Effective Dynamics of the domain wall

The width of the domanin wall is $L \sim \frac{1}{m}$ If we consider fluctuations of

the scalar filed with wave length >>L the dynamics of the will be Independent of of the details of the wall.

$$\phi(t,x,y,z) = \phi_{cl}(x) + \delta\phi(t,x,y,z)$$

The lagrangian up to quadratic fluctuations is

$$L = L(\phi_{cl}) - \frac{1}{2}\partial_i(\delta\phi)\partial^i(\delta\phi) - \frac{1}{2}(-\partial_x^2 - m^2 + 3m^2 tan^2h(\frac{(x-x_0)m}{2})(\delta\phi)^2$$

Let us do the separation of variables

$$\delta\phi(t,x,y,z) = X(t,y,z)Z(x)$$

Effective Dynamics of the domain wall

To study the small perturbations we we should study the eigenvalue problem

$$(-\partial_x^2 - m^2 + 3m^2 tan^2 h(\frac{(x-x_0)m}{2}))Z_n(x) = w_n Z_n(x)$$

Exits a zero mode $Z_0(x) = \phi'_{cl}(x)$

This zero mode corresponds to a massless excitation and it is associated with the broken translation invariance

The action for these fluctuations given by

$$S = -T \int dt dy dz (1 + \frac{1}{2}\partial_i X \partial^i X) \quad T = -\int dz L(\phi_{cl})$$

It describe the accion of a membrane, 2-brane, at low energies

Effective Dynamics of the domain wall

The membrane action to all orders is given by

$$S = -T \int d^3\xi \sqrt{-\det g}$$

where is teh determinat of the induced metric

$$g_{\mu\nu}(\xi) = \partial_\mu X^m \partial_\nu X^n \eta_{mn}$$

Supersymmetric domain wall

- WZ Action

$$S_{\text{kin}} = \int \mathrm{d}^4x \left[-\partial^\mu \bar{Z} \partial_\mu Z - \bar{\chi} \not{\partial} P_L \chi + \bar{F} F \right]$$

$$S_F = \int \mathrm{d}^4x \left[F W'(Z) - \tfrac{1}{2} \bar{\chi} P_L W''(Z) \chi \right]$$

W(Z) superpotential, arbitrary holomorphic function of Z $\quad S_{\bar{F}} = (S_F)^\dagger$

Complete action $\quad S = S_{\text{kin}} + S_F + S_{\bar{F}}$

F $\bar{F}$ Are not a dynamical field, their equations of motion are algebraic

$F = -\overline{W}'(\bar{Z})$ $\quad \bar{F} = -W'(Z)$ we can eliminate them

Domain wall ½ BPS

The susy transformations for the WZ model are

$$\begin{aligned}\delta Z &= \frac{1}{\sqrt{2}}\bar{\epsilon}P_L\chi\,,\\ \delta P_L\chi &= \frac{1}{\sqrt{2}}P_L(\not{\partial}Z + F)\epsilon\,,\\ \delta F &= \frac{1}{\sqrt{2}}\bar{\epsilon}\,\not{\partial}P_L\chi\,.\end{aligned}$$

$$\begin{aligned}\delta\bar{Z} &= \frac{1}{\sqrt{2}}\bar{\epsilon}P_R\chi\,,\\ \delta P_R\chi &= \frac{1}{\sqrt{2}}P_R(\not{\partial}\bar{Z} + \bar{F})\epsilon\\ \delta\bar{F} &= \frac{1}{\sqrt{2}}\bar{\epsilon}\,\not{\partial}P_R\chi\,.\end{aligned}$$

For the domain wall ansatz the transformation of the should be

$$0 = \delta\chi = \frac{1}{2}(\gamma^1\partial_x Z - W')\epsilon_R + \frac{1}{2}(\gamma^1\partial_x\bar{Z} - \bar{W}')\epsilon_l$$

Domain wall ½ BPS

This condition implies

$$\partial_x Z = W'$$

and

$$(\gamma_x - 1)\epsilon_R = (\gamma_x - 1)\epsilon_L = 0$$

As $\gamma_x^2 = 1$ and $\mathrm{tr}\ \gamma_x = 0$ the space of solutions for ϵ is 2-dimensional

Note that this supersymmetric calculation recovers the result of the bosonic BPS Calculation. Therefore the domain wall is ½ BPS

This result can be deduced from the anticommutator of spinorial charges

Classical Solutions of Supergravity

- The solutions of supergravity give the metric, vector fields and scalar fields.
- The preserved supersymmetry means some rigid supersymmetry

$$\delta(\epsilon)\,\mathrm{boson} = \epsilon\,\mathrm{fermion}\,, \qquad \delta(\epsilon)\,\mathrm{fermion} = \epsilon\,\mathrm{boson}$$

Killing Spinors and BPS Solutions

- N=1 D=4 supergravity

Flat metric with fermions equal to zero is a solution of supergravity with

$$g_{\mu\nu} = \eta_{\mu\nu}$$

Vacuum solution

The residual global transformations are determined by the conditions

$$\delta e_\mu^a = \frac{1}{2}\bar{\epsilon}\gamma^a\psi_\mu = 0\,, \qquad \delta\psi_\mu = D_\mu\epsilon = 0$$

The Killing spinors of the Minkowski background are the set of 4 independent constant Majorana spinors. We have D=4 Poincare Susy algebra

Killing vectors and Killing spinors

$$k_A = k_A^\mu \frac{\partial}{\partial x^\mu}, \quad \mathcal{L}_{k_A} g_{\mu\nu} = \nabla_\mu k_{\nu A} + \nabla_\nu k_{\mu A} = 0$$

$$[k_A\, ,\, k_B] = f_{AB}{}^C k_C$$

Killing Spinors and BPS Solutions

- The integrability condition for Killing spinors

$\text{Killing spinor condition } D_\mu \epsilon = 0$ Integrability condition

$$[D_\mu, D_\nu]\epsilon = \frac{1}{4} R_{\mu\nu ab}\gamma^{ab}\epsilon = 0$$

$Suppose\ that\ \epsilon\ and\ \epsilon'\ are\ both\ Killing\ spinors$

$$-\frac{1}{2}\bar{\epsilon}'\gamma^\rho\gamma^\nu R_{\mu\nu ab}\gamma^{ab}\epsilon = R_{\mu\nu}\bar{\epsilon}'\gamma^\rho\gamma^\nu\epsilon = 0$$

$$R_{\mu\nu}\bar{\epsilon}'\{\gamma^\rho, \gamma^\nu\}\epsilon = 2R^\rho_\mu\,\bar{\epsilon}'\epsilon = 0$$

A spacetime with Killing spinors satisfies $R_{\mu\nu} = 0$ only if $\bar{\epsilon}'\epsilon \neq 0$

Killing spinors for pp-waves

Ansatz for the metric

$$\mathrm{d}s^2 = 2H(u,x,y)\mathrm{d}u^2 + 2dudv + \mathrm{d}x^2 + \mathrm{d}y^2$$

For H=0 reduces to Minkowski spacetime in light-cone coordinates

$$u = (x-t)/\sqrt{2},\ v = (x+t)/\sqrt{2}.$$

Flat metric in these coordinates $\hat{\eta}_{ab}$, where $a, b = +, -, 1, 2$

$$\hat{\eta}_{+-} = \hat{\eta}_{-+} = \hat{\eta}_{11} = \hat{\eta}_{22} = 1$$

Note that $K = \partial/\partial v$ is a covariant constant null vector

$$k^M \frac{\partial}{\partial x^M} = \frac{\partial}{\partial v},\ k_{M;N} = 0,\ g_{MN}k^M k^N = 0$$

Killing spinors for pp-waves

The frame 1-forms are

$$e^- = du\,, \qquad e^+ = dv + H du\,, \qquad e^1 = dx\,, \qquad e^2 = dy$$

From the first Cartan structure equation we get the torsion free spin connection one forms

$$\mathrm{d}e^a + \omega^a{}_b \wedge e^b \equiv T^a$$

$$\omega^{+1} = H_x e^-\,, \qquad \omega^{+2} = H_y e^-$$

and from the second one $\mathrm{d}\omega^{ab} + \omega^a{}_c \wedge \omega^{cb} = \rho^{ab}$

$$\rho^{+1} = H_{xx} e^1 \wedge e^- + H_{xy} e^2 \wedge e^-, \quad \rho^{+2} = H_{yy} e^2 \wedge e^- + H_{xy} e^1 \wedge e^-$$

Killing spinors for pp-waves

The Killing spinor conditions are

$$D_\mu \epsilon = (\partial_\mu + \frac{1}{4}\omega_\mu{}^{ab}\gamma_{ab})\epsilon = 0$$

explicitely

$$\begin{aligned} D_u\epsilon &= (\partial_u - \tfrac{1}{2}H_x\gamma^1\gamma^- - \tfrac{1}{2}H_x\gamma^2\gamma^-)\epsilon = 0\,, \\ D_v\epsilon &= \partial_v\epsilon = 0\,, \qquad D_x\epsilon = \partial_x\epsilon = 0\,, \qquad D_y\epsilon = \partial_y\epsilon = 0\,. \end{aligned}$$

All conditions are verified if we take constant spinors with constraint

$$\gamma^-\epsilon = 0, \quad \gamma_0\gamma^1\epsilon = \epsilon$$

Since $(\gamma_0\gamma^1)^2, \mathrm{tr}\,\gamma_0\gamma^1 = 0$ there are two Killing spinors.

Killing spinors for pp-waves

Notice

$$2\bar{\epsilon}'\epsilon = \bar{\epsilon}'(\gamma^+\gamma^- + \gamma^-\gamma^+)\epsilon = 0$$

To complete the analysis we need the Ricci tensor. The non-trvial component is

$$R_{--} = R_{-1-}{}^{1} + R_{-2-}{}^{2} = -(H_{xx} + H_{yy}).$$

Therefore the pp-wave is Ricic flat if and only if H is harmonic in the variables x,y

pp-waves in D=11 supergravity

Eleven dimensional supergravity with bosonic fileds the metric and the four-form field strength F_4 has pp-wave solutions

$$ds^2 = 2dx^+dx^- + H(x^i, x^-)(dx^-)^2 + \sum_{i=1}^{9}(dx^i)^2$$

$$F_4 = dx^- \wedge \varphi$$

where $H(x^i, x^-)$ obeys $\triangle H = \frac{1}{12}|\varphi|^2$

$\triangle$ is the laplacian in the transverse euclidean space $\mathbb{E}^9$

3-form

pp-waves in D=11 supergravity

$\partial/\partial x^+$ is a covariantly constant null vector

If we choose

$$H(x^i, x^-) = \sum A_{ij} x^i x^j$$

where $A_{ij} = A_{ji}$ is a constant symmetric matrix

They have at least 16 Killing spinors. If one choose

$$A_{ij} = \begin{cases} -\frac{1}{9}\mu^2\delta_{ij} & i,j = 1,2,3 \\ -\frac{1}{36}\mu^2\delta_{ij} & i,j = 4,5,\ldots,9 \end{cases}$$

$$\varphi = \mu dx^1 \wedge dx^2 \wedge dx^3 ,$$

The number of Killing spinors is 32!, like $\mathrm{AdS}_4 \times S^7$ and $\mathrm{AdS}_7 \times S^4$

Spheres

the unit sphere S^2 is the surface $(x^1)^2+(x^2)^2+(x^3)^2 = 1$ embedded in flat Euclidean space $\mathbb{R}^3$

$$x^1 = \sin\theta^2 \sin\theta^1, \ x^2 = \sin\theta^2 \cos\theta^1, \ x^3 = \cos\theta^2$$

$$0 \leq \theta^1 \leq 2\pi, \quad 0 \leq \theta^2 \leq \pi$$

The metric of the sphere is obtained as induced metric of the flat $\mathbb{R}^3$

$$\begin{aligned} \mathrm{d}\Omega_2^2 &= (\mathrm{d}x^1)^2 + (\mathrm{d}x^2)^2 + (\mathrm{d}x^3)^2 \\ &= (\mathrm{d}\theta^2)^2 + \sin^2\theta^2(\mathrm{d}\theta^1)^2 \,. \end{aligned}$$

Spheres

Frame one forms

$$\bar{e}^2 = \mathrm{d}\theta^2 , \qquad \bar{e}^1 = \sin\theta^2 \mathrm{d}\theta^1$$

Spin connection. First structure equation

$$\bar{\omega}^{12} = \cos\theta^2 \mathrm{d}\theta^1$$

Curvature. Second curvature equation

$$\bar{\rho}^{ab} = \bar{e}^a \wedge \bar{e}^b$$

Constant positive curvature

Spheres

Recursive proocedure for higher dimensional sphres

$$x^{n+1}_{(n)} = \cos\theta^n\,, \qquad x^a_{(n)} = \sin\theta^n x^a_{(n-1)}\,, \qquad a \leq n$$

Frame and connection forms are

$$\begin{aligned}
\bar{e}^n_{(n)} &= \mathrm{d}\theta^n\,, \qquad \bar{e}^a_{(n)} = \sin\theta^n \bar{e}^a_{(n-1)}\,, \qquad a \leq n-1 \\
\bar{\omega}^{ab}_{(n)} &= \bar{\omega}^{ab}_{(n-1)}\,, \qquad \bar{\omega}^{an}_{(n)} = \cos\theta^n \bar{e}^a_{(n-1)}\,.
\end{aligned}$$

$$\bar{e}^a = \Big(\prod_{j=a+1}^{n} \sin\theta^j\Big)\mathrm{d}\theta^a\,, \qquad a \leq n\,,$$

$$\bar{\omega}^{ab} = \cos\theta^b\Big(\prod_{j=a+1}^{b-1} \sin\theta^j\Big)\mathrm{d}\theta^a\,, \qquad 1 \leq a < b \leq n$$

Spheres

- Coset structure

Anti-de Sitter space

AdS_D for the D-dimensional case simple solutions of supergravity

with negative constant solution

$$S = \frac{1}{2\kappa^2} \int \mathrm{d}^D x \sqrt{-g}\,(R - \Lambda)$$

$$\Lambda = -(D-1)(D-2)/L^2$$

$$R_{\mu\nu} = -\frac{D-1}{L^2} g_{\mu\nu}$$

AdS is an example of a maximally symmetric spacetime

$$R_{\mu\nu\rho\sigma} = k\,(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho})$$

k is a constant of dimension $1/(\mathrm{length})^2$ $\quad k = -1/L^2$

Anti-de Sitter space

Einstein metric

$$\begin{aligned} R_{\mu\nu} &= k(D-1)g_{\mu\nu} \\ R &= k\,D(D-1)\,. \end{aligned}$$

Ads as a coset space

$$[P_0, K] = iP_1$$

$$[P_1, K] = iP_0 \qquad\qquad K = J_{01}.$$

$$[P_0, P_1] = -i\frac{1}{R^2}K$$

$$g = e^{iP_1 x^1} e^{iP_0 x^0}$$

Anti-de Sitter space

- MC 1-form

$$\begin{aligned}\Omega &= P_0\ dx^0\ +\ P_1\ (dx^1 \cos\frac{x^0}{R})\ -\ K\ (\frac{dx^1}{R}\sin\frac{x^0}{R}) \\ &\equiv P_0\ e^0\ +\ P_1\ e^1\ +\ K\ \omega^{01}\ .\end{aligned}$$

Ads metric

$$ds^2 = -e^0 e^0\ +\ e^1 e^1\ =\ -dx^{0\,2}\ +\ dx^{1\,2}\ \cos^2\frac{x^0}{R}$$

Ads can be embedded in pseuo-Euclidean space

$$\eta_{ab} u^a u^b = -(u^0)^2 + (u^1)^2 - (u^2)^2 = -R^2$$

Anti-de Sitter space

$$u^0 = R \, \sin \frac{x^0}{R}$$

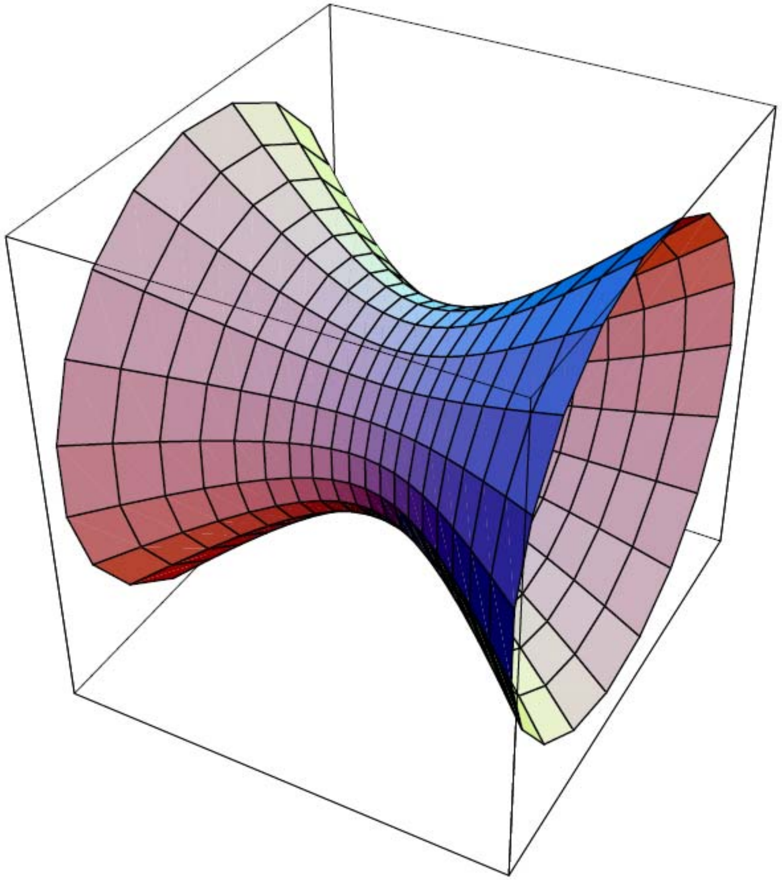
$$u^1 = R \, \cos \frac{x^0}{R} \, \sinh \frac{x^1}{R}$$

$$u^2 = R \, \cos \frac{x^0}{R} \, \cosh \frac{x^1}{R}$$

metric

$$ds^2 = -e^0 e^0 \, + \, e^1 e^1 \, = \, -dx^{0\,2} \, + \, dx^{1\,2} \, \cos^2 \frac{x^0}{R}$$

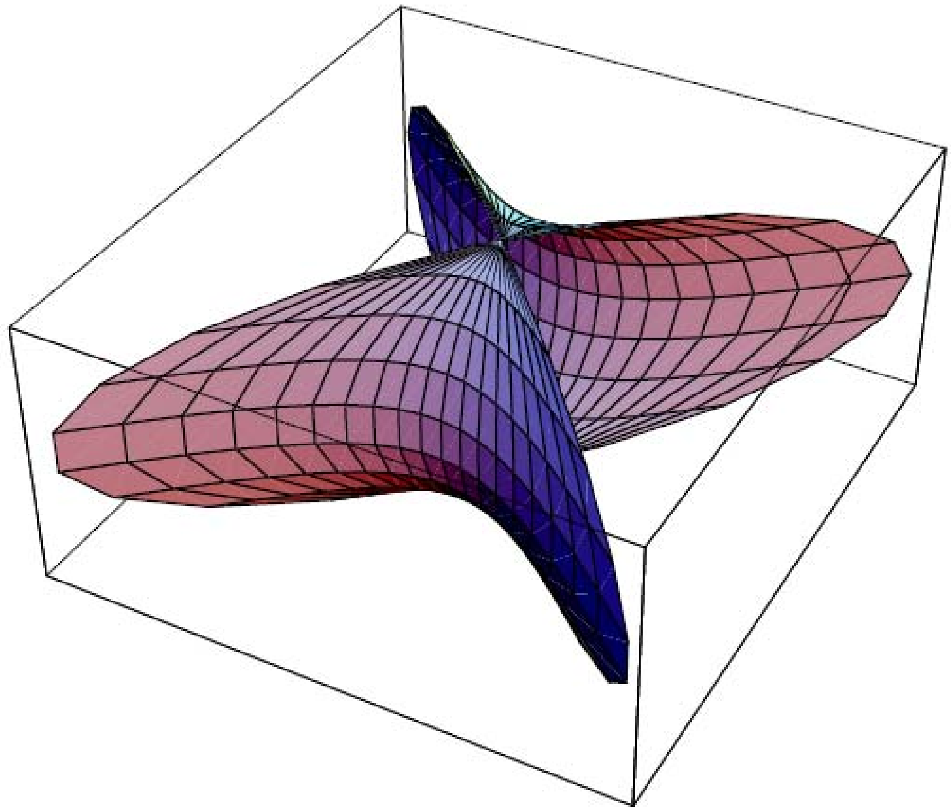
Anti-de Sitter space



Anti-de Sitter space

Note that u^0 varies in (-R.R) $x^0 \in (0, 2\pi R)$ i $x^1 \in (-\infty, \infty)$

Local parametrization



Global parametrization

$$g = e^{iP_0 x^0} e^{iP_1 x^1}$$

$$\begin{aligned}\Omega : \quad &= P_1\ dx^1\ +\ P_0\ (dx^0 \cosh \frac{x^1}{R})\ +\ K\ (\frac{dx^0}{R} \sinh \frac{x^1}{R}). \\ &\equiv P_0\ e^0\ +\ P_1\ e^1\ +\ K\ \omega^{01}\ .\end{aligned}$$

$$\begin{aligned}u^0 =& R\ \cosh \frac{x^1}{R} \sin \frac{x^0}{R} \\ u^1 =& R\ \sinh \frac{x^1}{R} \\ u^2 =& R\ \cosh \frac{x^1}{R} \cos \frac{x^0}{R}\ .\end{aligned}$$

$$ds^2 = -\ \cosh^2 \frac{x^1}{R} {dx^0}^2\ +\ {dx^1}^2$$

Anti-de Sitter space

$$Y^A \eta_{AB} Y^B = -(Y^0)^2 + \sum_{i=1}^{D-1} (Y^i)^2 - (Y^D)^2 = -L^2$$

the coordinates $Y'^A = \lambda^A{}_B Y^B$ provided that $\lambda^A{}_B$ is a matrix

$SO(D-1,2)$

Different embeddings

$$\begin{aligned} Y^i &= r\bar{x}^i \qquad \text{with} \qquad \sum_{i=1}^{D-1} (\bar{x}^i)^2 = 1\,, \\ Y^0 &= \sqrt{L^2+r^2}\sin(t/L) \qquad Y^D = \sqrt{L^2+r^2}\cos(t/L) \end{aligned}$$

$\bar{x}^i$ parameterizes the unit sphere S^{D-2}

Anti-de Sitter space

This coordinate system is global, covers the whole hyperbolid for

$0 \leq r < \infty, \quad 0 \leq t < 2\pi L$ the algunlar variables the whole S^{D-2}

New radial coordinate $\cosh(y/L) = \sqrt{1 + r^2/L^2}$

$$ds^2 = -\cosh^2(y/L)\mathrm{d}t^2 + \mathrm{d}y^2 + L^2 \sinh^2(y/L)\, \mathrm{d}\Omega^2_{D-2}$$

Another possibility $\cosh(y/L) = 1/\cos\rho \quad t = L\tau$

$$ds^2 = \frac{L^2}{\cos^2\rho}\left[-\mathrm{d}\tau^2 + \left(\mathrm{d}\rho^2 + \sin^2\rho\, \mathrm{d}\Omega^2_{D-2}\right)\right]$$

It is conformalto the direct product of the real line, time coordinate, times the Sphere in D-1 dimensions

Anti-de Sitter space

Poincaré patch

$$\begin{aligned} Y^0 &= Lux^0\,, \\ Y^i &= Lux^i\,, \qquad i = 1,\ldots,D-2 \\ Y^{D-1} &= \frac{1}{2u}\left(-1+u^2(L^2-x^2)\right)\,, \\ Y^D &= \frac{1}{2u}\left(1+u^2(L^2+x^2)\right)\,, \\ x^2 &= -(x^0)^2+\sum(x^i)^2\,. \end{aligned}$$

$$\mathrm{d}s^2 = L^2\left[\frac{\mathrm{d}u^2}{u^2}+u^2\left(-(\mathrm{d}x^0)^2+\sum_i(\mathrm{d}x^i)^2\right)\right]$$

Anti-de Sitter space

$$z = 1/u$$

$$\mathrm{d}s^2 = \frac{L^2}{z^2}\left[\mathrm{d}z^2 - (\mathrm{d}x^0)^2 + \sum_i (\mathrm{d}x^i)^2\right]$$

The metric is conformal to the positive region of D dimensional Minlowski space with coordinates (x^0, x^i, z)

Killing spinors for anti-de Sitter space

The bosonic action that leads to AdS space is

$$S = \frac{1}{2\kappa^2} \int \mathrm{d}^D x \sqrt{-g}\,(R - \Lambda)$$

Killing spinor are solutions of

$$\hat{D}_\mu \epsilon \equiv (D_\mu - \frac{1}{2L}\gamma_\mu)\epsilon = 0$$

Integrability condition

$$[\hat{D}_\mu\,,\,\hat{D}_\nu]\epsilon \;=\; \left(\frac{1}{4}R_{\mu\nu ab}\gamma^{ab} + \frac{1}{2L^2}\gamma_{\mu\nu}\right)\epsilon$$

If we insert $R_{\mu\nu ab} = -(e_{a\mu}e_{b\nu} - e_{a\nu}e_{b\mu})/L^2$ vanishes identically

It is a hint that AdS is a maximally supersymmetric space

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Killing spinors for anti-de Sitter space

We will study Killing spinors in the Poincaré patch of AdS_D

$$z = L\mathrm{e}^{-r/L}$$

$$\mathrm{d}s^2 = \mathrm{e}^{2r/L}\eta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu + \mathrm{d}r^2$$

Frame fields $$e^{\hat{\mu}} = \mathrm{e}^{r/L}\mathrm{d}x^\mu\,, \qquad e^r = \mathrm{d}r$$

Spin connection $$\omega^{\hat{\mu}r} = \frac{1}{L}e^{\hat{\mu}}\,, \qquad \omega^{\mu\nu} = 0$$

$$\begin{aligned}\hat{D}_r &= (\partial_r - \frac{1}{2L}\gamma_r)\epsilon = 0\,,\\ \hat{D}_\mu &= (\partial_\mu + \frac{1}{2L}(\gamma_r - 1))\epsilon = 0\end{aligned}$$

Killing spinors for anti-de Sitter space

we introduce constant spinors $\eta_\pm$ which satisfy $\gamma_r \eta_\pm = \pm\eta_\pm$

$$\begin{aligned}
\epsilon_+ &= \mathrm{e}^{r/2L}\eta_+ \, , \\
\epsilon_- &= (\mathrm{e}^{-r/2L} + \frac{1}{L}\mathrm{e}^{r/2L} x^\mu \gamma_{\hat{\mu}})\eta_-
\end{aligned}$$

The last term includes transverse indexes.