$\mathcal{N} = 1$ pure supergravity in 4 dimensions

- Basics

supersymmetry holds *locally* in a supergravity theory.

the spinor parameters
$\epsilon(x)$ are arbitrary functions of the spacetime coordinates.

The SUSY algebra $[\delta_1, \delta_2]\Phi(x) = -\frac{1}{2}\bar{\epsilon}_1\gamma^\mu\epsilon_2\,\partial_\mu\Phi(x)$

will then involve local translation parameters $\bar{\epsilon}_1\gamma^\mu\epsilon_2$

Therefore we have diffeomorphism. Thus local susy requires gravity

fields $e_\mu^a(x)$ $\Psi_\mu^i(x),\ i = 1, \ldots, \mathcal{N}$

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There are four major applications of supergravity

1) If there is some sort of broken global symmetry. N=1 D=4 supergravity coupled to chiral and gauge multiplets of global Susy could describe the physics of elementary particles

2) D=10 supergravity is the low energy limit of superstring theory. Solutions of SUGRA exhibit spacetime compactification

3) Role of D=11 supergravity for M-theory

4) AdS/CFT in the limit in which string theory is approximated by supergravity. correlations of the boundary gauge theory at strong coupling are available from weak coupling classical calculations in five and ten dimensional supergravity

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Toroidal reduction of D=11 Supergravity

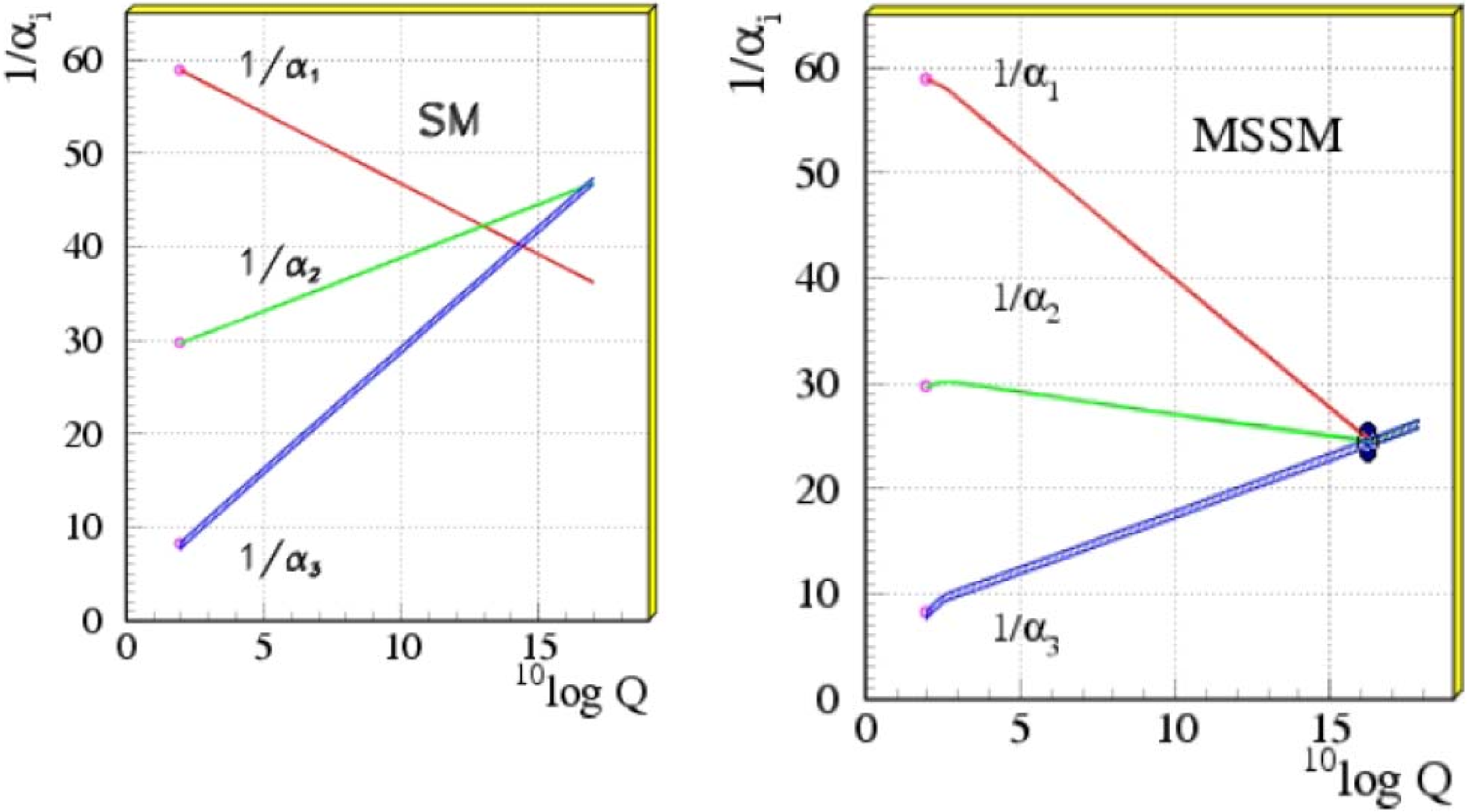
$$
\begin{aligned}
ds^2_{11} &= e^{2\alpha\phi} ds^2_{10} + e^{2\beta\phi}(dy + A_\mu dx^\mu)^2 \\
C^{11}_{(p)} &= C^{10}_{(p)} + C^{10}_{(p-1)} \wedge dy
\end{aligned}
$$

IIA SUGRA bosonic fields

$\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{7}, \mathbf{8}, \mathbf{9}.$ 5,6

Fermionic fields, non-chiral gravitino, non-chiral dilatino

Gauge coupling unification



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- The universal part of supergravity. Second order formalism

$$
\begin{aligned}
S &= S_2 + S_{3/2}\,, \\
S_2 &= \frac{1}{2\kappa^2}\int \mathrm{d}^D x\, e\, e^{a\mu} e^{b\nu} R_{\mu\nu ab}(\omega)\,, \\
S_{3/2} &= -\frac{1}{2\kappa^2}\int \mathrm{d}^D x\, e\, \bar{\psi}_\mu \gamma^{\mu\nu\rho} D_\nu \psi_\rho\,,
\end{aligned}
$$

$D_\nu\psi_\rho \equiv \partial_\nu\psi_\rho + \frac{1}{4}\omega_{\nu ab}\gamma^{ab}\psi_\rho\,.$ We not need to include the connection $\Gamma^\sigma_{\nu\rho}(g)\psi_\sigma$ due to symmetry properties

$\omega_{\nu ab}(e)$, Is the torsion-free spin connection

$\kappa^2 = 8\pi G_N$ is the gravitational coupling constant

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From Supersymmetry to SUGRA

GR can be viewed as a “gauge” theory of the Poincare group

$$\{M_{ab}\,,P_a\} \underset{\text{gauging}}{\longrightarrow} \mathcal{A}_\mu \equiv \tfrac{1}{2}\omega_\mu{}^{ab}M_{ab} + e_\mu{}^a P_a$$

$$R_{\mu\nu} \equiv 2\,\partial_{[\mu}\mathcal{A}_{\nu]} + [\mathcal{A}_\mu, \mathcal{A}_\nu] = \tfrac{1}{2}R_{\mu\nu}{}^{ab}M_{ab} + R_{\mu\nu}{}^a P_a$$

The components $R_{\mu\nu}{}^{ab}$ are the Lorentz curvature and $R_{\mu\nu}{}^{a}$ to the torsion

The action

$$S \sim \int d^4x\, e\, R(e,\omega)$$

Not invariant under local Poncaire. Torsion =0 by hand

Einstein ‘s vacuum equation $R_{\mu\nu}{}^a = 0\,,\qquad G_{\mu\nu} = 0$

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$$\{M_{ab}\,, P_a\,, Q^\alpha\} \underset{\text{gauging}}{\longrightarrow} \mathcal{A}_\mu \equiv \tfrac{1}{2}\omega_\mu{}^{ab} M_{ab} + e_\mu{}^a P_a + \bar{\psi}_{\mu\alpha} Q^\alpha$$

$$R_{\mu\nu} \equiv 2\,\partial_{[\mu}\mathcal{A}_{\nu]} + [\mathcal{A}_\mu, \mathcal{A}_\nu] = \tfrac{1}{2} R_{\mu\nu}{}^{ab} M_{ab} + R_{\mu\nu}{}^a P_a + \bar{R}_{\mu\nu\alpha} Q^\alpha$$

$$S \sim \int d^4x\, e\, \left\{ R(e,\omega) + \epsilon^{\mu\nu\rho\sigma}\, \bar{\psi}_\mu \gamma_5 \gamma_\nu \mathcal{D}_\rho(\omega)\psi_\sigma \right\}$$

$$\gamma_{\mu\nu\rho} = \mathrm{i}\varepsilon_{\mu\nu\rho\sigma}\gamma^\sigma\gamma_*$$

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- Transformation rules

$$
\begin{aligned}
\delta e_\mu^a &= \tfrac{1}{2}\bar{\epsilon}\gamma^a\psi_\mu\,, \\
\delta\psi_\mu &= D_\mu\epsilon(x) \equiv \partial_\mu\epsilon + \tfrac{1}{4}\omega_{\mu ab}\gamma^{ab}\epsilon
\end{aligned}
$$

$$\delta e_a^\mu = -\tfrac{1}{2}\bar{\epsilon}\gamma^\mu\psi_a\,, \qquad \delta e = \tfrac{1}{2}e\,(\bar{\epsilon}\gamma^\rho\psi_\rho).$$

The variation of the action consists of terms linear in ψ_μ From the frame field variation and the gravitino variation and cubic terms from the fiedd variation of the gravitino action

Variation of the gravitational action

$$\delta S_2 = \frac{1}{2\kappa^2}\int \mathrm{d}^D x\, e\, \left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right)(-\bar{\epsilon}\gamma^\mu\psi^\nu)$$

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- Transformation rules for gauge theory point of view

Gauge prameters

$$\epsilon = \epsilon^a P_a + \epsilon^{ab} M_{ab} + \bar{\epsilon}^\alpha Q_\alpha = \epsilon^a P_a + \epsilon^{ab} M_{ab} + \bar{Q}^\alpha \epsilon_\alpha$$

gauge transformations

$$\delta \mathcal{A}_\mu = \partial_\mu \epsilon + [\epsilon, \mathcal{A}_\mu]$$

$$\begin{aligned} \delta e_\mu^a &= \tfrac{1}{2} \bar{\epsilon} \gamma^a \psi_\mu \,, \\ \delta \psi_\mu &= D_\mu \epsilon(x) \equiv \partial_\mu \epsilon + \tfrac{1}{4} \omega_{\mu ab} \gamma^{ab} \epsilon \end{aligned}$$

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gravitino variation

In the second order formalism, partial integration is valid, so we compute $\delta\bar{\psi}_\mu$ by two

$$\delta S_{3/2} = -\frac{1}{\kappa^2}\int \mathrm{d}^D x\, e\,\bar{\epsilon}\overleftarrow{D}_\mu \gamma^{\mu\nu\rho} D_\nu \psi_\rho$$

$$= \frac{1}{\kappa^2}\int \mathrm{d}^D x\, e\,\bar{\epsilon}\gamma^{\mu\nu\rho} D_\mu D_\nu \psi_\rho = \frac{1}{8\kappa^2}\int \mathrm{d}^D x\, e\,\bar{\epsilon}\gamma^{\mu\nu\rho} R_{\mu\nu ab}\gamma^{ab}\psi_\rho$$

We now need some Dirac algebra to evaluate the product $\gamma^{\mu\nu\rho}\gamma^{ab}$

$$\gamma^{\mu\nu\rho}\gamma_{\sigma\tau} = \gamma^{\mu\nu\rho}{}_{\sigma\tau} + 6\gamma^{[\mu\nu}{}_{[\tau}\delta^{\rho]}{}_{\sigma]} + 6\gamma^{[\mu}\delta^{\nu}{}_{[\tau}\delta^{\rho]}{}_{\sigma]}$$

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$$\begin{aligned}\gamma^{\mu\nu\rho}\gamma^{ab}R_{\mu\nu ab} &= \gamma^{\mu\nu\rho ab}R_{\mu\nu ab} + 6R_{\mu\nu}{}^{[\rho}{}_b\gamma^{\mu\nu]b} + 6\gamma^{[\mu}R_{\mu\nu}{}^{\rho\nu]} \\ &= \gamma^{\mu\nu\rho ab}R_{\mu\nu ab} + 2R_{\mu\nu}{}^{\rho}{}_b\gamma^{\mu\nu b} + 4R_{\mu\nu}{}^{\mu}{}_b\gamma^{\nu\rho b} \\ &\quad +4\gamma^{\mu}R_{\mu\nu}{}^{\rho\nu} + 2\gamma^{\rho}R_{\mu\nu}{}^{\nu\mu}\,,\end{aligned}$$

First Bianchi identity without torsin

$$R_{\mu\nu\rho}{}^{a} + R_{\nu\rho\mu}{}^{a} + R_{\rho\mu\nu}{}^{a} = 0\ ,\ R_{\mu\nu\rho}{}^{a} = R_{\mu\nu b}{}^{a}e^{b}_{\rho} \qquad R_{\nu b} = R_{\mu\nu}{}^{\mu}{}_{b}$$

Finaly we have

$$\delta S_{3/2} = \frac{1}{2\kappa^2}\int \mathrm{d}^D x\, e\,(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R)(\bar{\epsilon}\gamma^{\mu}\psi^{\nu})$$

Therefore the linear terms cancel

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- Supersymmetry symmetry properties at the level of the equations of motion

Free super Maxwell eq of motion $\partial^\mu F_{\mu\nu} = 0 \quad \gamma^\mu \partial_\mu \lambda = 0$

Susy transformations $\delta A_\mu = \frac{1}{2}\bar{\epsilon}\gamma_\mu \lambda \qquad \delta\lambda = \frac{1}{4}\gamma^{\nu\rho}F_{\nu\rho}\epsilon$

The SUSY transform of the Dirac equation is

$$0 = \gamma^\mu \partial_\mu \delta\lambda = \frac{1}{4}\gamma^\mu \gamma^{\nu\rho}\partial_\mu F_{\nu\rho}\epsilon$$

For local SUSY transformationsin in the linear approximation

$$0 = \gamma^{\mu\nu\rho} D_\nu \delta\psi_\rho = -\frac{1}{2}(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R)\gamma_\nu \epsilon$$

the right side vanishes if the Einstein equation is satisfied

Buchdal problem

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The supersymmetry transform of the Einstein equation vanishes if the gravitino satisfies its equation of motion. For linear fluctuations about Minkowski this is true if the SUSY transformation of the metric

$$\delta g_{\mu\nu} = \frac{1}{2}\bar{\epsilon}(\gamma_\mu \psi_\nu + \gamma_\nu \psi_\mu)$$

the global limit in which ϵ and γ_μ are constant

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- First order formalism

We regard the spin connection as an independent variable. We want to get the Equations of motion for the spin connection

$$\delta S_{3/2} = -\frac{1}{8\kappa^2}\int \mathrm{d}^D x\, e\,(\bar{\psi}_\mu \gamma^{\mu\nu\rho}\gamma_{ab}\psi_\rho)\delta\omega_\nu{}^{ab}$$

valid for D=2,3,4,10, 11 where Majorana spinors exist

spinor bilinears of rank 3 are symmetric ,therefore we have

$$\bar{\psi}_\mu \gamma^{\mu\nu\rho}\gamma_{ab}\psi_\rho = \bar{\psi}_\mu \left(\gamma^{\mu\nu\rho}{}_{ab} + 6\gamma^{[\mu} e^\nu{}_{[b} e^{\rho]}{}_{a]}\right)\psi_\rho$$

The spin connection equation of motion is

$$\delta S_2 + \delta S_{3/2} = 0$$

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$$
\begin{aligned}
\delta S_2 &= \frac{1}{2\kappa^2}\int \mathrm{d}^D x\, e\, \left(-2K_{\rho\mu}{}^{\rho} e_a^{\mu} e_b^{\nu} \delta\omega_{\nu}{}^{ab} + T_{ab}{}^{\rho}\delta\omega_{\rho}{}^{ab}\right) \\
&= \frac{1}{2\kappa^2}\int \mathrm{d}^D x\, e\, \left(T_{\rho a}{}^{\rho} e_b^{\nu} - T_{\rho b}{}^{\rho} e_a^{\nu} + T_{ab}{}^{\nu}\right)\delta\omega_{\nu}{}^{ab}\,,
\end{aligned}
$$

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- First order formalism

$$T_{ab}{}^{\nu} = \tfrac{1}{2}\bar{\psi}_a \gamma^{\nu} \psi_b + \tfrac{1}{4}\bar{\psi}_{\mu} \gamma^{\mu\nu\rho}{}_{ab} \psi_{\rho}$$

The fifth rank tensor vanishes for D=4. For dimensions D>4 this term is not Vanishing and is one the complications of supergravity

The equivalent second order action of gravity is

$$S = \frac{1}{2\kappa^2} \int \mathrm{d}^4 x\, e\, \left[R(e) - \bar{\psi}_{\mu} \gamma^{\mu\nu\rho} D_{\nu} \psi_{\rho} + \mathcal{L}_{\mathrm{SG,torsion}} \right]$$

$$\mathcal{L}_{\mathrm{SG,torsion}} = -\frac{1}{16} \left[(\bar{\psi}^{\rho} \gamma^{\mu} \psi^{\nu})(\bar{\psi}_{\rho} \gamma_{\mu} \psi_{\nu} + 2\bar{\psi}_{\rho} \gamma_{\nu} \psi_{\mu}) - 4(\bar{\psi}_{\mu} \gamma \cdot \psi)(\bar{\psi}^{\mu} \gamma \cdot \psi) \right]$$

With

$$D_{\nu} \psi_{\rho} \equiv \partial_{\nu} \psi_{\rho} + \tfrac{1}{4} \omega_{\nu ab}(e) \gamma^{ab} \psi_{\rho}$$

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- Local supersymmetry transformations

The second order action for N=1 D=4 is supergravity is complete and it is local supersymmetry

$$\delta e_\mu^a = \tfrac{1}{2}\bar{\epsilon}\gamma^a\psi_\mu \,,$$

$$\begin{aligned}
\delta\psi_\mu &= D_\mu\epsilon \equiv \partial_\mu\epsilon + \tfrac{1}{4}\omega_{\mu ab}\gamma^{ab}\epsilon \,, \\
\omega_{\mu ab} &= \omega_{\mu ab}(e) + K_{\mu ab} \,, \\
K_{\mu\nu\rho} &= -\tfrac{1}{4}(\bar{\psi}_\mu\gamma_\rho\psi_\nu - \bar{\psi}_\nu\gamma_\mu\psi_\rho + \bar{\psi}_\rho\gamma_\nu\psi_\mu)
\end{aligned}$$

which includes the gravitino torsion

The variation of the action contains terms which are first, third and fifth order in the gravitino field. The terms are independent and must cancel separately

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In the first order formalism the fifth variation is avoided, but we need to specify the transformation of the spin connection. This procedure is complicated when matter multiplets are coupled to supergravity

The 1.5 order formalism

Consider an action which is a functional of three variables $S[e,\, \omega,\, \psi]$

but let us use the equation of motion for the spin connection and chain rule

$$\delta S[e,\, \omega(e) + K,\, \psi] = \int \mathrm{d}^D x \left[\frac{\delta S}{\delta e} \delta e + \frac{\delta S}{\delta \omega} \delta(\omega(e) + K) + \frac{\delta S}{\delta \psi} \delta \psi \right]$$

$$= \int \mathrm{d}^D x \left[\frac{\delta S}{\delta e} \delta e + \frac{\delta S}{\delta \psi} \delta \psi \right]$$

$\mathcal{N} = 1$ pure supergravity in 4 dimensions

- At the end substitute

$$\omega_{\mu ab} \;=\; \omega_{\mu ab}(e) + K_{\mu ab}$$

Local supersymmetry of $\mathcal{N}=1$, $D=4$ supergravity

- Let us work in the 1.5 formalism and rewrite the action of the Rarita Schwinger part

$$\begin{aligned} S &= S_2 + S_{3/2}\,, \\ S_2 &= \frac{1}{2\kappa^2}\int \mathrm{d}^D x\, e\, e^{a\mu} e^{b\nu} R_{\mu\nu ab}(\omega)\,, \\ S_{3/2} &= -\frac{1}{2\kappa^2}\int \mathrm{d}^D x\, e\, \bar\psi_\mu \gamma^{\mu\nu\rho} D_\nu \psi_\rho\,, \end{aligned}$$

Recall $\gamma^{abc} = -\mathrm{i}\varepsilon^{abcd}\gamma_*\gamma_d\,, \qquad \gamma^{\mu\nu\rho} = -\mathrm{i}e^{-1}\varepsilon^{\mu\nu\rho\sigma}\gamma_*\gamma_\sigma \quad \varepsilon^{0123} = -1$

valid only in 4 dimensions

$$S_{3/2} = \frac{\mathrm{i}}{2\kappa^2}\int \mathrm{d}^4 x\, \varepsilon^{\mu\nu\rho\sigma}\bar\psi_\mu \gamma_*\gamma_\sigma D_\nu \psi_\rho$$

Local supersymmetry of $\mathcal{N}=1,\ D=4$ supergravity

$$\delta S \ = \ \delta S_2 + \delta S_{3/2,e} + \delta S_{3/2,\psi} + \delta S_{3/2,\bar{\psi}}$$

We must obtain the ψ and $\bar{\psi}$ variations separately because

the presence of torsion. Also the Ricci tensor is not symmetric $R_{\mu\nu}(\omega)$

$$\delta S_2 = \frac{1}{2\kappa^2}\int \mathrm{d}^4x\, e\,(R_{\mu\nu}(\omega) - \frac{1}{2}g_{\mu\nu}R(\omega))(-\bar{\epsilon}\gamma^{\mu}\psi^{\nu})$$

$$\delta S_{3/2,e} = \frac{\mathrm{i}}{4\kappa^2}\int \mathrm{d}^4x\, \varepsilon^{\mu\nu\rho\sigma}(\bar{\epsilon}\gamma^{a}\psi_{\sigma})(\bar{\psi}_{\mu}\gamma_{*}\gamma_{a}D_{\nu}\psi_{\rho})$$

only due to the variation of γ_{σ}

Local supersymmetry of $\mathcal{N}=1$, $D=4$ supergravity

$$\begin{aligned}\delta S_{3/2,\psi} &= \frac{\mathrm{i}}{2\kappa^2}\int \mathrm{d}^4x\,\varepsilon^{\mu\nu\rho\sigma}\bar{\psi}_\mu\gamma_*\gamma_\sigma D_\nu D_\rho\epsilon \\ &= \frac{\mathrm{i}}{16\kappa^2}\int \mathrm{d}^4x\,\bar{\psi}_\mu\varepsilon^{\mu\nu\rho\sigma}\gamma_*\gamma_\sigma\gamma^{ab}R_{\nu\rho ab}(\omega)\epsilon\end{aligned}$$

we write the $\bar{\psi}_\mu$ variation and exchange the spinors $D_\mu\epsilon$ and ψ_ρ

$t_3 = 1$

$$\delta S_{3/2,\bar{\psi}} = \frac{\mathrm{i}}{2\kappa^2}\int \mathrm{d}^4x\,\varepsilon^{\mu\nu\rho\sigma}\bar{\psi}_\rho\overleftarrow{D}_\nu\gamma_*\gamma_\sigma D_\mu\epsilon$$

The left acting derivative $\bar{\psi}_\rho\overleftarrow{D}_\nu = \partial_\nu\bar{\psi}_\rho - \frac{1}{4}\bar{\psi}_\rho\omega_{\nu ab}\gamma^{ab}$

can be partially integrated and acts distributively

Local supersymmetry of $\mathcal{N}=1$, $D=4$ supergravity

$$\begin{aligned}\delta S_{3/2,\bar\psi} &= \frac{-\mathrm{i}}{2\kappa^2}\int \mathrm{d}^4x\, \varepsilon^{\mu\nu\rho\sigma}\bar\psi_\rho\gamma_*[(D_\nu\gamma_\sigma)D_\mu\epsilon+\gamma_\sigma D_\nu D_\mu\epsilon]\\ &= \frac{-\mathrm{i}}{2\kappa^2}\int \mathrm{d}^4x\, \varepsilon^{\mu\nu\rho\sigma}\bar\psi_\rho\gamma_*\left[\frac{1}{2}T_{\nu\sigma}{}^a\gamma_a D_\mu\epsilon-\frac{1}{8}\gamma_\sigma\gamma^{ab}R_{\mu\nu ab}(\omega)\epsilon\right]\end{aligned}$$

recall $$\nabla_\nu\gamma_\sigma = 0$$

$$\delta S_{3/2,\psi}+\delta S_{3/2,\bar\psi} = \frac{-\mathrm{i}}{2\kappa^2}\int \mathrm{d}^4x\, \varepsilon^{\mu\nu\rho\sigma}\bar\psi_\rho\gamma_*\left[\frac{1}{2}T_{\nu\sigma}{}^a\gamma_a D_\mu\epsilon-\frac{1}{4}\gamma_\sigma\gamma_{ab}R_{\mu\nu}{}^{ab}(\omega)\epsilon\right]$$

Last term using

$$\gamma_\sigma\gamma_{ab}=\gamma_{\sigma ab}+2e_{\sigma[a}\gamma_{b]}=\mathrm{i}e_\sigma^d\varepsilon_{abcd}\gamma_*\gamma^c+2e_{\sigma[a}\gamma_{b]}$$

Local supersymmetry of $\mathcal{N}=1,\ D=4$ supergravity

first term

$$\begin{aligned}\varepsilon^{\mu\nu\rho\sigma}\varepsilon_{abcd}e_\sigma^d R_{\nu\rho}{}^{ab}(\omega) &= -2e\left[e_a^\mu e_b^\nu e_c^\rho + e_b^\mu e_c^\nu e_a^\rho + e_c^\mu e_a^\nu e_b^\rho\right] R_{\nu\rho}{}^{ab}(\omega)\\ &= 4e\left[R_c{}^\mu(\omega) - \tfrac{1}{2}e_c^\mu R(\omega)\right]\,.\end{aligned}$$

plus $\bar{\psi}_\mu\gamma^c\epsilon = -\bar{\epsilon}\gamma^c\psi_\mu$,the result cancels the δS_2

The last term with the first Bianchi identity

$$R_{\mu\nu\rho}{}^a + R_{\nu\rho\mu}{}^a + R_{\rho\mu\nu}{}^a = -D_\mu T_{\nu\rho}{}^a - D_\nu T_{\rho\mu}{}^a - D_\rho T_{\mu\nu}{}^a$$

$$\varepsilon^{\mu\nu\rho\sigma}R_{\nu\rho\sigma b}(\omega) = -\varepsilon^{\mu\nu\rho\sigma}D_\nu T_{\rho\sigma b}$$

$$\delta S_2 + \delta S_{3/2,\psi} + \delta S_{3/2,\bar{\psi}} = \frac{-\mathrm{i}}{4\kappa^2}\int \mathrm{d}^4x\,\varepsilon^{\mu\nu\rho\sigma}\bar{\psi}_\mu\gamma_*\gamma_a\left[T_{\rho\sigma}{}^a D_\nu\epsilon + (D_\nu T_{\rho\sigma}{}^a)\,\epsilon\right]$$

Local supersymmetry of $\mathcal{N}=1,\ D=4$ supergravity

$$\delta S_{3/2,e} = \frac{\mathrm{i}}{4\kappa^2}\int \mathrm{d}^4x\, \varepsilon^{\mu\nu\rho\sigma}(\bar{\epsilon}\gamma^a\psi_\sigma)(\bar{\psi}_\mu\gamma_*\gamma_a D_\nu\psi_\rho)$$

Fierz rearrangement

$$(\bar{\epsilon}\gamma^a\psi_{[\sigma})(\bar{\psi}_{\mu]}\gamma_a\gamma_* D_\nu\psi_\rho) \quad = \quad -\tfrac{1}{4}\sum_A(-)^{r_A}(4-2r_A)(\bar{\epsilon}\Gamma_A\gamma_* D_\nu\psi_\rho)(\bar{\psi}_{[\mu}\Gamma^A\psi_{\sigma]})$$

the left hand side is antisymmetric in $[\mu\sigma]$ only the terms $r_A = 1 \text{ and } 2$ contribute

$$= \quad \tfrac{1}{2}(\bar{\epsilon}\gamma_a\gamma_* D_\nu\psi_\rho)(\bar{\psi}_\mu\gamma^a\psi_\sigma) = \quad (\bar{\epsilon}\gamma_*\gamma_a D_\nu\psi_\rho)T_{\mu\sigma}{}^a$$

$$\delta S_{3/2,e} = \frac{-\mathrm{i}}{4\kappa^2}\int \mathrm{d}^4x\, \varepsilon^{\mu\nu\rho\sigma} T_{\rho\sigma}{}^a\bar{\psi}_\mu\overleftarrow{D}_\nu\gamma_*\gamma_a\epsilon$$

Local supersymmetry of $\mathcal{N} = 1,\ D = 4$ supergravity

$$\delta S = \frac{-\mathrm{i}}{4\kappa^2} \int \mathrm{d}^4x\, \varepsilon^{\mu\nu\rho\sigma} \partial_\nu [T_{\rho\sigma}{}^b \bar{\psi}_\mu \gamma_* \gamma_b \epsilon] \equiv 0$$

$\mathcal{N} = 1,\ D = 4$ supergravity is locally supersymmetric