Local supersymmetry of $\mathcal{N}=1,\ D=4$ supergravity

The algebra of local supersymmetry

$$
\begin{aligned}
[\delta_1,\delta_2]\,e_\mu^a &= \tfrac{1}{2}\delta_1\bar\epsilon_2\gamma^a\psi_\mu-(1\leftrightarrow 2)=\tfrac{1}{2}\bar\epsilon_2\gamma^a\delta_1\psi_\mu-(1\leftrightarrow 2)\\
&= \tfrac{1}{2}\bar\epsilon_2\gamma^a D_\mu\epsilon_1-(1\leftrightarrow 2)\\
&= \tfrac{1}{2}(\bar\epsilon_2\gamma^a D_\mu\epsilon_1+D_\mu\bar\epsilon_2\gamma^a\epsilon_1)\\
&= D_\mu\xi^a\,,\qquad \xi^a=\tfrac{1}{2}\bar\epsilon_2\gamma^a\epsilon_1=-\tfrac{1}{2}\bar\epsilon_1\gamma^a\epsilon_2\,.
\end{aligned}
$$

Infinitesimal transformation of the frame field

$$\delta_\xi e_\mu^a=\xi^\rho\partial_\rho e_\mu^a+\partial_\mu\xi^\rho e_\rho^a$$

covariant form

$$
\begin{aligned}
\delta_\xi e_\mu^a &= \xi^\rho\nabla_\rho e_\mu^a-\xi^\rho\omega_\rho{}^a{}_b e_\mu^b+\xi^\rho\Gamma^\sigma_{\rho\mu}e_\sigma^a+\nabla_\mu\xi^\rho\,e_\rho^a-\Gamma^\rho_{\mu\sigma}\xi^\sigma e_\rho^a\\
&= \nabla_\mu\xi^\rho e_\rho^a-\xi^\rho\omega_\rho{}^a{}_b e_\mu^b+\xi^\rho T_{\rho\mu}{}^a\,.
\end{aligned}
$$

Local supersymmetry of $\mathcal{N}=1,\ D=4$ supergravity

$$\nabla_\rho e^a_\mu \;=\; 0 \qquad e^a_\rho \nabla_\mu \xi^\rho = D_\mu \xi^a \qquad \xi^\rho T_{\rho\mu}{}^a = \tfrac{1}{2}(\xi^\rho \bar{\psi}_\rho)\gamma^a \psi_\mu$$

$$[\delta_1, \delta_2]\, e^a_\mu \;=\; \left(\delta_\xi - \delta_{\hat{\lambda}} - \delta_{\hat{\epsilon}}\right) e^a_\mu$$

the susy parameter $\hat{\epsilon} = \xi^\rho \psi_\rho$

$$[\delta_1, \delta_2]\, \psi_\mu = \xi^\rho (D_\rho \psi_\mu - D_\mu \psi_\rho) + \dots .$$

$$[\delta_1, \delta_2]\, \psi_\mu \;=\; \left(\delta_\xi - \delta_{\hat{\lambda}} - \delta_{\hat{\epsilon}}\right) \psi_\mu + \dots .$$

the dots means a symmetric combination of the equations of motion

Local supersymmetry of $\mathcal{N} = 1$, $D = 4$ supergravity

- Generalizations

one can couple the gravity multiplet $(e^a_\mu,\, \psi_\mu)$

to gauge $(A^A_\mu,\, \lambda^A)$ and chiral $(z^\alpha,\, P_L\chi^\alpha)$ multiplets

Supergravity in dimensions different from four

D=10 supergravities Type IIA and IIB are the low energy limits of superstring theories of the same name

Type II A and gauged supergravities appear in ADS/CFT correspondence

D=11 low energy limit of M theory that it is not perturbative

Generalizations

The only non-vanishing part of

$$\{Q_\alpha, Q_\beta\} = -\tfrac{1}{2}(\gamma^a)_{\alpha\beta}P_a$$

is

$$\{(P_L Q)_\alpha, (P_R Q)_\beta\} = -\tfrac{1}{2}(P_L\gamma^a)_{\alpha\beta}P_a$$

Extendes superalgebras there are several supercharges $i = 1, \ldots, \mathcal{N}$

We introduce the notation $Q_i = P_L Q_i\,, \qquad Q^i = P_R Q^i$

The Majorana spinors are thus $Q^i + Q_i$

$$\left\{Q_{\alpha i}, Q_\beta{}^j\right\} = -\tfrac{1}{2}(P_L\gamma^a)_{\alpha\beta}\delta_i^j P_a$$

Generalizations

Central charges in 4 dimensions

N=2 SUSY

$$\{Q_{\alpha i}, Q_{\beta j}\} = \varepsilon_{ij} P_{L\alpha\beta} Z \,, \qquad \left\{Q_{\alpha}{}^{i}, Q_{\beta}{}^{j}\right\} = \varepsilon^{ij} P_{R\alpha\beta} \bar{Z}$$

The generators Z and its complex conjugate $\bar{Z}$ are central

'Central' charges in higher dimensions

$$\{Q_\alpha, Q_\beta\} = \gamma^{\mu}_{\alpha\beta} P_\mu + \gamma^{\mu\nu}_{\alpha\beta} Z_{\mu\nu} + \gamma_{\alpha\beta}^{\mu_1 \cdots \mu_5} Z_{\mu_1 \cdots \mu_5}$$

The 'central charges' Z are no longer Lorentz scalars

Generalizations

More supercharges $Q^{i\alpha},\ i = 1 \ldots N,$

Central charges

$$Q^{ij} = -Q^{ji}\,, \quad P^{ij} = -P^{ji}$$

$$\left\{Q^{i\alpha}\,, Q^{j\beta}\right\} = i\delta^{ij}(\gamma^a \mathcal{C}^{-1})^{\alpha\beta} P_a - i(\mathcal{C}^{-1})^{\alpha\beta} Q^{ij} - (\gamma_5 \mathcal{C}^{-1})^{\alpha\beta} P^{ij}$$

$$\mathcal{A}_\mu \equiv \tfrac{1}{2}\omega_\mu{}^{ab} M_{ab} + e_\mu{}^a P_a + \tfrac{1}{2} A^{ij}{}_\mu Q^{ij} + \bar{\psi}^i_{\mu\alpha} Q^{i\alpha}$$

Gauged SUGRA

$D = 11$ **supergravity**

$D \leq 11$ **from dimensional reduction**

Recall the Kaluza-Klein compactification on $\mathrm{Minkowski}_D \times S^1$

The Fourier modes of the symmetric tensor h_{MN} gives

symmetric tensor fields $h_{\mu\nu k}$, vector fields $h_{\mu D k}$ scalar fields h_{DDk}

More generally we study compactifications of a D' $M_{D'} = M_D \times X_d$;

X_d A compact d-dimensional space

KK compactification keeps the massless and massive modes

Dimensional reduction keeps only the massless modes. The truncation is consistent if the field equation of the heavy modes are not sourced by the light modes,

$D = 11$ supergravity

$D = 11$ is the maximal dimension We consider a toroidal compactification

$$M_4 \times T^{D'-4}$$

Assume D'=11, we have a 32 Majorana spinor

$$\begin{aligned} \Gamma^\mu &= \gamma^\mu \times \mathbb{1}\,, && \mu = 0,1,2,3\,, \\ \Gamma^i &= \gamma_* \times \hat{\gamma}^i\,, && i = 4,5,6,7,8,9,10 \end{aligned}$$

$\mathbb{1}$ is the 8×8 unit matrix, and $\gamma_* = \mathrm{i}\gamma_0\gamma_1\gamma_2\gamma_3$, while the $\hat{\gamma}^i$

generate the Clifford algebra in 7d euclidean space. In this basis the gravitino

$$\Psi_{M\,\alpha\,a} \qquad \alpha = 1,2,3,4 \qquad a = 1,\ldots 8$$

$D = 11$ supergravity

$\Psi_{\mu\,\alpha a}$ 8 gravitinos $\Psi_{i\,\alpha a}$ 7x8=56 spin 1/2

Note the total number of fermions 64 is the particle representation of N=8 susy Algebra. If D'>11 we will have in 4d spins >2 for which a consistent theory is not Know for finite number of fields. Vasiliev

Field content

128 bosons degrees of freedom
graviton $g_{\mu\nu}$, **44** of SO(9):

3-form $C_{\mu\nu\rho}$, $C_{(3)}$, **84** of SO(9):

128 fermions degrees of freedom

$D = 11$ supergravity

metric tensor g_{MN}, the 3-form potential A_{MNP}

Majorana vector-spinor Ψ_M

upon dimensional reduction on T^7

$g_{\mu\nu}$ 4d metric $g_{\mu i}$ 7 spin 1 particles g_{ij} 28 scalars

$A_{\mu\nu\rho}$ contain no degrees of freedom in $D = 4$ $A_{\mu ij}$ 21 vectors

A_{ijk} 35 scalars $A_{\mu\nu i}$ 7 scalars (dual)

Which is the field content of N=8 SUGRA in d=4

$D = 11$ supergravity

Construction of the action and transformation rules

Gauge transformation of 3-form

$$\begin{aligned}
\delta A_{\mu\nu\rho} &= 3\partial_{[\mu}\theta_{\nu\rho]} \equiv \partial_\mu\theta_{\nu\rho} + \partial_\nu\theta_{\rho\mu} + \partial_\rho\theta_{\mu\nu} \,, \\
F_{\mu\nu\rho\sigma} &= 4\partial_{[\mu}A_{\nu\rho\sigma]} \equiv \partial_\mu A_{\nu\rho\sigma} - \partial_\nu A_{\rho\sigma\mu} + \partial_\rho A_{\sigma\mu\nu} - \partial_\sigma A_{\mu\nu\rho} \,, \\
\partial_{[\tau}F_{\mu\nu\rho\sigma]} &\equiv 0 \,.
\end{aligned}$$

Bianchi dentity

Ansatz action

$$S = \frac{1}{2\kappa^2}\int \mathrm{d}^{11}x\, e\, \left[e^{a\mu}e^{b\nu}R_{\mu\nu ab} - \bar\psi_\mu\gamma^{\mu\nu\rho}D_\nu\psi_\rho - \frac{1}{24}F^{\mu\nu\rho\sigma}F_{\mu\nu\rho\sigma} + \ldots \right]$$

Initially we use second order formalism with torsion-free spin connection $\omega_{\mu ab}(e)$

$D = 11$ supergravity

- Ansatz transformations

$$
\begin{aligned}
\delta e_\mu^a &= \tfrac{1}{2}\bar{\epsilon}\gamma^a\psi_\mu\,, \\
\delta\psi_\mu &= D_\mu\epsilon + \left(a\,\gamma^{\alpha\beta\gamma\delta}{}_\mu + \,b\,\gamma^{\beta\gamma\delta}\delta_\mu^\alpha\right)F_{\alpha\beta\gamma\delta}\epsilon\,, \\
\delta A_{\mu\nu\rho} &= -c\,\bar{\epsilon}\gamma_{[\mu\nu}\psi_{\rho]} = -\tfrac{1}{3}c\,\bar{\epsilon}(\gamma_{\mu\nu}\psi_\rho + \gamma_{\nu\rho}\psi_\mu + \gamma_{\rho\mu}\psi_\nu)
\end{aligned}
$$

Useful relations

$$\bar{\chi}\gamma^{\mu_1\mu_2\ldots\mu_r}\lambda = t_r\bar{\lambda}\gamma^{\mu_1\mu_2\ldots\mu_r}\chi\,, \qquad t_0 = t_3 = 1\,, \quad t_1 = t_2 = -1\,, \quad t_{r+4} = t_r$$

To determine the constants we consider the free action (global susy)

$$S_0 = \frac{1}{2}\int \mathrm{d}^{11}x\,\left[-\bar{\psi}_\mu\gamma^{\mu\nu\rho}\partial_\nu\psi_\rho - \frac{1}{24}F^{\mu\nu\rho\sigma}F_{\mu\nu\rho\sigma}\right]$$

$D = 11$ supergravity

- transformations

$$\delta\bar{\psi}_\mu = \bar{\epsilon}\left(-a\gamma^{\alpha\beta\gamma\delta}{}_\mu + b\gamma^{\beta\gamma\delta}\delta^\alpha_\mu\right)F_{\alpha\beta\gamma\delta}$$

$$\delta S_0 = \int \mathrm{d}^{11}x\,\bar{\epsilon}\left[\left(a\gamma^{\alpha\beta\gamma\delta}{}_\mu - b\gamma^{\beta\gamma\delta}\delta^\alpha_\mu\right)F_{\alpha\beta\gamma\delta}\gamma^{\mu\nu\rho}\partial_\nu\psi_\rho - \tfrac{1}{6}c\gamma_{\nu\rho}\psi_\sigma\partial_\mu F^{\mu\nu\rho\sigma}\right]$$

$$= \int \mathrm{d}^{11}x\bar{\epsilon}\left[\left(-a\gamma^{\alpha\beta\gamma\delta}{}_\mu + b\gamma^{\beta\gamma\delta}\delta^\alpha_\mu\right)\partial_\nu F_{\alpha\beta\gamma\delta}\gamma^{\mu\nu\rho}\psi_\rho - \tfrac{1}{6}c\gamma_{\nu\rho}\psi_\sigma\partial_\mu F^{\mu\nu\rho\sigma}\right]$$

using

$$\begin{aligned}
\gamma^{\alpha\beta\gamma\delta}{}_\mu\gamma^{\mu\nu\rho}F_{\alpha\beta\gamma\delta} &= (D-6)\gamma^{\alpha\beta\gamma\delta\nu\rho}F_{\alpha\beta\gamma\delta} + 8(D-5)\gamma^{\alpha\beta\gamma[\nu}F^{\rho]}{}_{\alpha\beta\gamma} \\
&\quad -12(D-4)\gamma^{\alpha\beta}F_{\alpha\beta}{}^{\nu\rho}\,, \\
\gamma^{\beta\gamma\delta}\gamma^{\mu\nu\rho}F_{\mu\beta\gamma\delta} &= -\gamma^{\nu\rho\alpha\beta\gamma\delta}F_{\alpha\beta\gamma\delta} - 6\gamma^{\alpha\beta\gamma[\nu}F^{\rho]}{}_{\alpha\beta\gamma} + 6\gamma^{\alpha\beta}F_{\alpha\beta}{}^{\nu\rho}
\end{aligned}$$

$D = 11$ supergravity

- transformations

and Bianchi identity we get $a = c/216, \; b = -8a$

$$\begin{aligned}\delta\psi_\mu &= \partial_\mu\epsilon + \frac{c}{216}\left(\gamma^{\alpha\beta\gamma\delta}{}_\mu - 8\gamma^{\beta\gamma\delta}\delta^\alpha_\mu\right)F_{\alpha\beta\gamma\delta}\epsilon \\ \delta A_{\mu\nu\rho} &= -c\bar{\epsilon}\gamma_{[\mu\nu}\psi_{\rho]}\,.\end{aligned}$$

To determine c we compute the commutator of two susy transformations

$$[\delta_1, \delta_2]A_{\mu\nu\rho} = -\tfrac{4}{9}c^2\bar{\epsilon}_1\gamma^\sigma\epsilon_2 F_{\sigma\mu\nu\rho}$$

If $c^2 = 9/8$

With gauge transformation given the parameter $\theta_{\mu\nu} = -\bar{\epsilon}_1\gamma^\sigma\epsilon_2 A_{\sigma\mu\nu}$

$$[\delta_1, \delta_2] = \delta_{susy} + \delta_{gauge}$$

$D = 11$ supergravity

- transformations

the conserved Noether current is (coefficient of $D_\nu \epsilon$

$$\mathcal{J}^\nu = \frac{\sqrt{2}}{96}\left(\gamma^{\alpha\beta\gamma\delta\nu\rho} F_{\alpha\beta\gamma\delta} + 12\gamma^{\alpha\beta} F_{\alpha\beta}{}^{\nu\rho}\right)\psi_\rho$$

Ansatz for the action and transformations in the interacting case. We introduce the frame field $e^a_\mu(x)$ and a gauge susy parameter $\epsilon(x)$

$$\begin{aligned}
\delta e^a_\mu &= \frac{1}{2}\bar{\epsilon}\gamma^a\psi_\mu\,, \\
\delta\psi_\mu &= D_\mu\epsilon + \frac{\sqrt{2}}{288}\left(\gamma^{\alpha\beta\gamma\delta}{}_\mu - 8\gamma^{\beta\gamma\delta}\delta^\alpha_\mu\right)F_{\alpha\beta\gamma\delta}\epsilon \\
\delta A_{\mu\nu\rho} &= -\frac{3\sqrt{2}}{4}\bar{\epsilon}\gamma_{[\mu\nu}\psi_{\rho]}\,,
\end{aligned}$$

$D = 11$ supergravity

- Action

$$
\begin{aligned}
S &= \frac{1}{2\kappa^2}\int \mathrm{d}^{11}x\, e\, \Big[e^{a\mu} e^{b\nu} R_{\mu\nu ab} - \bar{\psi}_\mu \gamma^{\mu\nu\rho} D_\nu \psi_\rho - \frac{1}{24} F^{\mu\nu\rho\sigma} F_{\mu\nu\rho\sigma} \\
&\qquad - \frac{\sqrt{2}}{96} \bar{\psi}_\nu \left(\gamma^{\alpha\beta\gamma\delta\nu\rho} F_{\alpha\beta\gamma\delta} + 12 \gamma^{\alpha\beta} F_{\alpha\beta}{}^{\nu\rho} \right) \psi_\rho + \dots \Big] \\
&= \frac{1}{\kappa^2} \int \mathrm{d}^{11}x\, e\; L
\end{aligned}
$$

We need to find the dots

$D = 11$ supergravity

- Action

$$\delta L_{FF} + \delta L_{\bar\psi F\psi} \;=\; -\frac{1}{16\cdot 144}\bar\epsilon\gamma^{\alpha'\beta'\gamma'\delta'\alpha\beta\gamma\delta\rho}\psi_\rho\, F_{\alpha'\beta'\gamma'\delta'}F_{\alpha\beta\gamma\delta}\;;$$

We need to cancel this term of rank 9. Recall

$$\gamma^{\alpha'\beta'\gamma'\delta'\alpha\beta\gamma\delta\rho} \;=\; -\frac{1}{2e}\,\varepsilon^{\alpha'\beta'\gamma'\delta'\alpha\beta\gamma\delta\rho\mu\nu}\,\gamma_{\nu\mu}$$

$$\begin{aligned} e\left(\delta L_{FF} + \delta L_{\bar\psi F\psi}\right) &= \frac{1}{32\cdot 144}\varepsilon^{\alpha'\beta'\gamma'\delta'\alpha\beta\gamma\delta\rho\mu\nu}\bar\epsilon\gamma_{\nu\mu}\psi_\rho\, F_{\alpha'\beta'\gamma'\delta'}F_{\alpha\beta\gamma\delta}\,, \qquad (1(\\ &= \frac{4}{3\sqrt{2}\cdot 32\cdot 144}\varepsilon^{\alpha'\beta'\gamma'\delta'\alpha\beta\gamma\delta\rho\mu\nu}(\delta A_{\mu\nu\rho})\, F_{\alpha'\beta'\gamma'\delta'}F_{\alpha\beta\gamma\delta} \end{aligned}$$

Suggest to introduce a term in the action

$D = 11$ supergravity

- Action

$$
\begin{aligned}
S_{\mathrm{C-S}} &= -\frac{\sqrt{2}}{(144\kappa)^2}\int \mathrm{d}^{11}x\,\varepsilon^{\alpha'\beta'\gamma'\delta'\alpha\beta\gamma\delta\mu\nu\rho}F_{\alpha'\beta'\gamma'\delta'}F_{\alpha\beta\gamma\delta}A_{\mu\nu\rho}\,,\\
&= -\frac{\sqrt{2}}{6\kappa^2}\int F^{(4)}\wedge F^{(4)}\wedge A^{(3)}\,,
\end{aligned}
$$

Full action

$$
\begin{aligned}
S = \frac{1}{2\kappa^2}\int \mathrm{d}^{11}x\,e\,\Big[& e^{a\mu}e^{b\nu}R_{\mu\nu ab}(\omega) - \bar{\psi}_\mu\gamma^{\mu\nu\rho}D_\nu(\tfrac{1}{2}(\omega+\hat{\omega}))\psi_\rho - \frac{1}{24}F^{\mu\nu\rho\sigma}F_{\mu\nu\rho\sigma}\\
&-\frac{\sqrt{2}}{192}\bar{\psi}_\nu\left(\gamma^{\alpha\beta\gamma\delta\nu\rho}+12\gamma^{\alpha\beta}g^{\gamma\nu}g^{\delta\rho}\right)\psi_\rho(F_{\alpha\beta\gamma\delta}+\hat{F}_{\alpha\beta\gamma\delta}) \qquad (10.26)\\
&-\frac{2\sqrt{2}}{(144)^2}\varepsilon^{\alpha'\beta'\gamma'\delta'\alpha\beta\gamma\delta\mu\nu\rho}F_{\alpha'\beta'\gamma'\delta'}F_{\alpha\beta\gamma\delta}A_{\mu\nu\rho}\Big]\,. \qquad (10.27)
\end{aligned}
$$

$D = 11$ supergravity

$$
\begin{aligned}
\omega_{\mu ab} &= \omega_{\mu ab}(e) + K_{\mu ab}\,, \\
\hat{\omega}_{\mu ab} &= \omega_{\mu ab}(e) - \tfrac{1}{4}(\bar{\psi}_\mu \gamma_b \psi_a - \bar{\psi}_a \gamma_\mu \psi_b + \bar{\psi}_b \gamma_a \psi_\mu)\,, \\
K_{\mu ab} &= -\tfrac{1}{4}(\bar{\psi}_\mu \gamma_b \psi_a - \bar{\psi}_a \gamma_\mu \psi_b + \bar{\psi}_b \gamma_a \psi_\mu) + \tfrac{1}{8}\bar{\psi}_\nu \gamma^{\nu\rho}{}_{\mu ab}\psi_\rho\,, \\
\hat{F}_{\mu\nu\rho\sigma} &= 4\,\partial_{[\mu} A_{\nu\rho\sigma]} + \tfrac{3}{2}\sqrt{2}\,\bar{\psi}_{[\mu}\gamma_{\nu\rho}\psi_{\sigma]}\,.
\end{aligned}
$$

This action is invariant under the transformation rules

$$
\begin{aligned}
\delta e_\mu^a &= \frac{1}{2}\bar{\epsilon}\gamma^a \psi_\mu\,, \\
\delta \psi_\mu &= D_\mu(\hat{\omega})\epsilon + \frac{\sqrt{2}}{288}\left(\gamma^{\alpha\beta\gamma\delta}{}_\mu - 8\gamma^{\beta\gamma\delta}\delta_\mu^\alpha\right)\hat{F}_{\alpha\beta\gamma\delta}\epsilon \\
\delta A_{\mu\nu\rho} &= -\frac{3\sqrt{2}}{4}\bar{\epsilon}\gamma_{[\mu\nu}\psi_{\rho]}\,.
\end{aligned}
$$

$D = 11$ supergravity

The algebra of $D = 11$ supergravity

$$[\delta_Q(\epsilon_1),\, \delta_Q(\epsilon_2)] \;=\; \delta_{\rm gct}(\xi^\mu) + \delta_L(\lambda^{ab}) + \delta_Q(\epsilon_3) + \delta_A(\theta_{\mu\nu})\,,$$

$$\begin{aligned}
\xi^\mu &= \tfrac{1}{2}\epsilon_2\gamma^\mu\epsilon_1\,,\\
\lambda^{ab} &= -\xi^\mu\hat{\omega}_\mu{}^{ab} + \frac{1}{288}\sqrt{2}\bar{\epsilon}_1\left(\gamma^{ab\mu\nu\rho\sigma}\hat{F}_{\mu\nu\rho\sigma} + 24\gamma_{\mu\nu}\hat{F}^{ab\mu\nu}\right)\epsilon_2\,,\\
\epsilon_3 &= -\xi^\mu\psi_\mu\,,\\
\theta_{\mu\nu} &= -\xi^\rho A_{\rho\mu\nu} + \tfrac{1}{4}\sqrt{2}\bar{\epsilon}_1\gamma_{\mu\nu}\epsilon_2\,.
\end{aligned}$$

The spinor bilinears $\bar{\epsilon}_1\Gamma^{(2)}\epsilon_2$ and $\bar{\epsilon}_1\Gamma^{(6)}\epsilon_2$ have a special role. They are non-vanishing fot the classical BPS M2 and M5 solutions .

$D = 11$ supergravity

Lagrangian

$$L = \sqrt{-g}\{R - \frac{1}{2}(F_{(4)})^2 + \cdots\} + C_{(3)} \wedge dC_{(3)} \wedge dC_{(3)} + \cdots fermions$$

$F_{(4)} = \partial C_{(3)}$, 4-form field strength.

Equations of motion of the metric gives the Einstein's equations.

Equation of the 3-form are

$$d(*F_{(4)} + C_{(3)} \wedge dC_{(3)}) = 0$$

which can be written in a first order form, duality relation, if we introduce the dual form, doubled formalism[2], $C_{(6)}$

$$*F_{(4)} = dC_{(6)} - C_{(3)} \wedge dC_{(3)} = F_{(6)}$$

[2]E. Cremmer,B. Julia, H. Lu, C. Pope hep-th/0112071

$D = 11$ supergravity

The duality relation is invariant under gauge transformations

$$\begin{aligned} \delta C_{(3)} &= d\lambda_{(2)} = \Lambda_{(3)} \\ \delta C_{(6)} &= d\lambda_{(5)} + C_{(3)} \wedge d\lambda_{(2)} = \Lambda_{(6)} + C_{(3)} \wedge \Lambda_{(3)} \end{aligned}$$

$\Lambda_{(3)}, \Lambda_{(6)}$ are closed
The gauge transformations are non-abelian

$$[3, 3] = 6$$

Let us call the 3-form the fundamental form.
We introduce the level as the number of times it is appearing the "fundamental" form 3 in the multiple commutators. ***3*** is at level 1 and ***6*** is at level 2.

Superspace

8-dimensional 'supermanifold'   Coordinates of superspace

commuting x^μ plus 4 anti-commuting coordinates θ_α

Majorana spinor chiral projections $P_L\theta$ and $P_R\theta$

Supersymmetry transformations are 'motions' in the superspace

$$x^\mu \;\to\; x'^\mu = x^\mu + \tfrac{1}{4}\bar{\epsilon}\gamma^\mu\theta,\ \theta \;\to\; \theta' = \theta - \epsilon$$

real superfield $\qquad \Phi(x,\theta) = \bar{\Phi}(x,\theta)$

$$\begin{aligned}\Phi(x,\theta) \quad = \quad & C + \tfrac{1}{2}\mathrm{i}\bar{\theta}\gamma_*\zeta - \tfrac{1}{8}\bar{\theta}P_L\theta\mathcal{H} - \tfrac{1}{8}\bar{\theta}P_R\theta\bar{\mathcal{H}} - \tfrac{1}{8}\mathrm{i}\bar{\theta}\gamma_5\gamma^\mu\theta B_\mu \\ & -\tfrac{1}{8}\mathrm{i}\bar{\theta}\theta\bar{\theta}\gamma_*\left(\lambda + \tfrac{1}{2}\partial\!\!\!/\zeta\right) + \tfrac{1}{32}\bar{\theta}P_L\theta\bar{\theta}P_R\theta\left(D + \tfrac{1}{2}\Box C\right)\end{aligned}$$

Superspace

Supersymmetry transformations generated by the vector fields

$$\mathbb{Q}_\alpha = \frac{\overrightarrow{\partial}}{\partial\bar{\theta}^\alpha} - \frac{1}{4}(\gamma^\mu\theta)_\alpha\frac{\partial}{\partial x^\mu}\,,$$

$$\overline{\mathbb{Q}}^\alpha \equiv \mathcal{C}^{\alpha\beta}\mathbb{Q}_\beta = -\frac{\overrightarrow{\partial}}{\partial\theta_\alpha} + \frac{1}{4}(\bar{\theta}\gamma^\mu)^\alpha\frac{\partial}{\partial x^\mu}$$

$$\{\mathbb{Q}_\alpha, \overline{\mathbb{Q}}^\beta\} = \tfrac{1}{2}(\gamma^\mu)_\alpha{}^\beta\partial_\mu$$

Note the sign difference with respect to SUSY algebra

The variation of a superfield is defined as

$$\delta\Phi \equiv \bar{\epsilon}\mathbb{Q}\,\Phi = \overline{\mathbb{Q}}\epsilon\,\Phi$$

$$[\delta(\epsilon_1), \delta(\epsilon_2)]\,\Phi = \bar{\epsilon}_2\{\mathbb{Q}_\alpha, \overline{\mathbb{Q}}^\beta\}\epsilon_{1\beta}\,\Phi = \tfrac{1}{2}\bar{\epsilon}_2\gamma^\mu\epsilon_1\partial_\mu\Phi$$

Superspace

- Transformation in components

$$\delta\Phi \equiv \bar{\epsilon}\mathbb{Q}\,\Phi \;=\; \delta C + \tfrac{1}{2}\mathrm{i}\bar{\theta}\gamma_*\delta\zeta + \ldots\,.$$

$$\delta C = \tfrac{1}{2}\mathrm{i}\bar{\epsilon}\gamma_*\zeta \qquad \delta\zeta = -\tfrac{1}{2}\mathrm{i}\gamma_*\partial\!\!\!/ C\epsilon + \ldots,$$

Covariant derivative

$$\mathbb{D} = \frac{\overrightarrow{\partial}}{\partial\bar{\theta}} + \frac{1}{4}\gamma^\mu\theta\frac{\partial}{\partial x^\mu}$$

the algebra of covariant derivatives is the same that the original susy algebra

Superspace

chiral superfield

$P_R \mathbb{D}\Phi = 0$ $\theta = 0$ this constraint implies that the linear term

$P_L\theta$

Consider $x_+^\mu = x^\mu + \frac{1}{8}\bar{\theta}\gamma_*\gamma^\mu\theta$

$$P_L\mathbb{Q} = P_L \frac{\overrightarrow{\partial}}{\partial\bar{\theta}}\,, \qquad P_R\mathbb{Q} = P_R\left(\frac{\overrightarrow{\partial}}{\partial\bar{\theta}} - \frac{1}{2}\gamma^\mu\theta\frac{\partial}{\partial x_+^\mu}\right)$$

$$P_L\mathbb{D} = P_L\left(\frac{\overrightarrow{\partial}}{\partial\bar{\theta}} + \frac{1}{2}\gamma^\mu\theta\frac{\partial}{\partial x_+^\mu}\right)\,, \qquad P_R\mathbb{D} = P_R\frac{\overrightarrow{\partial}}{\partial\bar{\theta}}$$

$$\Phi(x_+,\theta) = Z(x_+) + \frac{1}{\sqrt{2}}\bar{\theta}P_L\chi(x_+) + \frac{1}{4}\bar{\theta}P_L\theta F(x_+)$$

Superspace

Superaction | Integration of Grassman variables

D term

$$\int \mathrm{d}^4x\,\mathrm{d}^4\theta\,\Phi(x,\theta) = \frac{1}{8}\int \mathrm{d}^4x\,D(x)$$

F term

$$\int \mathrm{d}^4x\,\mathrm{d}^2 P_L\theta\,\Phi(x,\theta)$$

Bogomol’ny bound

- Consider an scalar field theory in 4d flat space time

$$L = -\frac{1}{2}(\partial_\mu \phi)^2 - V(\phi), \quad V(\phi) = \phi^2 (m + g\phi)^2$$

There are two vacua at $\phi = 0, \quad \phi = -\frac{m}{g}$

We expect a domain wall separating the region of two vacua

We look for an static configuration connecting the two vacua

$$\phi(x) \longrightarrow -\frac{m}{g}, \quad x \to \infty$$

$$\phi(x) \longrightarrow 0, \quad x \to -\infty$$

Bogomol’ny bound

- BPS procedure

The potential V can be wriiten in terms of superpotential W

$$W = \frac{1}{2}\phi^2 + \frac{1}{3}g\phi^3 \qquad\qquad V = W'^2$$

Energy density in terms energy momentum tensor

$$\begin{aligned} T_{00} &= |\dot{\phi}|^2 + \nabla\phi \cdot \nabla\phi + |W'|^2 \\ &= |\partial_x \phi|^2 + |W'|^2 \qquad \text{for domain wall} \end{aligned}$$

Total energy

$$\begin{aligned} \mathcal{E} &= \int_{-\infty}^{\infty} dx \left[|\partial_x \phi|^2 + W'^2 \right] \\ &= \int_{-\infty}^{\infty} dx \, (\partial_x \phi - W')^2 + 2\Delta W \end{aligned}$$

Bogomol’ny bound

where $$\Delta W = \left[W\left(\phi(x)\right)\right]_{x=-\infty}^{x=\infty}$$

We have an energy bound

$$\mathcal{E} \geq 2|\Delta W|$$

which is saturated if the first order equation, BPS equation is verified

$$\partial_x \phi = W'$$

In this case the energy is $$\mathcal{E} = \frac{m^3}{3g^2}.$$

Domain wall as a BPS solution

$$\phi(x) = -\frac{\mu}{2g}\left[\tanh\left(\mu\left(x - x_0\right)/2\right) + 1\right]$$

One can prove that this BPS solution is also a solution of the second order equations of motion

Notice that the domain wall is non-perturbative solution of the equations of motion

If the theory can be embbed in a supersymmetric theory, the solutions of the BPS equations will preserve some supersymmetry