu = \partial_\mu e^a_\nu + \omega_\mu{}^a{}_b e^b_\nu - \Gamma^{\sigma}_{\mu\nu} e^a_\sigma = 0$

# First structure equation

- Partial integration We have

$$\partial_\mu \sqrt{-g} = \sqrt{-g}\,\Gamma^{\rho}_{\rho\mu}(g)$$ from which

$$\int \mathrm{d}^D x \,\sqrt{-g}\,\nabla_\mu V^\mu = \int \mathrm{d}^D x\,\partial_\mu\left(\sqrt{-g}\,V^\mu\right) - \int \mathrm{d}^D x\,\sqrt{-g}\,K_{\nu\mu}{}^{\nu} V^\mu$$

The second term shows the violation of the manipulations of the integration by Parts in the case of torsion

$$K_{\nu\mu}{}^{\nu} = -T_{\nu\mu}{}^{\nu}$$

# Second structure equation

- Curvature tensor

spin connection $\omega_{\mu ab}$ transforms as a YM gauge potential for the Group O(D-1,1)

$$R_{\mu\nu ab} \equiv \partial_\mu \omega_{\nu ab} - \partial_\nu \omega_{\mu ab} + \omega_{\mu ac}\omega_\nu{}^c{}_b - \omega_{\nu ac}\omega_\mu{}^c{}_b$$

YM field strength. We define the curvature two form

$$\rho^{ab} = \frac{1}{2} R_{\mu\nu}{}^{ab}(x) \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \,.$$

Second structure equation

$$\mathrm{d}\omega^{ab} + \omega^a{}_c \wedge \omega^{cb} = \rho^{ab} \,.$$

# Bianchi identities

$$\rho^{ab} \wedge e_b = \mathrm{d}T^a + \omega^{ab} \wedge T_b \,,$$

$$\mathrm{d}\rho^{ab} + \omega^a{}_c \wedge \rho^{cb} - \rho^{ac} \wedge \omega_c{}^b = 0$$

$using\ R_{\mu\nu\rho}{}^a = R_{\mu\nu b}{}^a e^b_\rho$ we have

$$R_{\mu\nu\rho}{}^a + R_{\nu\rho\mu}{}^a + R_{\rho\mu\nu}{}^a = -D_\mu T_{\nu\rho}{}^a - D_\nu T_{\rho\mu}{}^a - D_\rho T_{\mu\nu}{}^a$$

First Bianchi identity, it has no analogue in YM

$$D_\mu R_{\nu\rho}{}^{ab} + D_\nu R_{\rho\mu}{}^{ab} + D_\rho R_{\mu\nu}{}^{ab} = 0$$

usual Bianchi identity for YM

useful relation $\delta R_{\mu\nu ab} = D_\mu \delta\omega_{\nu ab} - D_\nu \delta\omega_{\mu ab}$

# Ricci identities and curvature tensor

Commutator of covariant derivatives

$$
\begin{aligned}
[D_\mu, D_\nu]\Phi &= \tfrac{1}{2} R_{\mu\nu ab} M^{ab} \Phi \\
[D_\mu, D_\nu]V^a &= R_{\mu\nu}{}^a{}_b V^b \,, \\
[D_\mu, D_\nu]\Psi &= \tfrac{1}{4} R_{\mu\nu ab} \gamma^{ab} \Psi \,.
\end{aligned}
$$

$$
[\nabla_\mu, \nabla_\nu] V^\rho = R_{\mu\nu}{}^\rho{}_\sigma V^\sigma - T_{\mu\nu}{}^\sigma \nabla_\sigma V^\rho
$$

Curvature tensor

$$
R_{\mu\nu}{}^\rho{}_\sigma = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\tau} \Gamma^\tau_{\nu\sigma} - \Gamma^\rho_{\nu\tau} \Gamma^\tau_{\mu\sigma}
$$

$$
R_{\mu\nu}{}^\rho{}_\sigma = R_{\mu\nu ab} e^{a\rho} e^b_\sigma
$$

Second Bianchi identity

$$
\nabla_\mu R_{\nu\rho}{}^{\sigma\tau} + \nabla_\nu R_{\rho\mu}{}^{\sigma\tau} + \nabla_\rho R_{\mu\nu}{}^{\sigma\tau} = T_{\mu\nu}{}^\xi R_{\xi\rho}{}^{\sigma\tau} + T_{\nu\rho}{}^\xi R_{\xi\mu}{}^{\sigma\tau} + T_{\rho\mu}{}^\xi R_{\xi\nu}{}^{\sigma\tau}
$$

# Ricci tensor

Ricci tensor $$R_{\mu\nu} = R_{\mu}{}^{\sigma}{}_{\nu\sigma}$$

Scalar curvature R= $g^{\mu\nu}R_{\mu\nu}$

If there is no torsion $$R_{\mu\nu} = R_{\nu\mu} \quad R_{\mu\nu\rho\sigma} = R_{\rho\sigma\mu\nu}$$

Useful relation $$\delta R_{\mu\nu}{}^{\rho}{}_{\sigma} = \nabla_{\mu}\delta\Gamma^{\rho}_{\nu\sigma} - \nabla_{\nu}\delta\Gamma^{\rho}_{\mu\sigma}$$

Hilbert action

$$\frac{1}{(D-2)!}\int \varepsilon_{abc_1\ldots c_{(D-2)}} e^{c_1} \wedge \cdots \wedge e^{c_{(D-2)}} \wedge \rho^{ab} = \int \mathrm{d}^D x \sqrt{-g} R$$

# Dimensional analysis and Planck units

$$\frac{Gm_p}{c^2} = \frac{\hbar}{m_p c}$$

$$m_p = \sqrt{\frac{\hbar c}{G}} \qquad \simeq 10^{-5}\,\mathrm{grams} \simeq 10^{19}\,\mathrm{GeV}$$

$$l_p = \sqrt{\frac{G\hbar}{c^3}} = \sqrt{\frac{\kappa^2}{8\pi}} \qquad \simeq 10^{-33}\,\mathrm{cm}$$

$$t_p = \sqrt{\frac{G\hbar}{c^5}} = \sqrt{\frac{\kappa^2}{8\pi c^2}} \quad \simeq 10^{-44}\,\mathrm{seconds}$$

# The first and second order formulations of general relativity

- Second order formalism

Field content, $g_{\mu\nu}(x)$ $\phi(x)$ $A_\mu(x)$

Action

$$S = \int \mathrm{d}^D x \sqrt{-\det g} \left( \frac{1}{2\kappa^2} g^{\mu\nu} R_{\mu\nu}(g) + L \right)$$

$$L = -\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{1}{4} g^{\mu\rho} g^{\nu\sigma} F_{\mu\nu} F_{\rho\sigma} .$$

$\kappa^2 = 8\pi G_N$ is the gravitational coupling constant

$M_P \equiv M_{\text{Planck}}/\sqrt{8\pi} \sim 2 \times 10^{18}$ GeV

$\partial_\mu \phi = \nabla_\mu \phi$ $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)$

# The first and second order formulations of general relativity

- Variation of the action

$$\delta S = \frac{1}{2\kappa^2}\int \mathrm{d}^D x[\delta(\sqrt{-\det g}g^{\mu\nu})R_{\mu\nu} + \sqrt{-\det g}g^{\mu\nu}\delta R_{\mu\nu} + \ldots]$$

Last term total derivative due $\delta R_{\mu\nu} \;=\; \nabla_\rho \delta\Gamma^\rho_{\mu\nu} - \nabla_\mu \delta\Gamma^\rho_{\nu\rho}$ plus no torsion

Einstein equations

$$G_{\mu\nu} \;\equiv\; R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \kappa^2 T_{\mu\nu}\,,$$

$$T_{\mu\nu} \;\equiv\; -2\frac{1}{\sqrt{-\det g}}\frac{\delta(\sqrt{-\det g}\,L)}{\delta g^{\mu\nu}} \;=\; \partial_\mu\phi\partial_\nu\phi + F_\mu{}^\rho F_{\nu\rho} + g_{\mu\nu}L$$

The Einstein equations are consistent only if the matter tensor $\nabla^\mu T_{\mu\nu} = 0$

# The first and second order formulations of general relativity

- Conservation energy-momentum tensor

The invariance under diff of the matter action

$$\delta \int \mathrm{d}^D x \sqrt{-\det g}\, L =$$

$$\int \mathrm{d}^D x \, \sqrt{-\det g} \left[ \frac{1}{2} T^{\mu\nu} (\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu) + \frac{\delta L}{\delta \phi} \mathcal{L}_\xi \phi + \frac{\delta L}{\delta A_\mu} \mathcal{L}_\xi A_\mu \right] = 0$$

if $\phi$ and $A_\mu$ do satisfy the equations of motion

$$\int \mathrm{d}^D x \sqrt{-\det g} T^{\mu\nu} \nabla_\mu \xi_\nu = 0$$

which implies $$\nabla^\mu T_{\mu\nu} = 0$$

# The first and second order formulations of general relativity

- Scalar and gauge field equations

$$
\begin{aligned}
\partial_\mu(\sqrt{-\det g}\, g^{\mu\nu}\partial_\nu\phi) &= \sqrt{-\det g}\, g^{\mu\nu}\, \nabla_\mu \partial_\nu \phi = 0 \\
\partial_\mu(\sqrt{-\det g}\, F^{\mu\nu}) &= \sqrt{-\det g}\, \nabla_\mu F^{\mu\nu} = 0\,.
\end{aligned}
$$

Ricci form of Einstein field equation

$$
R_{\mu\nu} = \kappa^2 \left[ T_{\mu\nu} - \frac{1}{D-2} g_{\mu\nu} T^\rho_\rho \right]
$$

# The first and second order formulations of general relativity

## Useful relations

$$
\begin{aligned}
\delta g^{\mu\nu} &= -g^{\mu\rho}\delta g_{\rho\sigma}g^{\sigma\nu}\ , \\
\delta\sqrt{-\det g} &= \frac{1}{2}\sqrt{-\det g}\, g^{\mu\nu}\delta g_{\mu\nu} \;=\; -\frac{1}{2}\sqrt{-\det g} g_{\mu\nu}\,\delta g^{\mu\nu}
\end{aligned}
$$

Particular case of $\delta \det M = (\det M)\,\mathrm{Tr}(M^{-1}\delta M)$

# The first and second order formulations of general relativity

- Matter scalars

L= $-\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)$

$$R_{\mu\nu} \;=\; \kappa^2\left[\partial_\mu\phi\partial_\nu\phi + g_{\mu\nu}\frac{2}{D-2}V\right]$$

# The first and second order formulations of general relativity

## Gravitational fluctuations of flat spacetime

In absence of matter $R_{\mu\nu} = 0$ Solution $g_{\mu\nu} = \eta_{\mu\nu}$

fluctuations $g_{\mu\nu}(x) = \eta_{\mu\nu} + \kappa h_{\mu\nu}(x)$

$$\Gamma^{\rho\,\mathrm{Lin}}_{\mu\nu} = \tfrac{1}{2}\kappa\,\eta^{\rho\sigma}(\partial_\mu h_{\sigma\nu} + \partial_\nu h_{\mu\sigma} - \partial_\sigma h_{\mu\nu})$$

$$R^{\mathrm{Lin}}_{\mu\nu} = -\tfrac{1}{2}\kappa\,[\Box h_{\mu\nu} - \partial^\rho(\partial_\mu h_{\rho\nu} + \partial_\nu h_{\mu\rho}) + \partial_\mu\partial_\nu h^\rho_\rho]$$

The gauge transformations are obtained linearizing the diff transformations

$$\begin{aligned}
\delta g_{\mu\nu} &= \kappa\,(\nabla_\mu\xi_\nu + \nabla_\nu\xi_\mu)\,, \\
\delta R_{\mu\nu} &= \kappa\,(\xi^\rho\nabla_\rho R_{\mu\nu} + \nabla_\mu\xi^\rho R_{\rho\nu} + \nabla_\nu\xi^\rho R_{\mu\rho})
\end{aligned}$$

# The first and second order formulations of general relativity

$$\begin{aligned} \delta h_{\mu\nu} &= \partial_\mu \xi_\nu + \partial_\nu \xi_\nu \\ \delta R^{\text{Lin}}_{\mu\nu} &= 0\,. \end{aligned}$$

Degrees of freedom. Choose the gauge $\partial^i h_{i\mu} = 0$

Fixes completely the gauge from

$$\delta(\partial^i h_{i\mu}) = \nabla^2 \xi_\mu + \partial^i \partial_\mu \xi_i = 0$$

Let $\mu \to j$ and contract with $\partial^j$ to learn that $\nabla^2 \partial^i \xi_i = 0$

We have $\xi_\mu(x) \equiv 0$

The equations of motion become

$$\Box h_{\mu\nu} - \partial^0(\partial_\mu h_{0\nu} + \partial_\nu h_{\mu 0}) + \partial_\mu \partial_\nu (h_{ii} - h_{00}) = 0\,.$$

# The first and second order formulations of general relativity

- Degree of freedom

$$\mu = \nu = 0 \qquad : \quad \nabla^2 h_{00} + \partial_0^2 h_{ii} = 0\,,$$
$$\mu = \nu = i \qquad : \quad 2\nabla^2 h_{ii} - \partial_0^2 h_{ii} - \nabla^2 h_{00} = 0$$

$\nabla^2 h_{ii} = 0, \text{ so that } h_{ii} = 0$ $\qquad h_{00} \equiv 0$ $\qquad h_{0i} \equiv 0$ $\qquad$ oi terrm

The non-trivial equations are $\qquad \Box h_{ij} = 0$

Constraints $\qquad \partial^i h_{ij} = 0 \qquad h_{ii} = 0.$

Since $\qquad D(D-1)/2 - (D-1) - 1 = D(D-3)/2$

The number of on-shell degrees of freedom, helicities is $\quad D(D-3)/2$

$\square\square$ Symmetric traceless representation

# The first and second order formulations of general relativity

## Second order formalism for gravity and fermions

Field content, frame field $e^a_\mu(x)$ massless Dirac field $\Psi(x)$

Spin connection dependent quantity $\omega_\mu{}^{ab}(e)$

$$S = S_2 + S_{1/2} = \int \mathrm{d}^D x\, e\, \left[ \frac{1}{2\kappa^2} e^\mu_a e^\nu_b R_{\mu\nu}{}^{ab}(e) - \frac{1}{2}\bar{\Psi}\gamma^\mu \nabla_\mu \Psi + \frac{1}{2}\bar{\Psi}\overleftarrow{\nabla}_\mu \gamma^\mu \Psi \right]$$

$$R_{\mu\nu ab} \;=\; \partial_\mu \omega_{\nu ab} - \partial_\nu \omega_{\mu ab} + \omega_{\mu ac}\omega_\nu{}^c{}_b - \omega_{\nu ac}\omega_\mu{}^c{}_b$$

$S_2(e,\omega)$ $$\left.\frac{\delta S_2}{\delta \omega_{\mu ab}}\right|_{\omega=\omega(e)} \;=\; 0$$

# The first and second order formulations of general relativity

$$\begin{aligned}\nabla_\mu \Psi &= D_\mu \Psi = (\partial_\mu + \tfrac{1}{4}\omega_\mu^{ab}\gamma_{ab})\Psi \\ \bar{\Psi}\overleftarrow{\nabla}_\mu &= \bar{\Psi}\overleftarrow{D}_\mu = \bar{\Psi}(\overleftarrow{\partial}_\mu - \tfrac{1}{4}\omega_\mu^{ab}\gamma_{ab})\end{aligned}$$

The total covariant derivative and the Lorentz covariant derivative coincide for spinor field but not for the gravitino

# The first and second order formulations of general relativity

- Curved space gamma matrices

Constant gamma matrices verify $\{\gamma^a, \gamma^b\} = 2\eta^{ab}$

Frame fields are used to transform frame *vector* indices to a coordinate basis.

$$\gamma^\mu = e^\mu_a \gamma^a = g^{\mu\nu}\gamma_\nu$$

$$\{\gamma^\mu, \gamma^\nu\} = g^{\mu\nu}$$

The curved gamma matrices transforms a vector under coordinate transformations
But they have also spinor indexes

$$\begin{aligned}\nabla_\mu \gamma_\nu &= \partial_\mu \gamma_\nu + \frac{1}{4}\omega^{ab}_\mu [\gamma_{ab}, \gamma_\nu] - \Gamma^\rho_{\mu\nu}\gamma_\rho \\ &= \gamma^a(\partial_\mu e_{a\nu} + \omega_{\mu ab} e^b_\nu - \Gamma^\rho_{\mu\nu} e_{a\rho})\end{aligned}$$

$$\nabla_\mu \gamma_\nu = 0$$

holds for any affine connection with or without torsion

# The first and second order formulations of general relativity

- Fermion equation of motion

$$\gamma^\mu \nabla_\mu \Psi \;=\; 0$$

$$\gamma^\mu \nabla_\mu \gamma^\nu \nabla_\nu \Psi \;=\; \left(g^{\mu\nu} \nabla_\mu \nabla_\nu \;-\; \tfrac{1}{4} R\right) \Psi$$

Steps in the derivation of Einstein equation

$$\begin{aligned}\delta S \;&=\; \int \mathrm{d}^D x\, e\, \Bigg[\frac{1}{\kappa^2}\left(e^{b\nu} R_{\mu\nu ab} - \frac{1}{2} e_{a\mu} R\right)\delta e^{a\mu} \\ &\qquad -\frac{1}{2}\bar{\Psi}\gamma^a \overleftrightarrow{\nabla}_\mu \Psi \delta e^{a\mu} - \frac{1}{8}\bar{\Psi}\{\gamma^\mu, \gamma^{ab}\}\Psi \delta\omega_{\mu ab}\Bigg]\end{aligned}$$

We drop a term proportional to the fermion lagrangian because we use the equations of motion for the fermion

# The first and second order formulations of general relativity

$$\delta S = \int \mathrm{d}^D x \left[ \frac{1}{\kappa^2} \left( e^{b\nu} R_{\mu\nu ab} - \frac{1}{2} e_{a\mu} R \right) \delta e^{a\mu} - \frac{1}{4} \bar{\Psi} \left[ \gamma_a \overleftrightarrow{\nabla}_\mu + \gamma_\mu e_a^\rho \overleftrightarrow{\nabla}_\rho \right] \Psi \delta e^{a\mu} \right]$$

From which we deduce the Einstein equation

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \;=\; \kappa^2 T_{\mu\nu} \;\equiv\; -\kappa^2 \frac{1}{4} \bar{\Psi} \left[ \gamma_\mu \overleftrightarrow{\nabla}_\nu + \gamma_\nu \overleftrightarrow{\nabla}_\mu \right] \Psi$$

The stress tensor is the covariant version of flat Dirac symmetric stress tensor

$$\nabla^\mu T_{\mu\nu} = 0, \text{ and symmetry, } T_{\mu\nu} = T_{\nu\mu}$$

Follows from the matter being invariant coordinate and local Lorentz transformations

# The first and second order formulations of general relativity

- The first order formalism for gravity and fermions

Field content frame field $e^a_\mu$ and spin connection $\omega_{\mu ab}$

Fermion field $\Psi(x)$ Same form of the action asin the second order formalism but now vielbein and spin conn independent

Variation of the gravitational action

$$\delta S_2 = \frac{1}{2\kappa^2} \int \mathrm{d}^D x \, e \; e^\mu_a e^\nu_b \left( D_\mu \delta\omega_\nu{}^{ab} - D_\nu \delta\omega_\mu{}^{ab} \right)$$

We have used $\delta R_{\mu\nu ab} = D_\mu \delta\omega_{\nu ab} - D_\nu \delta\omega_{\mu ab}$

$$\delta S_2 = \frac{1}{2\kappa^2} \int \mathrm{d}^D x \, e \; e^\mu_a e^\nu_b \left( 2\nabla_\mu \delta\omega_\nu{}^{ab} + T_{\mu\nu}{}^\rho \delta\omega_\rho{}^{ab} \right)$$

# The first and second order formulations of general relativity

- The first order formalism for gravity and fermions

Integration by parts

$$\int \mathrm{d}^D x\,\sqrt{-g}\,\nabla_\mu V^\mu = \int \mathrm{d}^D x\,\partial_\mu\left(\sqrt{-g}\,V^\mu\right) - \int \mathrm{d}^D x\,\sqrt{-g}\,K_{\nu\mu}{}^\nu V^\mu$$

$$\begin{aligned}\delta S_2 &= \frac{1}{2\kappa^2}\int \mathrm{d}^D x\, e\,\left(-2K_{\rho\mu}{}^\rho e_a^\mu e_b^\nu \delta\omega_\nu{}^{ab} + T_{ab}{}^\rho \delta\omega_\rho{}^{ab}\right)\\ &= \frac{1}{2\kappa^2}\int \mathrm{d}^D x\, e\,\left(T_{\rho a}{}^\rho e_b^\nu - T_{\rho b}{}^\rho e_a^\nu + T_{ab}{}^\nu\right)\delta\omega_\nu{}^{ab}\,,\end{aligned}$$

Form the fermion action

$$\begin{aligned}\delta S_{1/2} &= -\tfrac{1}{8}\int \mathrm{d}^D x\, e\,\bar{\Psi}\{\gamma^\nu\,,\,\gamma_{ab}\}\Psi\,\delta\omega_\nu^{ab}\\ &= -\tfrac{1}{4}\int \mathrm{d}^D x\, e\,\bar{\Psi}\,\gamma^\nu{}_{ab}\,\Psi\,\delta\omega_\nu^{ab}\,.\end{aligned}$$

# The first and second order formulations of general relativity

- The first order formalism for gravity and fermions

The equations of motion of the spin connection gives

$$T_{ab}{}^{\nu} - T_{a\rho}{}^{\rho}\, e_b^{\nu} + T_{b\rho}{}^{\rho}\, e_a^{\nu} \; = \tfrac{1}{2}\kappa^2 \bar{\Psi}\, \gamma_{ab}{}^{\nu}\, \Psi$$

the right hand side is traceless therefore also the torsion is traceless

$$T_{ab}{}^{\nu} = \tfrac{1}{2}\kappa^2 \bar{\Psi}\, \gamma_{ab}{}^{\nu}\, \Psi = -2K^{\nu}{}_{ab}$$

If we substitute $\omega = \omega(e) + K$

$$S = \frac{1}{2\kappa^2}\int \mathrm{d}^D x\, e \;\Big[R(g) - \kappa^2 \bar{\Psi}\gamma^{\mu}\overleftrightarrow{\nabla}_{\mu}\Psi \qquad (7$$

$$-2\nabla_{\mu}K_{\nu}{}^{\nu\mu} + K_{\mu\nu\rho}K^{\nu\mu\rho} - K_{\rho}{}^{\rho}{}_{\mu}K_{\sigma}{}^{\sigma\mu} - \tfrac{1}{2}\bar{\Psi}\gamma_{\mu\nu\rho}\Psi\, K^{\mu\nu\rho}\Big]$$

# The first and second order formulations of general relativity

- The first order formalism for gravity and fermions

The physical equivalent second order action is

$$S = \frac{1}{2}\int \mathrm{d}^D x\, e\,\left[\frac{1}{\kappa^2}R(g) - \bar{\Psi}\gamma^\mu \overset{\leftrightarrow}{\nabla}_\mu \Psi + \frac{1}{16}\kappa^2(\bar{\Psi}\gamma_{\mu\nu\rho}\Psi)(\bar{\Psi}\gamma^{\mu\nu\rho}\Psi)\right]$$

Physical effects in the fermion theories with torsion and without torsion
Differ only in the presence of quartic fermion term.
This term generates 4-point contact diagrams .