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Thermodynamics of Graviton Condensate and the Kiselev Black Hole

by

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Santiago - Chile
Agosto - 2020

Acknowledgements

I would like to thank my advisor, Jorge Alfaro, for his constant support in this thesis and my academic career. I have benefited from his enormous knowledge of physics, and his committed attitude towards his graduate students.

I am also very grateful to the creators of the following websites: arXiv, Inspired HEP, and "other" open sources. Thanks to them, I have been able to benefit significantly from various reviews and scientific publications. Their work takes enormous importance in uncertain times like the present, where leaving home is not an option.

I would also like to thank my parents for their unwavering support for my academic career. To my mother for always being willing to listen to my wildest dreams, and to my father for supporting me, no matter how unlikely my dreams look to him.

Finally, I want to dedicate this thesis to Paulina Peña. Throughout my master's degree, you have been a stabilizing force for my extreme way of life. Thank you for understanding my intellectual interests, and for being patient with my frequent motivational speeches. Please, never forget that nobody knows who will be next.

Abstract

In this thesis, we will present the thermodynamic study of a model that considers the black hole as a condensation of gravitons (55) (56). We will obtain a correction to the Hawking temperature and a negative pressure for a black hole of mass M. In this way, the graviton condensate, which is assumed to be at the critical point defined by the condition μ_{ch}=0, will have well-defined thermodynamic quantities P, V, T_h, S, and U as any other Bose-Einstein condensate. We will also discuss the Kiselev black hole, which has the capacity to parametrize the most well-known spherically symmetric black holes. We will show that this is true, even at the thermodynamic level. Finally, we will present a new metric, which we will call the BEC-Kiselev black hole, that will allow us to extend the graviton condensate to the case of solutions with different types of the energy-momentum tensor.

Contents

Chapter 1

Introduction

Nowadays, we are living a new golden age for gravitational physics, both in theoretical and experimental physics. In 1998, the accelerated expansion of the universe was discovered from supernovae measurements. This discovery requires an unknown form of energy called dark energy, which has the property of a repulsion action with a negative pressure distributed homogeneously in the universe. We also have the dark matter paradigm. Many astrophysical observations lead to the idea that there is more matter than we can see. The dark matter does not interact with the Standard Model fields, so we cannot see it in a conventional sense. However, we know about the existence of dark matter through its gravitational effects. An example of this evidence is the galaxy rotation curve. The extreme of the spiral galaxies' arms rotates faster than the Newtonian gravity predicts. To explain this problem with gravity, we require more matter, the dark matter. There are as many theoretical models of dark matter as there is experimental evidence of their existence.

More recently, in 2016, the discovery of gravitational waves has opened many possibilities both in theoretical and experimental physics to search for new physics out there in our universe. Before 2016, all observations in astronomy, and cosmology were made only by exploring the electromagnetic spectrum. This fact is astonishing by itself. Only studying the light, humankind has been able to discover incredible physical phenomena from the scale of our neighborhood, the milk galaxy, until cosmic scale with the discovery of the CMB spectrum in 1965 by Penzias and Wilson. Nowadays, we have a new eyeglass to detect physical phenomena never seen before. Some signals of gravitational waves are explained by the merge of monster black holes in some distant region of our universe. The gravitational waves' discovery supports the existence of one of the most extreme objects that we can imagine, the black holes.

Quantum Gravity: The holy grail of all sciences. After more than 100 years that we have the general theory of relativity, we still have not been able to obtain its definitive quantum version. For the moment, black holes are our best option to discover a new fundamental principle that allows us to achieve the long-awaited theory of quantum gravitation. Thanks to the physical intuition of Jacob Bekenstein, we know that black holes have entropy. Nevertheless, this would not be entirely true if it were not for Hawking's calculation of a black hole's thermal spectrum. Hawking radiation is the only quantum gravity computation that we have, but this leads to the information paradox. According to quantum mechanics, the information must be conserved in order to respect the unitary evolution of any quantum system. However, the event horizon of a black hole encloses a spacetime region that is causally disconnected from the outside. This fact leads to the possibility that when a black hole finishes evaporating, the information ceases to exist, implying a violation of the conservation of information. This paradox and related ones have made black hole physics a hot topic of research today. Thanks to the holographic principle, there have been enormous advances in the direction of solving this historical paradox. Nevertheless, there is still much to unravel in this minefield of contradictions and paradoxes.

One possibility to solve the paradox mentioned above and other related problems is that black holes are actually Bose-Einstein condensates. This thesis will explore the possibility of describing black hole thermodynamics in terms of a graviton condensate. This thesis is organized as follows. In Chapter 2, we will present the background of black holes physics. We will discuss the nature of the event horizon and how Hawking temperature establishes the black holes' thermal character. Chapter 3 will introduce the pressure and volume term for a black hole in two different approaches. In the first one, the negative cosmological constant plays the role of a pressure term. In the second one, we will discuss a local program to the black hole thermodynamics called the Horizon thermodynamics approach. Chapter 4 will begin with the Black Hole N-portrait proposal discussion, where one considers that the Hawking radiation is due to the leakage of particles from a graviton condensate. Then, we will discuss a model, which gives a geometrical interpretation of the N-portrait proposal. After this, we will obtain the thermodynamic behavior of this geometrical model. Chapter 5, we will introduce the Kiselev black hole, which can parameterize the most well-known spherically symmetric black holes. We will obtain its thermodynamic behavior, and we will discuss its possible interpretation as a graviton condensate. Finally, we will add a series of appendices with two purposes in mind: to avoid excessive lengthening the central part of this thesis, and to be self-contained to serve as an introductory reference to people outside the field of black holes.

Chapter 2

Black Holes Thermodynamics

This chapter has been written with two purposes in mind. The first one is to establish the basis of black hole physics that we will use throughout this thesis. The second purpose is that this work to be self-contained to serve as an introductory reference for people outside the field of the black hole [BH]. To avoid overstretching this chapter, we have left long calculations and technical details in appendices A, B, C, and D so that we will refer only to the final results.

2.1 Notation and Conventions

This section aims to establish the notation and conventions for the more relevant tensors in General Relativity [GR], which will use throughout the whole thesis. Our signature is $(-,+,+,+)$, and we will limit ourselves to work in 4 dimensions. Besides, Greek indices α, β, ..., etc. will run over 0,1,2,3, and Latin indices a, b, ..., etc. will run over 1,2,3. Also, we will use natural units $[G = c = k_b = \hbar = 1]$, except on some special occasions where we will explicitly state the unit system used. With that said, we start with the Einstein-Hilbert action coupled with matter fields

$$S = \int \left[\frac{1}{8\pi} R + \mathcal{L}_M \right] \sqrt{-g} d^4x \tag{2.1}$$

$\mathcal{L}_M$ represents the "matter" Lagrangian density for any non-gravitational field present, and R is the Ricci scalar defined as

$$R = g^{\mu\nu} R_{\mu\nu} \tag{2.2}$$

The Ricci tensor $R_{\mu\nu}$ comes from the contraction of the metric and the Riemann tensor in the following way

$$R_{\mu\nu} = g^{\alpha\beta} R_{\mu\alpha\nu\beta} \tag{2.3}$$

The Riemann tensor in terms of the metric connection is given by

$$R^{\mu}_{\ \nu\alpha\beta} = \partial_\alpha \Gamma^{\mu}_{\nu\beta} - \partial_\beta \Gamma^{\mu}_{\nu\alpha} + \Gamma^{\lambda}_{\nu\beta} \Gamma^{\mu}_{\lambda\alpha} - \Gamma^{\lambda}_{\nu\alpha} \Gamma^{\mu}_{\lambda\beta} \tag{2.4}$$

Finally, the metric connection is defined as follows

$$\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2} g^{\mu\lambda} \left(-\partial_\lambda g_{\alpha\beta} + \partial_\alpha g_{\lambda\beta} + \partial_\beta g_{\alpha\lambda} \right) \tag{2.5}$$

In honor of the brevity, we will skip the explicit variation of the action, which can be found in any intermediate-level textbook of GR. The Einstein equation from the variational principle (2.1) is given by

$$G^{\mu\nu} = 8\pi T^{\mu\nu} \tag{2.6}$$

The Einstein tensor involved the metric and its first and second derivatives through the Ricci tensor and Ricci scalar in the following way

$$G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R \tag{2.7}$$

The energy-momentum tensor can be obtained from the "matter" Lagrangian density using the following expression

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\partial(\sqrt{-g}\mathcal{L}_M)}{\partial g^{\mu\nu}} \tag{2.8}$$

One can write the Einstein equation in an alternative way if one notes that $R = -T$. Using this relation one obtains the alternative expression for the Einstein equation

$$R^{\mu\nu} = 8\pi(T^{\mu\nu} - \frac{1}{2}g^{\mu\nu}T) \tag{2.9}$$

In the Einstein-Hilbert action, we intentionally omitted the Gibbons-Hawking-York boundary term, which allows us to have a well-defined variational principle. We did this for simplicity, and because this term does not play any role throughout this thesis. To see the detail of this boundary term, we recommend a brilliant textbook called Gravitation, Foundations, and Frontiers (1). To see the details of the variation of (2.1) clearly and pedagogically, we recommend the introductory textbook called General Relativity, an introduction for physicists (2). In this section, we have written the definitions of the more used tensors of GR to establish a unique notation and conventions for all of them. To see the properties of these tensors, the same textbooks mentioned above are recommended.

2.2 Continuity of the Metric at the Event Horizon

In 1916 Karl Schwarzschild found the Einstein equation's first exact solution for a static spherical symmetry, just a few months after that Einstein got the final form of his equations. This solution represents the outside of a spherical mass M such a non-rotating star. Einstein's equations are so difficult to solve that it was almost 50 years before Roy Kerr was able to find the solution with stationary axial symmetry, representing the exterior of a rotating star in 1963. Apart from the technical difficulty of finding these solutions, it also was a conceptual challenge to understand them. If the system's radius R were smaller than a certain radius r_h, the system would undergo a gravitational collapse from which even light could not escape. The radius r_h mark the no return zone, and we call it the event horizon radius.

In the more natural coordinate system for spherically symmetric black holes, the metric is singular at r_h. Incredibly, for more than 40 years, the nature of the singularity at r_h was a total mystery. Solely after the works of Finkelstein, Kruskal, and Szekeres around 1960, the physicists understood that the event horizon radius was not a real singularity.

It can be removed doing coordinate transformations. Still more important, nowadays, we understand the metric is smoothly continuous at r_h as a natural consequence of the equivalence principle. The golden age of black holes physics started after the revelation of the metric's well-behavior at the event horizon. The goal of this section is to study the behavior of the metric around the event horizon.

A static spherically symmetric metric can be written as

$$ds^2 = -f(r)dt^2 + \frac{1}{h(r)}dr^2 + r^2 d\Omega \tag{2.10}$$

Here $d\Omega \equiv d\theta^2 + \sin^2(\theta)d\phi^2$ is the unit 2-sphere. We will call $f(r)$ the lapse function throughout this work. The components of the metric connection and the Einstein tensor are given in Appendix A. We will focus, for now, on the Schwarzschild solution where $f = h$. The lapse function is

$$f(r) = 1 - \frac{2M}{r} \tag{2.11}$$

We can see two places where the metric is singular, one is $r_s = 0$, and the other is $r_h = 2M$ where g_{rr} diverges. First, r_s is a real singularity of the BH; we cannot avoid it with the current understanding of BH physics. Second, r_h is a coordinate singularity that appears due to the wrong choice of coordinates. To remove the singularity at r_h, we use the following coordinate transformation

$$u = t + r + 2M \ln\left|\frac{r}{2M} - 1\right| \tag{2.12}$$

$$\Rightarrow du = dt + \left(1 - \frac{r}{2M}\right)^{-1} dr$$

Therefore, the metric takes the simple form

$$ds^2 = -\left(1 - \frac{2M}{r}\right) du^2 + 2dudr + r^2 d\Omega^2 \tag{2.13}$$

This line element is the Schwarzschild solution in the so-called Advanced Eddington-Finkelstein coordinates. Taking the range of the radial coordinate as $0 < r < \infty$, we see now the metric is regular at $r = r_h$.

A more useful coordinate system is Kruskal-Szekeres coordinates, which has the following transformations

$$\kappa X = e^{\kappa r^*} \cosh(\kappa t), \quad \kappa T = e^{\kappa r^*} \sinh(\kappa t) \tag{2.14}$$

Where the intermediate coordinate r^*, called the tortoise coordinate, is defined as

$$r^* = \int \frac{dl}{f(l)} \tag{2.15}$$

In the Schwarzschild case, the tortoise coordinate is given by

$$\Rightarrow r^* = r + 2M \ln\left|\frac{r}{2M} - 1\right|$$

While the constant κ is defined as

$$\kappa \equiv \frac{1}{2} f'(r_h) \tag{2.16}$$

$$\Rightarrow \kappa = \frac{1}{4M}$$

Later, this constant called the surface gravity will play a fundamental role in black hole thermodynamics. In the meantime, using these coordinate transformations, the line element takes the form

$$ds^2 = \frac{4r_h^3}{r} e^{-r/r_h} \left(-dT^2 + dX^2\right) + r^2 d\Omega^2 \tag{2.17}$$

Where r is an implicit function of X and T determined by the relation

$$\left(\frac{r}{r_h} - 1\right) e^{r/r_h} = X^2 - T^2 \tag{2.18}$$

The line element (2.17) is diagonal, also it is regular at $r = r_h$. In the Kruskal-Szekeres coordinates, the radial light rays $[d\Omega^2 = 0, ds^2 = 0]$ make 45 degree lines in $X - T$ sector of the metric just like in the Minkowski flat spacetime. We can see this in the following diagram

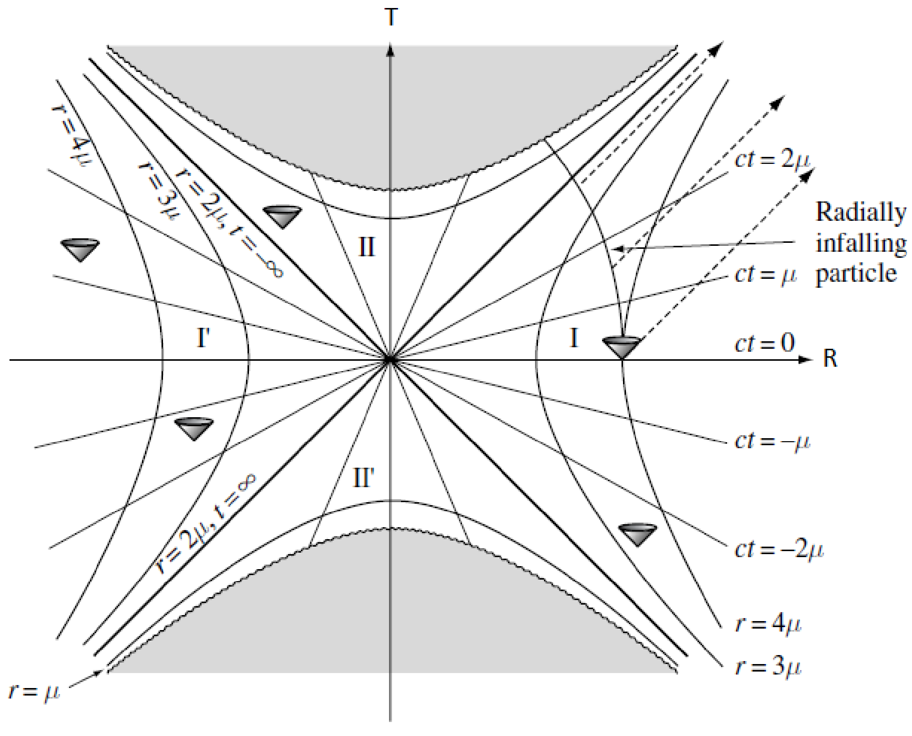

Figure 2.1: Spacetime diagram of the extended Schwarzschild geometry in Kruskal-Szekeres coordinates. Here $\mu \equiv \frac{2MG}{c^2}$. The region I is our universe. The region II is the interior of the black hole. We can see how a light-cone is radially infalling. The broken-line arrows show escaping signals. The region I' is another asymptotic flat spacetime connected through a wormhole to I, and the region II' is the white hole. Our discussion does not consider the regions I' and II [Image extracted from (2)]

2.2.1 The Near-horizon Metric

In this subsection, we will use results from Appendix B, where the Rindler frame for the Minkowski flat spacetime has been described. We expand an arbitrary lapse function near r_h as

$$f(r) \approx f(r_h) + f'(r_h)(r - r_h) \Rightarrow f(r) = 2\kappa l \tag{2.19}$$

Where we use the definition (2.16) of κ and we define $l \equiv r - r_h$. Using this expansion the line element becomes

$$ds^2 \approx -2\kappa l dt^2 + \frac{1}{2\kappa l} dl^2 + dL^2 \tag{2.20}$$

Here dL^2 denotes the metric on t =constant, r =constant. The horizon is now located at $l = 0$. The sector $t - l$ is exactly the Rindler frame described by (B.5), where κ plays the role of acceleration. The metric is still singular at $l = 0$, but we know that geometry does not have any singularity here because the near-horizon metric is the Minkowski flat spacetime. This result is an amazing one; it is the equivalence principle in action [To see the details of how one arrives explicitly at the Minkowski line element go to Appendix B]. Moreover, the equation (2.19) is for any arbitrary lapse function with the only condition of not having null first derivative at $r = r_h$ [$l = 0$]. This result applies to a wide range of metrics. From this point of view, there is nothing special at the event horizon. However, the horizon still marks the zone of no return for any object or photon falling into the BH. The interior of the BH is causally disconnected from the exterior just as an observer who accelerates uniformly is from some patches of Minkowski flat spacetime.

At the classical level, the event horizon is well-understood. It is a null surface which divides the spacetime into zones causally disconnected. In some sense, it is a perfect sample of the marriage between the relativity principle with the equivalence principle. However, at the quantum level, different observers doing quantum physics around the event horizon may have inconsistent results. For instance, a free-falling observer called Alice feels nothing special at the horizon [a super-massive BH can eliminate all tidal forces because these forces fall as $\sim r^3$, so at the horizon, fall as $\sim M^3$]. On the contrary, the external observer called Bob will observe how Alice is falling into a hellish region of extreme temperature, being thermalized, and eventually re-emitted as Hawking radiation (3).

2.3 Black Hole Mechanics

The objective of this section is to establish the first law of black hole mechanics (11). For this, we will define the Komar integral (12) and the Smarr relation (13) based on the references (1) and (5). We will require the use of the killing vector and the concept of surface gravity, whose definitions and properties we will adjunct in Appendix C. We will calculate the Komar integral and the Smarr relation in their general form for a static spherical solution. The first law of black hole mechanics is similar to the first law of thermodynamics. However, at the classical level, this similarity is still a mathematical peculiarity because, classically, BH does not radiate thermally.

2.3.1 The Komar Integral

Let ξ^α be a Killing vector, then we defined the following current vector

$$Q^\alpha \equiv \xi_\beta R^{\alpha\beta} \tag{2.21}$$

This vector has a zero divergence, and it allows us to define integral expressions for conserved quantities. First, we verified its zero divergence

$$\nabla_\gamma Q^\alpha = \xi_\beta \nabla_\gamma R^{\alpha\beta} + R^{\alpha\beta}\nabla_\gamma \xi_\beta$$

From Einstein equation we know $\nabla_\alpha G^{\alpha\beta} = 0$, so $\nabla_\alpha R^{\alpha\beta} = \frac{1}{2} g^{\gamma\beta}\partial_\gamma R$. Besides, due to ξ^α is a killing vector the quantity $\nabla_\alpha \xi_\beta$ is antisymmetric (C.1). Therefore

$$\Rightarrow \nabla_\alpha Q^\alpha = \frac{1}{2}\xi^\beta \partial_\beta R \Rightarrow \nabla_\alpha Q^\alpha = 0$$

In the last step, we use the fact that any geometrical scalars field has zero directional derivatives when ξ^β is a killing vector. Now, we define the following integral

$$I \equiv -\frac{1}{4\pi}\int_\Sigma d^3\Sigma_\alpha Q^\alpha \tag{2.22}$$

Here Σ is the hypersurface of integration. Using the definition of the current vector (2.21) and the alternative form of the Einstein equation (2.9), we can express this integral I in terms of quantities of the energy-momentum tensor

$$I = -2\int_\Sigma d^3\Sigma_\beta \xi^\alpha (T^\beta_{\ \alpha} - \frac{1}{2}\delta^\beta_\alpha T) \tag{2.23}$$

This integral is a global conserved quantity when Σ is a spacelike hypersurface. So, the integrating measure $d^3\Sigma_\beta$ is spacelike. The physical meaning of this quantity depends on the nature of the killing vector used.

Now, we are going to show another form of this integral I. First, we have from the equation (C.3) a relation between the Killing vector and Ricci tensor

$$\nabla_\alpha \nabla^\alpha \xi^\beta = -R^\beta_{\ \mu}\xi^\mu$$

The demonstration of this relation is in Appendix C. We replace this equation in the definition of I (2.22)

$$I = \frac{1}{4\pi}\int_\Sigma \nabla_\beta \nabla^\beta \xi^\alpha d^3\Sigma_\alpha$$

Now, using the divergence theorem in this expression, we obtain

$$I = \frac{1}{8\pi}\int_S \nabla^\beta \xi^\alpha d^2\Sigma_{\alpha\beta} \tag{2.24}$$

Here S is the boundary of the spacelike hypersurface Σ. When $\xi^\mu_{(t)}$ is a timelike killing vector the only non-null term of $\nabla^\beta\xi^\alpha$ is $\nabla^r \xi^t_{(t)} = g^{rr}\Gamma^t_{tr}$ [See Appendix A for non-null components of the connection]. Besides, the two-dimensional measure is

$$d^2\Sigma_{\alpha\beta} = \frac{1}{2!}\sqrt{h}\epsilon_{\alpha\beta\gamma\lambda}dx^\gamma \wedge dx^\lambda \Rightarrow d^2\Sigma_{\alpha\beta} = \epsilon_{\alpha\beta\theta\phi}R^2 sin(\theta)d\theta d\phi$$

Where θ and ϕ are not indices, they are the spherical coordinates related to 2-sphere and h the induced metric. In this case, we have $h = \sqrt{g_{\theta\theta}g_{\phi\phi}} = R^2 sin(\theta)$ from (A.1). We use R instead of r to indicate a fixed radius of the spherical surface outside the event horizon. Finally, we define the integral $I \equiv E_k$ when we use a time Killing vector $\xi^{\mu}_{(t)}$. Thus we obtain

$$E_k = \frac{1}{4\pi}\int_S \nabla^r \xi^t_{(t)} R^2 sin(\theta) d\theta d\phi$$

Here we have $\nabla^r \xi^t_{(t)} = g^{rr}\Gamma^t_{tr}$, and using equation (A.2), one obtains $\nabla^r \xi^t = \frac{1}{2}f'(r)$. Then, the Komar mass in static spherical solution is given by

$$E_k = \frac{R^2}{2}f'(R) \tag{2.25}$$

The Komar mass E_k is a global conserved quantity of the spacetime. This equation should be valid for all lapse functions in static spherical solution. We are going to apply this result for two well-famous solutions. First, the lapse function for Schwarzschild solution is $f(r) = 1 - \frac{2M}{r}$, then its first derivative is $f'(r) = \frac{2M}{r^2}$. Therefore, we obtain

$$E_k = \frac{R^2}{2}\frac{2M}{R^2} \Rightarrow E_k = M$$

This result is independent of the value of R and it is a well-known result for the energy of the Schwarzschild solution. The lapse function for Reissner-Nordstrom solution is $f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}$, so its derivative is $f'(r) = \frac{2}{r^2}\left(M - \frac{Q^2}{r}\right)$. Hence, we obtain

$$E_k(R) = M - \frac{Q^2}{R} \Rightarrow E_k = M$$

The result depends on R, but taking the limit $R \to \infty$, the Komar mass does not depend anymore on R. The Komar mass depends on the asymptotic behavior of the lapse function. This feature will be relevant for the cases of not asymptotically flat spacetime, such as spacetime with cosmological constant or the Kiselev black hole [This BH will be the topic of a chapter 5].

2.3.2 The Smarr Relation

In the presence of a BH, the hypersurface Σ must be taken between the outermost horizon and infinity. The interior of BH is not considered for computing global integral. In these cases, we denote equation (2.23) as $I \equiv I_{bh}$

$$I_{bh} = -2\int_{\Omega}\int_{r_+}^{R} d^3\Sigma_t (T^t{}_t - \frac{1}{2}T)$$

Here r_+ denotes the outermost horizon and Ω the two-dimensional surface of the 2-sphere. We have already considered the spacelike measure $d^3\Sigma_t$ and the Killing vector $\xi^{\mu}_{(t)}$. Using the divergence theorem at infinity, we have the Komar mass E_k given by (2.25), and using the same theorem on the horizon one has the following expression

$$M_H = -\frac{1}{8\pi}\int_H \nabla^{\beta}\xi^{\alpha} d^2\Sigma_{\alpha\beta} \tag{2.26}$$

This equation is the same that (2.24), where the negative sign appears because we took the radial infinity as a positive direction, so toward the horizon, the normal to the null surface is negative-directed. The boundary surface H is the event horizon. To distinguish this integral from the Komar mass, we call it M_H. Summarizing, we have for the Komar mass E_k, the surface integral over the event horizon M_H, and the "matter" integral I_{bh} the following mathematical identity

$$E_k - M_H = -2\int_{\Omega}\int_{r_+}^{R} d^3\Sigma_t(T^t{}_t - \frac{1}{2}T) \tag{2.27}$$

We can relate the surface integral M_H with the horizon area A in a direct way. To do this, we remind the definition of the surface gravity from equation (C.5), which is

$$\xi^\mu \nabla_\mu \xi^\alpha \equiv \kappa \xi^\alpha$$

Using this equation, and the fact that $d^2\Sigma_{\alpha\beta}$ is a two-dimensional null measure, one gets for the expression inside of the surface integral (2.26)

$$\nabla^\beta \xi^\alpha d^2\Sigma_{\alpha\beta} = \nabla^\beta \xi^\alpha (l_\alpha \xi_\beta - l_\beta \xi_\alpha) dA \Rightarrow \nabla^\beta \xi^\alpha d^2\Sigma_{\alpha\beta} = 2\kappa \xi^\alpha l_\alpha dA$$

Where l^α is an auxiliary null vector [$l^a l_\alpha$=0]. We remind from equation (C.4) that on the horizon the killing vector $\xi^\alpha_{(t)}$ vanishes [$\xi^a_{(t)}\xi_{\alpha(t)} = 0$], so one chooses the auxiliary null vector such that $l^\alpha \xi_{\alpha(t)} = -1$ to expand the two-dimensional null surface in these null vectors. To recall the integral expressions over null surfaces see, section 5.5 of (1). Therefore, one gets for M_H that

$$M_H = -\frac{1}{8\pi}\int_H 2\kappa \xi^\alpha l_\alpha dA \Rightarrow M_H = \frac{1}{4\pi}\kappa A$$

Finally, replacing this result in equation (2.27) we obtain a mathematical identity called the Smarr relation

$$E_k = \frac{1}{4\pi}\kappa A - 2\int_{\Omega}\int_{r_h}^{R} d^3\Sigma_t(T^t{}_t - \frac{1}{2}T) \tag{2.28}$$

This result holds in a static spherical solution of the Einstein equation.

For instance, in the case of the Reissner-Nordstrom BH one has $T^t{}_t = \frac{-Q^2}{8\pi r^4}$ and $T = 0$ [See, for instance, (2)] . Applying the equation (2.28), we obtain

$$M = \frac{1}{4\pi}\kappa A - 2\int_{\Omega}\int_{r_h}^{R} \frac{(-Q^2)}{8\pi r^4} r^2 \sin(\theta) dr d\theta d\phi \Rightarrow M = \frac{1}{4\pi}\kappa A - \frac{Q^2}{R} + \frac{Q^2}{r_h}$$

Recognizing the electrical potential evaluated at the horizon as $\phi_Q = \frac{Q}{r_h}$, and taking the limit $R \to \infty$ one arrives at the Smarr relation for the Reissner-Nordstrom BH

$$M = \frac{1}{4\pi}\kappa A + \phi_Q Q \tag{2.29}$$

If one takes the mass parameter M as a function of the area A and the charge Q, i.e., $M = M(A, Q)$, one can obtain a first-order differentials relation as follows

$$dM = \frac{\partial M}{\partial A} dA + \frac{\partial M}{\partial Q} dQ \tag{2.30}$$

For the Reissner-Nordstrom BH, the outermost horizon is located at $r_+ = M + \sqrt{M^2 - Q^2}$. From this, it is pretty easy to verify the equation (2.30) and to find the values for $\frac{\partial M}{\partial A}$ and $\frac{\partial M}{\partial Q}$. However, for BH with a more complicated lapse function solving $f(r_+) = 0$ analytically can be intractable. Luckily, we have generalized Euler theorem. Let W a function such $W(x, y, .., z) \to W(\alpha^p x, \alpha^q y, .., , \alpha^t z) = \alpha^s W(x, y, ..., z)$, the Euler theorem implies

$$sW(x, y, ..., z) = p\left(\frac{\partial W}{\partial x}\right)x + q\left(\frac{\partial W}{\partial y}\right)y + ... + t\left(\frac{\partial W}{\partial z}\right)z \tag{2.31}$$

Using r_+ as enlargement factor, we recognize the following scaling relations $M \sim r_+$, $S \sim r_+^2$ and $Q \sim r_+$, and applying the Euler theorem for Reissner-Nordstrom BH, we obtain

$$M = 2\frac{\partial M}{\partial A}A + \frac{\partial M}{\partial Q}Q \tag{2.32}$$

Comparing equation (2.29) with (2.32), one obtains for the partial derivatives that

$$\frac{\partial M}{\partial A} = \frac{\kappa}{8\pi}, \quad \frac{\partial M}{\partial Q} = \phi_Q \tag{2.33}$$

Therefore, we obtain the following first-order differentials relation among parameters of the black hole

$$dM = \frac{1}{8\pi}\kappa dA + \phi_Q dQ \tag{2.34}$$

This relation is the first law of the Black Hole Mechanics exemplified for the case of Reissner-Nordstrom BH. Combining the Smarr relation with generalized Euler relation is a powerful tool to explicitly avoid finding the lapse function's roots. Having the Smarr relation is equivalent to have the first law of the Black Hole Mechanics. It is crucial noticing that all the results of this section are obtained at the classical level. There are no reasons yet to believe that black holes are thermal systems, even though the equation (2.34) looks the same as the first law of thermodynamics, where the factor $\phi_Q dQ$ has the same structure as the electrical work used in thermodynamics.

2.4 Black Hole As Thermal Quantum System

The idea to assign entropy to a black hole can be traced to 1970 when John Archibald Wheeler and his graduate student Jacob Bekenstein joked about the perfect crime. Imagine someone who let drop a cup of tea into the black hole; if nothing can escape from the inside of a black hole, the universe would lose the entropy of the cup of tea violating the second law of thermodynamics. This anecdote inspired Bekenstein to actually assigning entropy to black holes to avoid such criminal acts [See (14), (15) for the original works and (29) for a historical review of entropy since Carnot to Bekenstein]. Bekenstein connected the entropy with the area of the black hole encouraged by the Hawking area theorem, which states that the black hole area can never decrease, behavior similar to the second law of thermodynamics. However, at that time, there were no physical reasons to associate a temperature to a black hole. At the classical level, the black holes do not emit thermal radiation.

In 1975, in a seminal work, Stephen Hawking revealed the black holes' thermal nature. He computed thermal radiation of the BH using a test scalar field on curved spacetime. Quantum effects are the key to allow a black hole radiates and evaporate losing mass. Some quantum modes fall inside the horizon. Other quantum modes reach infinity, which constitutes the Hawking radiation (16). All of this is possible because the null energy condition is violated at the quantum level [To see the energy conditions go to Appendix D]. This condition is one of the Hawking area theorem's hypotheses. In this way, the first law of the black hole mechanics become precisely in the first law of thermodynamics. The BH area is decreasing during the evaporation, so its entropy is decreasing too. However, the sum of the entropy of the Hawking radiation and the entropy of the black hole itself can never decrease. This idea is the generalized second law (GLS) of black holes thermodynamics (17).

In Appendix G, we will compute the Hawking temperature of an eternal black hole using QFT on curved spacetimes. In this section, in honor of brevity, we will use the Euclidean trick to obtain the Hawking temperature and establish the Hawking-Bekenstein relation for the black hole entropy. To start, we will use another form for the near-horizon metric from equation (B.6). Thus, we have near horizon that

$$ds^2 = -(\kappa r)^2 dt^2 + dr^2 + dL^2$$

Using the Euclidean time $t = it_E$ and ignoring the sector dL^2, one has

$$ds^2 = (\kappa r)^2 dt_E^2 + dr^2$$

$$\Rightarrow ds^2 = dr^2 + r^2 d(\kappa t_E)^2$$

This 2-dimensional line element is simply the polar coordinate system with κt_E acting as the angular coordinate. So, demanding periodicity to avoid a conical singularity, one has $t_E \to t_E + \frac{2\pi}{\kappa}$. We also notice that the Euclidean black holes do not have an interior (30). We exemplify this in the following figure for the Euclidean Schwarzschild black hole

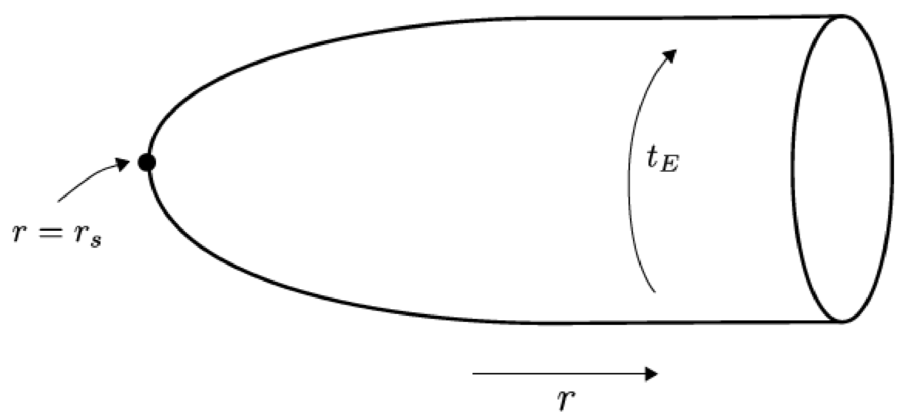

Figure 2.2: The Euclidean Schwarzschild black hole. The Euclidean time and the radial direction have the geometry of a cigar, which is smooth at the tip $r = r_{r_h}$. At each point we also have a sphere of radius r. [Image and caption extracted from (30)]

The quantum system's temperature is the inverse of the period [see (7) for a pedagogical discussion of the Euclidean trick]. From this, we obtain the Hawking relation

$$T_h = \frac{\kappa}{2\pi} \tag{2.35}$$

Given the Hawking temperature, one can fix the proportional constant between the area A and the entropy S of BH, so one arrives at the Bekenstein-Hawking relation

$$S_{bh} = \frac{A}{4} \tag{2.36}$$

In 1993, Robert Wald proved that the black hole entropy is the Noether charge arising from a diffeomorphism invariant Lagrangian. See details in (18).

Using these results in the case of the Reissner-Nordstrom BH, one obtains the Smarr relation and the first law of BH thermodynamics as follows

$$M = 2T_h S + \phi_Q Q \tag{2.37}$$

$$dM = T_h dS + \phi_Q dQ \tag{2.38}$$

The Smarr relation acts as the Euler relation for the standard thermodynamic systems. We must notice that the BH entropy is proportional to area, not to the volume of BH. This feature is distinctive of the BH thermodynamics. Another point is that the Hawking temperature is measured by someone static at infinity. Using a quasi-local analysis (19), one can see that the Hawking temperature, in spherically symmetric configuration, satisfies the Tolman Law (20)

$$T_{proper}(r) = \frac{T_h}{\sqrt{-g_{tt}}}, \quad (r > r_+) \tag{2.39}$$

Thus, for an static observer at infinity in asymptotic flat spacetime both temperatures become equal $T_{proper} = T_h$ [$r \to \infty$ implies $\sqrt{-g_{tt}} \to 1$]. From this, we can see that a static observer just outside of the horizon would measure an unusually high temperature because of here, the red-shift factor $1/\sqrt{-g_{tt}}$ becomes infinity. A static observer just outside the horizon would require a powerfully infinite engine to keep static. This fact is the conceptual reason for such unusual high temperatures.

So far, we have not explicitly given the Hawking temperature for the Reissner-Nordstrom BH. Doing this is quite simple, thanks to our general construction. Using the general result under spherical symmetry of surface gravity from equation (C.7) plus the Hawking relation (2.35), one obtains that

$$T_h = \frac{f'(r_+)}{4\pi} \tag{2.40}$$

Thanks to this equation, it is pretty easy to obtain the Hawking temperature and the proper temperature for any spherical BH. For the Reissner-Nordstrom BH, one has

$$T_h = \frac{\sqrt{M^2 - Q^2}}{2\pi(M + \sqrt{M^2 - Q^2})^2} \tag{2.41}$$

$$T_{proper} = \frac{\sqrt{M^2 - Q^2}}{2\pi(M + \sqrt{M^2 - Q^2})^2} \frac{1}{\sqrt{1 - \frac{2M}{r} + \frac{Q^2}{r^2}}} \tag{2.42}$$

The extreme Reissner-Nordstrom solution is defined when $M = Q$, so one has $T_h = T_{proper} = 0$. It is a cold black hole [its entropy does not vanish]. The Schwarzschild solution is recovered by taking $Q = 0$, so the temperatures become

$$T_h = \frac{1}{8\pi M}$$

$$T_{proper} = \frac{1}{8\pi M} \frac{1}{\sqrt{1 - \frac{2M}{r}}}$$

The heat capacity for the Schwarzschild case is given by

$$C = -8\pi M^2 \Leftrightarrow C = -2S$$

Where we used the Hawking-Bekenstein entropy $S = 4\pi M^2$ and the definition of the heat capacity $C \equiv T\frac{\partial S}{\partial T}$. The Schwarzschild BH is thermodynamically unstable due to its negative heat capacity. Self-gravitational systems with negative heat capacity are known in astrophysics since 1967 (21).

2.4.1 Thermodynamics from Textbook

Black holes thermodynamics have some peculiarities which we must highlight. First, they are highly non-extensive thermodynamic systems. To see this, we remind the first law of the textbook thermodynamics

$$dU = TdS - PdV + \mu dN \tag{2.43}$$

Only in this subsection T will denote the system's temperature, no trace of the energy-momentum tensor. The internal energy U is a function of the extensive variables S, V, and N. An extensive variable is proportional to the "size" of the system. For instance, in the case of internal energy we have

$$U(\alpha S, \alpha V, \alpha N) = \alpha U(S, V, N)$$

On the other hand, the intensive variables T, P, and μ do not change with the enlargement factor α. All extensive variables are homogeneous functions of the first-order. On the contrary, all intensive variables are homogeneous functions of the zero-order. Based on this scaling relation for extensive and intensive variables, one can use the generalized Euler theorem (2.31) and arrive at the Euler formula for textbook thermodynamics

$$U = TS - PV + \mu N \tag{2.44}$$

Taking differential to this equation, one arrives at

$$\Rightarrow dU = SdT + TdS - PdV - VdP + \mu dN + Nd\mu$$

This result does not coincide with the first law of TD (2.43). We need to impose the following constraint called the Gibbs-Duhem relation

$$0 = SdT - VdP + Nd\mu \tag{2.45}$$

This equation must always be fulfilled for textbook thermodynamics and expresses that the intensive variables are not independent among them. Equation (2.44) and equation (2.45) are no longer true for BH thermodynamics. The enlargement factor for BH is the horizon radius r_+, which implies a lack of a clear limitation between extensive and intensive variables. For instance, for the Schwarzschild BH, the entropy scales as $S_{bh} \sim r_+^2$, and the Hawking temperature scales as $T_h \sim r_+^{-1}$. The notion of extensive and intensive variables is lost. The Smarr relation replaces the Euler formula for textbook thermodynamics. However, the Gibbs-Duhem relation is lost for BH thermodynamics (22). The moral is that in undergraduate thermodynamics [TD], it assumes other hypotheses apart from TD's four basic laws. One must be careful about these auxiliary hypotheses because they cannot be true in particular quantum systems such as BH. The lack of extensivity in thermodynamic systems is a hot topic of research; it is not a feature that belongs exclusively to BH. Modern textbook about this topic are (26), (27), (28). In these references, one can find several systems with peculiar behavior due to their lack of extensivity.

Due to the peculiarities of the black holes thermodynamics, many authors do not believe in it. To see an interesting discussion on this debate, look at the works of Wallace (31), (32), and (33). In these papers, he discusses BH thermodynamics, BH Statistical mechanics, and the information paradox. On the other hand, for many theoretical physicists, the Hawking relation (2.35) and Bekenstein-Hawking relation (2.36) are almost experimental facts. Quantum black hole physics is not a complete theory; it has inconsistencies and paradoxes that, to this day, are not resolved. In order to resolve these paradoxes, practically all fields of research in physics have been used. Today, black hole physics encompasses gravity theory, quantum theory, information theory, and condensed matter theory. The physics of quantum black holes has become a fascinating research area of constant exchange of ideas between different areas of physics. We are in exciting times to be a relativist.

Chapter 3

The Pressure and Volume for BH Thermodynamics

In chapter 2, we focused on the origin of the black hole thermodynamics, and we stated the Hawking relation between the temperature T_h and surface gravity κ and the Bekenstein-Hawking relation between the entropy S_{bh} and the BH area A. Now, we will introduce the pressure and volume term for a black hole in two different approaches. In the context of black holes, which are asymptotically AdS, the negative cosmological constant is a natural candidate to be the thermodynamic pressure term (38). In the second approach called Horizon Thermodynamics [HZ] proposed by Padmanabhan in 2002, the Einstein equation is considered like a thermodynamic identity, and any content of matter contribute to the pressure term (44). Incredibly, in both approaches, the thermodynamic volume coincides with the so-called naive geometrical volume, i.e., consider the black holes as a sphere of radius r_h. This result is not expected because the differential volume for spherical symmetry in GR is given by $dV = \sqrt{g_{rr}g_{\theta\theta}g_{\phi\phi}}drd\theta d\phi$, where in general $g_{rr} \neq 1$. Then, $dV \neq r^2 \sin(\theta)drd\theta d\phi$.

3.1 AdS Black Holes Thermodynamics

To exemplify the AdS Black Holes Thermodynamics, we will use the AdS Reissner-Nordstrom BH, which has the following lapse function

$$f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{\Lambda}{3}r^2 \tag{3.1}$$

Where the cosmological constant is negative $\Lambda < 0$. This solution is not asymptotically flat. So, when one computes the Komar integral, this diverges. To see this, we use the equation (2.25)

$$E_k = \frac{R^2}{2}f'(R) \Rightarrow E_k(R) = M - \frac{Q^2}{R} - \frac{\Lambda R^3}{3} \tag{3.2}$$

$$\Rightarrow E_k(\infty) = \infty, \quad (R \to \infty)$$

The divergence of the Komar integral is one of the reasons why the physicists delayed many years to develop the AdS Black Holes Thermodynamics. Another reason that blocked to consider the cosmological constant as a thermodynamic variable is that Λ is a constant. Many physicists could ask themselves: Why does a constant would be a thermodynamic quantity? After all, no one would believe that the speed of light c or the Newtonian constant G could be taken as a thermodynamic quantity. In the later '80s, Teitelboim and York proposed that Λ might be itself a dynamical variable (34), (35), then in 1995 Mann and Creighton incorporated the corresponding thermodynamic variable for Λ, though without given any interpretation to its conjugate variable (36).

3.1.1 Smarr Relation for AdS Reissner-Nordstrom BH

In 2009, Kastor and coworkers found the Smarr Relation for AdS BH using the Hamiltonian formalism for GR (37). In this way, they were able to interpret that Λ was proportional to positive pressure and obtained the thermodynamic volume, which is $V = \frac{4\pi r_+^3}{3}$. This volume coincides with naive geometrical volume that one could expect for a spherical object. Also, in the presence of Λ, the mass M is no longer the internal energy, M becomes the Enthalpy of the system.

We are going to obtain the results aforementioned using the equation (2.28) with Hawking relation (2.35) and Bekenstein-Hawking relation (2.36) already included. Consequently, we have for static spherical symmetry that

$$E_k = 2T_h S - 2\int_{\Sigma_{BH}} d^3\Sigma_t (T^t{}_t - \frac{1}{2}T) \tag{3.3}$$

We remind that Σ_{BH} is just the volume between the outermost horizon and the infinity. T_h is the Hawking temperature and T the trace of the energy-momentum tensor. Also, we are going to use the variable R for the asymptotic zone, and just at the end of the calculation, we will take the limit $R \to \infty$. Additionally, we recall the useful components of the energy-momentum tensor for our computation (2), which are

$$T^{\nu}{}_{\mu(\Lambda)} = \frac{-\Lambda}{8\pi}\delta^{\nu}{}_{\mu}, \quad T_{(\Lambda)} = \frac{-4\Lambda}{8\pi}$$

$$T^t{}_{t(EM)} = -\frac{Q^2}{8\pi r^4}, \quad T_{(EM)} = 0$$

Then, the integral of interest is

$$I_{bh} \equiv -2\int_{\Sigma_{BH}} d^3\Sigma_t (T^t{}_t - \frac{1}{2}T)$$

$$\Rightarrow I_{bh} = -8\pi \int_{r_h}^{R} dr \cdot r^2 (T^t{}_{t(\Lambda)} + T^t{}_{t(EM)} - \frac{1}{2}T_{(\Lambda)})$$

$$\Rightarrow I_{bh} = -8\pi \int_{r_h}^{R} dr \left(\frac{Q^2}{8\pi r^2} - \frac{\Lambda r^2}{8\pi} \right)$$

$$\Rightarrow I_{bh} = -\frac{Q^2}{R} + \frac{Q^2}{r_h} - \frac{\Lambda R^3}{3} + \frac{\Lambda r_h^3}{3}$$

Using this result on RHS of equation (3.3) with the result for the Komar mass (3.2) on the LHS, one arrives at

$$\Rightarrow M - \frac{Q^2}{R} - \frac{\Lambda R^3}{3} = 2T_h S - \frac{Q^2}{R} + \frac{Q^2}{r_h} - \frac{\Lambda R^3}{3} + \frac{\Lambda r_h^3}{3}$$

$$\Rightarrow M = 2T_h S + \frac{Q^2}{r_h} + \frac{\Lambda r_h^3}{3} \tag{3.4}$$

The last expression is independent of R, so taking $R \to \infty$ is harmless even when we are not in asymptotically flat spacetime. This result is gratifying; we are using the same equation of the '80s to compute a new result from 2009. Indeed, when one already knows the answer, it is not difficult to find other ways to perform the same calculation.

In the equation (3.4) one can recognize the electrostatic potential $\frac{Q^2}{r_r} = \phi_Q Q$. Also, one knows that the pressure associated to the cosmological constant is $P_\Lambda = \frac{-\Lambda}{8\pi}$ because it is a perfect fluid with equation of state $\epsilon_\Lambda = -P_\Lambda$. This pressure is positive due to $\Lambda < 0$. So, using these results in (3.4), one obtains

$$\Rightarrow M = 2T_h S + \phi_Q Q - \frac{8\pi r_h^3}{3} P_\Lambda$$

We are going to apply the generalized Euler theorem from the subsection 2.3.2. We need the scaling relation of the thermodynamic quantities and the system's natural size, given by r_+. We can see this scaling relation directly from the lapse function, and they are $M \sim r_+$, $Q \sim r_+$, $S \sim r_+^2$, and $\Lambda \sim r_+^{-2}$. Then, applying the generalized Euler theorem (2.31), one arrives at

$$\Rightarrow M = 2\frac{\partial M}{\partial S} S + \frac{\partial M}{\partial Q} Q - 2\frac{\partial M}{\partial P_\Lambda} P_\Lambda$$

Comparing the last two equations, one finds that

$$\frac{\partial M}{\partial S} = T_h, \quad \frac{\partial M}{\partial Q} = \phi_Q, \quad \frac{\partial M}{\partial P_\Lambda} = \frac{4\pi r_+^3}{3} \tag{3.5}$$

The conjugate variable for the pressure is the volume, and satisfactorily, our volume is $V = \frac{4\pi r_+^3}{3}$. This volume is defined in a thermodynamic way, not in a geometrical way. This distinction is really important, the geometrical measure for spherical symmetry in GR is given by $dV = \sqrt{g_{rr} g_{\theta\theta} g_{\phi\phi}} dr d\theta d\phi$, where $g_{rr} = 1/f(r)$, from this it follows that there is not a natural geometrical definition for the volume of a black hole because $g_{rr} \to \infty$ when $r \to r_+$ [Here I considered the line element given by (A.1)]. This concept of the thermodynamic volume will be pretty useful for later chapters. Finally, the Smarr relation for AdS Reissner-Nordstrom BH is given by

$$M = 2T_h S + \phi_Q Q - 2V P_\Lambda \tag{3.6}$$

Taking $M = M(S, Q, P)$ is easy to obtain the first law of the AdS Black Hole Thermodynamics

$$dM = T_h dS + \phi_Q dQ + V dP_\Lambda \tag{3.7}$$

The Hawking temperature can be calculated using equation (2.40) and is given by

$$T_h = \frac{1}{2\pi r_+^2}\left(M - \frac{Q^2}{r_+} - \frac{\Lambda}{3} r_+^3\right) \tag{3.8}$$

In this way, we have given the expressions for all thermodynamic quantities. We notice that in equation (3.7), one has the term VdP_Λ, not $P_\Lambda dV$. Doing the following Legendre transformation $M = E + P_\Lambda V$ to Smarr relation and the first law, we obtain the following expressions

$$E = 2T_h S - 3P_\Lambda V + \phi_Q Q, \quad E \equiv M - P_\Lambda V \tag{3.9}$$

$$dE = T_h dS - P_\Lambda dV + \phi_Q dQ \tag{3.10}$$

With these expressions, we conclude that the mass of the BH is the Enthalpy of the system $M = H(S, Q, P_\Lambda)$, not the internal energy. Using the equation of state $\epsilon_\Lambda = -P_\Lambda$ for a cosmological fluid, one arrives at

$$E = M + E_\Lambda, \quad E_\Lambda \equiv \epsilon_\Lambda V \tag{3.11}$$

The total internal energy E can be interpreted as the sum of the black hole mass M and the energy associated with the cosmological constant E_Λ. This quantity is the total vacuum energy, and it is negative because of $\Lambda < 0$ [$P_\Lambda > 0$]. For this reason, M is no longer the internal energy (38). At this point, one can wonder why we cannot do the same for the electric charge Q? Well, the term $\phi_Q Q$ is not strange for textbook thermodynamics. Hence, from this perspective, it is unnecessary to pursue this purpose. However, we will discuss this point in section 3.2, under the context of the Horizon Thermodynamics approach.

3.1.2 Heat Capacities and the Volume Term

In the particular case where $Q = 0$, we can compute the heat capacity at constant pressure. To do this, we remind its definition

$$C_P \equiv \frac{T}{\frac{\partial T}{\partial s}|_P} \tag{3.12}$$

The Hawking temperature is a function of M, r_+, and P_Λ, and is given by

$$T_h = \frac{1}{2\pi r_+^2}\left(M - \frac{\Lambda}{3} r_+^3\right) \tag{3.13}$$

We must eliminate the parameter M in this expression to calculate the heat capacity. From the lapse function evaluated at r_+, we have

$$1 - \frac{2M}{r_+} - \frac{\Lambda r_+^2}{2} = 0 \Rightarrow M = \frac{r_+}{2} - \frac{\Lambda r_+^3}{6} \tag{3.14}$$

By combining the last two results, we have the temperature only as a function of Λ, which is the same as a function of the pressure P_Λ

$$T_h = \frac{1}{2\pi r_+}\left(1 - \Lambda r_+^2\right) \tag{3.15}$$

Using the definition of C_P (3.12), straightforwardly, one obtains

$$C_P = -2S \frac{(1 - \Lambda r_+^2)}{(1 + \Lambda r_+^2)} \tag{3.16}$$

Finally, expressing the result in terms of the pressure $P_\Lambda = \frac{-\Lambda}{8\pi}$, one arrives at the desired result

$$C_P = 2S\frac{(8P_\Lambda S + 1)}{(8P_\Lambda S - 1)} \tag{3.17}$$

If $P_\Lambda = 0$, we recover $C = -2S$ for the Schwarzschild case, which implies a thermodynamic instability. However, here if $8P_\Lambda S > 1$, the heat capacity is positive. The inequality in terms of the cosmological constant is $-\Lambda r_+^2 > 1$, which is well-defined thanks to $\Lambda < 0$. The result that black holes can be thermodynamically stable in Anti-de Sitter spacetime is a well-known one (23). The heat capacity C_P diverges when $-\Lambda r_+^2 = 1$, or when $r_+^{(min)} = \sqrt{\frac{1}{|\Lambda|}}$. At this point, the minimum Hawking temperature for which the BH is stable is given by

$$T_{min} = \sqrt{\frac{2P}{\pi}} \tag{3.18}$$

Where we evaluated equation (3.15) at the critical radius $r_+^{(min)} = \sqrt{\frac{1}{|\Lambda|}}$. Below this minimum temperature the black hole is unstable. However, from equation (3.14) we can establish the transition between AdS Schwarzschild and pure AdS spacetime

$$M = \frac{r_+}{2} - \frac{\Lambda r_+^3}{6}, \quad M = 0 \Rightarrow r_+^{HP} = \sqrt{\frac{3}{|\Lambda|}}$$

Evaluating equation (3.15) at r_+^{HP}, we obtain the so-called Hawking-Page temperature T_{HP} (23). This temperature is the critical point where the transition between AdS Schwarzschild and pure AdS spacetime happens, and it is given by

$$T_{HP} = \sqrt{\frac{8P}{3\pi}} \tag{3.19}$$

Nevertheless, we have $T_{HP} > T_{min}$, so we cannot reach the minimum temperature before entering the transition from AdS Schwarzschild to pure AdS spacetime. This result is satisfying for thermodynamic stability.

We can play the same game with heat capacity at constant volume

$$C_V \equiv \frac{T}{\frac{\partial T}{\partial s}|_P} \Longleftrightarrow C_V \equiv T\left.\frac{\partial S}{\partial T}\right|_V \tag{3.20}$$

However, we must notice that both volume $V = \frac{4\pi r_+^3}{3}$ and entropy $S = \pi r_+^2$ are function of r_+ only. Therefore, if we demand constant volume, we will have constant entropy. Accordingly, the heat capacity at constant volume is zero. Moreover, we can write $S = S(V)$ and $V = V(S)$, which implies that they cannot be considered independent thermodynamic quantities. However, this is the result of being in spherical symmetry. In the stationary axisymmetric solutions, these thermodynamic variables become independent (39). For instance, this happens in the case of the AdS Kerr-Newman BH. To sum up, the coincidence between the thermodynamic volume and the naive geometrical volume is a mathematical artifact due to the spherical symmetry.

We must also notice that because we have a pressure and volume term in the AdS Black hole thermodynamics, we can compute the Carnot cycle and study its efficiency, which was made in (40)(41). Besides, one can consider the consequences of this extension of BH thermodynamics in AdS/CFT correspondence and discuss the meaning of the variable Λ. In honor of brevity, we are not going to deal with those remarkable results here. A recent and complete review for AdS Black Hole Thermodynamics and its consequences is (42).

3.2 Horizon Thermodynamics (HT)

So far, we have been working on the standard approach of black hole thermodynamics. Our calculations are based on integral quantities involved in the Smarr relation (3.3). Notably, we used the global conserved Komar mass (2.25) for black holes asymptotically flat. In the cases of black holes that are asymptotically Anti-de Sitter, the Komar mass diverges; however, we still use the Smarr relation, which implies a global notion of spacetime because we are doing an integral between the outermost horizon to infinity. Horizon Thermodynamics (HT) proposed by Padmanabhan in 2002 is based on local physics in opposition to the standard approach.

The key idea of the HT approach is that all horizons must be treated on equal footing because all of them imply a causal disconnection between different patches of the spacetime. Each observer has the right to do physics in her patch, and associate temperature with her notion of horizon (43). The Rindler spacetime has a temperature $T_R = \frac{a}{2\pi}$, where a is the acceleration (24), and AdS/Flat black holes have a temperature $T_R = \frac{\kappa}{2\pi}$ with κ the gravity surface. Even more, the de Sitter spacetime also has a temperature due to its cosmological horizon (25). For the last case, the lapse function is

$$f(r) = 1 - \frac{r^2}{l^2} \tag{3.21}$$

Using the Euclidean trick described in section 2.4 is pretty easy to obtain the corresponding temperature, which is

$$T_{dS} = \frac{l}{2\pi} \tag{3.22}$$

The key point of all this discussion is that we likely live in a de Sitter universe. A black hole in our universe would have the temperature associated with its event horizon besides the cosmological temperature T_{dS} at infinity. Multiple horizons put in a struggle the standard approach of black hole thermodynamics. A local description becomes one desired feature to do BH thermodynamics. We will describe the HT approach only for spherical symmetry without using all its theoretical power. The HT approach is applied to standard general relativity and more general theories, such as Lanczos-Lovelock gravity theories. To see a complete review about this approach see (44)(45). Somehow, the HT approach takes the Einstein equation as a thermodynamic identity. It is not the first time that someone considers the Einstein equation in this way. In 1995, Ted Jacobson "re-derived" the Einstein equation as a thermodynamic equation from the Clausius relation (46).

In the particular case of spherical symmetry, the approach of HT is quite simple. Under this symmetry, the most important result is that we can identify the thermodynamic pressure of black hole as $P \equiv T^r{}_r|_{r_h}$, where r_h represents each horizon or roots of the lapse function. Spherical symmetry allows us to obtain the first law of the HT in a simple way that we are going to replicate from the original references (47)(48) [See also chapter 16 of (1)]. For our purpose, it is enough to consider the outermost radius r_+ and to ignore the other interior horizons. First, from equation (A.7) we have the radial mixed component of the Einstein equation for the case $f = h$ which is

$$G^r{}_r = 8\pi T^r{}_r \Rightarrow \frac{f(r) - 1 + rf'(r)}{r^2} = 8\pi T^r{}_r$$

Evaluating this equation at $r = r_+$ where $f(r_+) = 0$ and taking $T^r{}_r|_{r_+} \equiv P$. Then, multiplying the whole equation with dr_+ we obtain

$$\frac{0 - 1 + r_+ f'(r_+)}{r_+^2} dr_+ = 8\pi P dr_+ \Rightarrow f'(r_+) r_+ dr_+ - dr_+ = 8\pi P r_+^2 dr_+$$

Reorganizing the differential in a suggestive way

$$\Rightarrow \frac{f'(r_+)}{4\pi} d(\pi r_+^2) - d(r_+/2) = P d\left(\frac{4}{3}\pi r_+^3\right)$$

Immediately, we recognize from equation (2.40) that $T_h = \frac{f'(r_+)}{4\pi}$ as result of being working with static spherical solutions. Besides, from the Bekenstein-Hawking relation (2.36), one has that $S = \frac{A}{4} = \pi r_+^2$. The HZ approach assumes the validity of the Euclidean trick (44) as a hypothesis. For this reason, the entropy and temperature do not change with respect to the standard thermodynamics approach. Finally, recognizing the volume as $V = \frac{4}{3}\pi r_+^3$, one obtains

$$\Rightarrow d(r_+/2) = T_h dS - P dV$$

This equation tells us that the internal energy must be $U = \frac{r_+}{2}$. We can interpret this result as the Misner-Sharp mass (49) evaluated at r_+, which is defined as follows

$$f(r) \equiv \left(1 - \frac{2U(r)}{r}\right) \tag{3.23}$$

Thus, when we evaluate this definition at r_+, so that $f(r_+) = 0$, we obtain $U(r_+) = \frac{r_+}{2} \equiv U$. This result justified the previous connection between the internal energy and the Misner-Sharp mass. Therefore, the first law of HT can be summarized as

$$dU = T_h dS - P dV \tag{3.24}$$

$$T_h \equiv \frac{\kappa}{2\pi}, \quad S \equiv \frac{A}{4} \tag{3.25}$$

$$V \equiv \frac{4\pi r_+^3}{3}, \quad P \equiv T^r{}_r|_{r_+}, \quad U \equiv \frac{r_+}{2} \tag{3.26}$$

One more time, one has that the thermodynamic volume coincides with the naive geometrical volume. The Misner-Sharp mass is a quasi-local definition of energy that, in spherical symmetry, is well-established and has useful properties (50). In a vacuum solution, that is to say, for the Schwarzschild BH, the standard BH thermodynamics approach coincides with the HT approach. The pressure vanishes, and the internal energy is $U = M = E_k$.

3.2.1 AdS Reissner-Nordstrom BH in HT approach

We will illustrate the HT approach in the case of AdS Reissner-Nordstrom BH. We recall its lapse function, which is

$$f(r) = \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{\Lambda r^2}{3}\right)$$

The radial mixed component of the energy-momentum tensor is the simple sum of electric field component and the cosmological perfect fluid component ${T^r}_r = -\frac{1}{8\pi}\frac{Q^2}{r^4} - \frac{\Lambda}{8\pi}$. From this expression, we can obtain the total pressure

$$\Rightarrow P = -\frac{1}{8\pi}\frac{Q^2}{r_+^4} - \frac{\Lambda}{8\pi}$$

$$P = P_\Lambda + P_{em}, \quad P_\Lambda \equiv -\frac{\Lambda}{8\pi}, \quad P_{em} \equiv -\frac{1}{8\pi}\frac{Q^2}{r_+^4} \tag{3.27}$$

The pressure term P_Λ is the same as the standard thermodynamics approach in AdS BH. The unusual term is the negative electrical pressure P_{em}. Using the definition of the Misner-Sharp mass (3.23), one obtains

$$U = M - \frac{Q^2}{2r_+} + \frac{\Lambda r_+^3}{6} \tag{3.28}$$

Which of course is equivalent to $U = \frac{r_+}{2}$. It is enough to use the lapse function evaluated at the horizon to see that. For completeness, we recall the Hawking temperature

$$T_h = \frac{1}{2\pi r_+^2}\left(M - \frac{Q^2}{r_+} - \frac{\Lambda}{3}r_+^3\right)$$

These thermodynamic quantities satisfy the first law $dU = TdS - PdV$ with entropy $S = \pi r_+^2$ and volume $V \equiv \frac{4\pi r_+^3}{3}$.

In the standard approach, $E = M + \epsilon_\Lambda V$ is the internal energy for AdS Reissner-Nordstrom BH (3.11), but in this approach, U is the internal energy, and they are not equal. A reasonable question is what is the origin of this difference. The physical insight tells us that U should include electric field energy plus cosmological energy $\epsilon_\Lambda V$. We can write (3.28) for the total internal energy as follows

$$U = M + E_{em} + E_\Lambda \tag{3.29}$$

$$E_{em} \equiv -\frac{Q^2}{2r_+}, \quad E_\Lambda = \epsilon_\Lambda V = -P_\Lambda V \tag{3.30}$$

The electrostatic energy E_{em} and electrical pressure P_{em} are negative quantities. We also notice from the electrical pressure (3.27) that we can establish the following relation

$$P_{em}V = \frac{1}{3}\left(-\frac{Q^2}{2r_+}\right) \Rightarrow P_{em} = \frac{1}{3}\frac{E_{em}}{V} \tag{3.31}$$

This equation is a well-known result of electromagnetic radiation. The pressure is one-third of energy density $\frac{E_{em}}{V}$ (51). We notice that this is the "thermodynamic" energy density that is different from the local energy density $\rho_{em} = -T^t{}_t = \frac{Q^2}{8\pi r^4}$ [See equation (D.12) and last part of Appendix A for quantities in the Local's frame].

We can try to trace the origin of electrostatic energy using the following definition for the electrical energy E^*_{em}

$$E^*_{em} = \frac{1}{2}\int_\Omega \int_{r_+}^{\infty} |\vec{E}|^2 dV$$

This equation is the standard energy for a electrostatic field in natural units. Also, we propose taking $dV = r^2\sin(\theta)drd\phi$ as the differential volume inspired by the thermodynamic volume. This proposal could fail easily because $g_{rr} \neq 1$. However, we will obtain almost the right result. We recall that $T^{t(em)}_{\ \ t} = -\frac{1}{2}|\vec{E}|^2$ (2)

$$\Rightarrow E^*_{em} = 4\pi\int_{r_+}^{\infty}\frac{1}{2}|\vec{E}|^2 r^2 dr \Rightarrow E^*_{em} = -4\pi\int_{r_+}^{\infty} T^{t(em)}_{\ \ t} r^2 dr$$

$$\Rightarrow E^*_{em} = \frac{Q^2}{2}\int_{r_+}^{\infty}\frac{1}{r^2}dr \Rightarrow E^*_{em} = \frac{Q^2}{2r_+}$$

This result has two disturbing things. First, we have made an integral from outermost horizon to infinity that is more related to a global than a local vision of black hole thermodynamics. Second, we have obtained the opposite sign of the electrostatic energy that appears in equation (3.30). If we did the integral from infinity to the event horizon, we would get the correct sign, but it does not seem quite natural to do this discussion.

3.2.2 Equivalence Between both Approaches

Conceptually, the standard approach is different from the HT approach. However, mathematically they are equivalent. We are going to show this in the particular case of AdS Reissner-Nordstrom BH. Taking the differential of U from (3.28)

$$dU = dM - \frac{Q}{r_+}dQ + \frac{Q^2}{2r_+^2}dr_+ + \frac{r_+^3}{6}d\Lambda + \frac{\Lambda r_+^2}{2}dr_+$$

$$\Rightarrow dU = dM - \phi_Q dQ - VdP_\Lambda + \left(\frac{Q^2}{2r_+^2} + \frac{\Lambda r_+^2}{2}\right)dr_+ \tag{3.32}$$

Where we have recognized the electrical potential $\phi_Q = \frac{Q}{r_+}$. Besides, we have obtained the term VdP_Λ from the following consideration

$$\frac{r_+^3}{6}d\Lambda = \frac{-8\pi r_+^2}{6}d\left(\frac{-\Lambda}{8\pi}\right) \Rightarrow \frac{r_+^3}{6}d\Lambda = -VdP_\Lambda$$

Now we reorganize the term PdV from the fist law of HT (3.24), and taking accounts the total pressure $P \equiv P_{em} + P_\Lambda$ from equation (3.27), we obtain that

$$-PdV = \left(\frac{\Lambda}{8\pi} + \frac{Q^2}{r_+^4}\right)(4\pi r_+^2 dr_+) \Rightarrow -PdV = \left(\frac{\Lambda r_+^2}{2} + \frac{Q^2}{2r_+^2}\right) dr_+$$

Using this result in the LHS of the first law of HT (3.24), one has

$$\Rightarrow T_h dS - PdV = T_h dS + \left(\frac{\Lambda r_+^2}{2} + \frac{Q^2}{2r_+^2}\right) dr_+ \tag{3.33}$$

Equating equations (3.32) with (3.33), we recover the standard form of the first law of BH thermodynamics (3.7)

$$dM = T_h dS + \phi_Q dQ + V dP_\Lambda$$

Both approaches are mathematically equivalent, but conceptually they are different. This difference brings consequences when doing thermodynamic processes. In the standard approach, the term $\phi_Q dQ$ will change when the black hole interacts with a charged test particle, whereas it will not change for a neutral particle. The HT approach does not make this difference by having the PdV term instead of $\phi_Q dQ$ for a charged black hole.

3.2.3 The Euler-Smarr Relation for HT

One natural question at this point is, is there a similar relation to Smarr relation for the HT approach? The Smarr relation plays the role of the Euler formula of textbook thermodynamics. Therefore, any consistent proposal of black holes thermodynamics must have some replacement for the Euler formula. Combining the generalized Euler theorem (2.31) with the experiences obtained in the previous sections, we propose the following relation that we will call the "Euler-Smarr relation" for HT approach

$$U = 2T_h S + 3V(-P) \tag{3.34}$$

We have used the following scaling relation, $S \sim r_+^2$ and $V \sim r_+^3$ to propose this relation. The idea behind this proposal is that if we use the Euler theorem, we obtain

$$U = 2\frac{\partial U}{\partial S} S + 3\frac{\partial U}{\partial V} V$$

Comparing the last two equations, we obtain the expected thermodynamic relation

$$\frac{\partial U}{\partial S} = T_h, \quad \frac{\partial U}{\partial V} = -P$$

Then, taking the internal energy as $U = U(T, P)$, we obtain

$$dU = \frac{\partial U}{\partial S} dS + \frac{\partial U}{\partial V} dV$$

Therefore, we recover the first law of the HT approach (3.24). In the case of AdS Reissner-Nordstrom BH, it is pretty quickly verified that the Euler-Smarr relation (3.34) is satisfied. Even more, we can prove its equivalence with standard Smarr relation. For this, we use (3.29) on the RHS of equation (3.34)

$$U = 2T_hS - 3VP \Rightarrow M + E_{em} + E_\Lambda = 2T_hS - 3(P_{em} + P_\Lambda)V$$

Where on the LHS, we have used equation (3.30) to replace U. Now, we use the result for each pressure (3.27) on the RHS, so we obtain

$$\Rightarrow M + E_{em} + E_\Lambda = 2T_hS + 3\epsilon_\Lambda V - 3V\frac{E_{em}}{3V}$$

Using that $E_\Lambda = \epsilon_\Lambda V$, and simplifying terms, we arrive at

$$\Rightarrow M = 2T_hS + 2\epsilon_\Lambda V - 2E_{em}$$

Finally, we recover the standard Smarr relation (3.6) after using the definition of each component of the energy given by (3.30)

$$\Rightarrow M = 2T_hS - 2P_\Lambda V + \phi_Q Q$$

Thus we concluded the proof. The Euler-Smarr relation (3.34) plays the role of Euler formula for textbook thermodynamics in the HT approach. This result is quite useful because we can immediately obtain the first law of the HT approach from (3.34). As far as we know, no other authors have explored the "Smarr relation" in the case of HT. Thus, the Euler-Smarr relation (3.34) would be a contribution of this thesis to the HT approach.

Chapter 4

Thermodynamics of the Graviton Condensate

The purpose of this chapter is to establish the Schwarzschild black hole's thermodynamics, considering it as a Bose-Einstein condensate [BEC] made of gravitons. We will call this proposal the geometrical BEC-BH model. In the first section, we will discuss the motivations and features of the geometrical BEC-BH model. In the second section, the thermodynamics of this model will be presented. As the main results, we obtain a small correction to the Hawking temperature and a negative pressure term associated with graviton condensate. We will also prove how the line element for the geometrical BEC-BH model is formally equivalent to that of the Letelier spacetime, which has an energy-momentum tensor based on a cloud of string surrounding the black hole (67).

4.1 Black Hole as Condensate of Gravitons

In this section, we will discuss the Dvali and Gomez's proposal that black hole physics can be understood as the physics of graviton condensate (52). Then, we will present the Alfaro and coworkers' geometrical model for a BEC of gravitons on the Schwarzschild BH background (55). Finally, we will introduce some improvements to the geometrical BEC-BH model in order to have a more consistent proposal with the black hole physics described in chapter 2. These improvements will allow us to extract the thermodynamics of this model.

4.1.1 Black Hole N-Portrait Proposal

In a series of engaging papers, Dvali and Gomez have proposed that black holes perhaps could be understood as a graviton condensate at the critical point of a quantum phase transition. Somehow, black holes are self-tuned and always stay at the critical point, a distinctive feature of these systems from other quantum systems (54). In this proposal, the Hawking radiation would be explained due to the quantum depletion of the gravitons from condensate. The key idea is that the whole black holes physics can be explained in terms of just one number N, the number of "off-shell" gravitons contained in the BEC (52), (53).

We start with the strength of graviton-graviton interaction which is measured by a dimensionless coupling α as follows

$$\alpha \sim \frac{L_P^2}{\lambda^2} \tag{4.1}$$

Where λ is the characteristic wavelength of the graviton. The Planck length is given by $L_P^2 = \hbar G$ $[c \equiv 1]$. From this equation, we can see that if gravitons' wavelength is large, the interaction among gravitons is extremely weak. Consequently, gravitons behave as free for practical purposes. According to this proposal, the number of gravitons is given by

$$N \sim \frac{M r_g}{\hbar} \tag{4.2}$$

Where r_g is the gravitational radius. For maximal N the wavelength is such that $r_g = \lambda$, then one has that $N\alpha \sim 1$. Black holes must always satisfy this critical condition. They cannot enter into the strong coupling regime $[N\alpha >> 1]$. Using equation (4.2) for the Schwarzschild BH one has $N \sim \sqrt{M}$. Then, we can write

$$\frac{dM}{dt} \approx \frac{1}{\sqrt{N}}\frac{dN}{dt}\sqrt{\frac{\hbar}{8G}}$$

The rate of variation of N can be estimated as follows: The surface $4\pi r_g^2$ times the density of mode n times the velocity $c = 1$. Now, we assume that for a given of r_g, only one mode can get out $[n \sim V^{-1}]$ (55). Then, we arrive at the following expression

$$\Rightarrow \frac{dM}{dt} \approx -\frac{1}{2\sqrt{N}} \cdot (4\pi r_g^2) \cdot 1 \cdot \frac{3}{4\pi r_g^3}$$

Using that $r_g = 2MG$ and equation (4.2), we obtain

$$\Rightarrow \frac{dM}{dt} \approx -\frac{3}{8}\frac{M_P^2}{GM^2} \approx -\frac{M_P^3}{L_P M^2}$$

In the last approximation, we have omitted unimportant numerical factors and used the Planck mass $M_P^2 = \frac{\hbar}{G}$. The Hawking's radiation power formula (8) is

$$\frac{dM}{dt} = -\frac{\Gamma_{GF}}{15360\pi M^2} \Rightarrow \frac{dM}{dt} \sim -T_h^2$$

Where Γ_{GF} is the gray factor, which is described in Appendix G in conjunction with the demonstration of this formula. Comparing the last two results, we recover the Hawking temperature

$$\Rightarrow T_h \approx \frac{1}{M}$$

With this argument, Dvali and Gomez's argues that the Hawking radiation can be understood as the quantum depletion of the BEC (52). Finally, expressing the Hawking temperature and the BH entropy in terms of the number of gravitons N, we obtain

$$T_h \sim \frac{1}{\sqrt{N}}, \quad S_{bh} \sim N \tag{4.3}$$

Using condensed matter arguments, the authors of this proposal have achieved computing the black hole's entropy like the quasi-degenerate nature of the BEC state (54).

The black hole N-Portrait proposal is quite intriguing, but it is not geometrical at all. Also, it is based on a more qualitative analysis than a quantitative one. In the next subsection, we will discuss a more geometrical model inspired by this proposal.

4.1.2 The Geometrical BEC-BH Model

Alfaro and coworkers have proposed a more geometrical model to understand the graviton condensate better (55), (56). In this model, the background spacetime is given, and over it, there is a Bose-Einstein graviton condensate. To achieve this, the metric is split as follows: $g_{\mu\nu} = \tilde{g}_{\mu\nu} + h_{\mu\nu}$. Where $\tilde{g}_{\mu\nu}$ is the background metric and $h_{\mu\nu}$ the quantum fluctuation that will describe the graviton condensate. Throughout this chapter, the background metric will be the Schwarzschild black hole. From now on, we will call this proposal the geometrical BEC-BH model. The number of gravitons is proposed to be built from the quantum fluctuation $h_{\mu\nu}$ as follows

$$N = \int_0^{r_h} \eta dV, \quad \eta \equiv \frac{1}{2r_h} h_{\alpha\beta} h^{\alpha\beta} \tag{4.4}$$

Where η is the number density of gravitons, and the differential volume is taken as $dV = r^2 \sin(\theta) dr d\theta d\phi$. This construction of the total number of gravitons N is not covariant. However, from the discussion of chapter 3, the volume taken is reasonable because dV could be understood as the differential of the thermodynamic volume. In order to respect the underlying symmetry of GR, the simplest form to introduce a term related to the BEC, at the level of action, is

$$S_{BEC} = -\frac{1}{8} \int d^4x \sqrt{-\tilde{g}} \mu(x) h_{\alpha\beta} h^{\alpha\beta} \tag{4.5}$$

Where the scalar field $\mu(x)$ would represent the chemical potential of the BEC. The idea behind this construction is that the Gross–Pitaevskii equation, which describes a BEC under certain conditions (57), is a nonlinear equation such as the Einstein equation. Thus, both equations could be considered as analogous to describe a graviton condensate. Then, it is proposed the following variational principle

$$S = \int d^4x \sqrt{-g} \frac{1}{8\pi} R - \frac{1}{8} \int d^4x \sqrt{-\tilde{g}} \mu(x) h_{\alpha\beta} h^{\alpha\beta} \tag{4.6}$$

The resulting equation of motion from this action is given by

$$G_{\alpha\beta}(\tilde{g} + h) = \Sigma \cdot \mu(x)(h_{\alpha\beta} - h_{\alpha\sigma} h^{\sigma}{}_{\beta}), \quad \Sigma \equiv \sqrt{\frac{-\tilde{g}}{-g}} \tag{4.7}$$

The computation of this variation is given in Appendix E, where we state a more general version of the action (4.5) with future purposes in mind. We must remark that indices of all tensor are raised and lowered with full metric $g_{\mu\nu}$, not with background metric $\tilde{g}_{\mu\nu}$. Besides, the $h_{\mu\nu}$ part of the metric is not a priori a small quantity. We will let the self-consistent computation of the metric to establishes its magnitude. The factor Σ can be absorbed in the definition of $\mu(x)$, so it does not play any role in this model. We also notice that the RHS of equation (4.7) is built from the metric fluctuation, which implies a quite restrictive form for the effective energy-momentum. Hence, the only mathematically consistent solution for the equation (4.7) might be $h_{\nu\mu} = 0$. However, it was found the following line element

$$ds^2_{BEC} = -\frac{1}{(1-B)}\left(1 - \frac{2M}{r}\right)dt^2 + \frac{1}{(1-B)\left(1-\frac{2M}{r}\right)}dr^2 + r^2 d\Omega^2 \tag{4.8}$$

Here B is a constant only defined inside of the BH. The magnitude of B is between 0 to 1, otherwise, it could change the sign between g_{tt} and g_{rr}. We will call this line element, the BEC-Schwarzschild solution, which describes the graviton condensate of the geometrical BEC-BH model. The metric also can be written as follows

$$g_{\alpha\beta} = diag\left(\frac{1}{(1-B)}\tilde{g}_{tt}, \frac{1}{(1-B)}\tilde{g}_{rr}, \tilde{g}_{\phi\phi}, \tilde{g}_{\theta\theta}\right) \tag{4.9}$$

This mathematical structure will be useful to extend this kind of solution to different black holes later. We highlight that only the temporal and radial part of the metric has changed. The fluctuation part of the metric has a simple mathematical structure

$$h^t{}_t = h^r{}_r = B \tag{4.10}$$

Therefore, the nonzero mixed components of $h_{\mu\nu}$ are constant. Thanks to this, we can see from equation (4.4) that the number density of gravitons η is a constant. This result is quite striking and will be useful to extend this construction later. We provide the result for the factor Σ and the scalar field $\mu(x)$, which are

$$\Sigma = (1-B), \quad \mu = -\frac{1}{(1-B)^2 r^2} \tag{4.11}$$

The nonzero mixed components of the Einstein tensor are

$$G^t{}_t = G^r{}_r = -\frac{B}{r^2} \tag{4.12}$$

Therefore, the nonzero mixed components of the effective energy-momentum tensor are given by

$$T^t{}_t = T^r{}_r = -\frac{B}{8\pi r^2} \tag{4.13}$$

In principle, this solution is valid only inside the event horizon, which is still located at $r_h = 2M$. The outside of the black hole is the standard Schwarzschild solution. Thus, the constant B is zero outside of the event horizon. This setting was chosen because if B is different from zero outside of the horizon, the metric would not be asymptotically flat.

Finally, using equation (4.4) we can compute the number of gravitons contained in the BEC

$$N = \frac{4\pi}{3}4M^2B^2 \Rightarrow M \sim \sqrt{N} \tag{4.14}$$

The constant B allows us to explicitly connect the number of gravitons with the mass M of the black hole. Even more remarkable, we can write the Hawking Temperature and the entropy of the black hole in terms of the number of gravitons as follows

$$T_h \sim \frac{1}{\sqrt{N}}, \quad S_{bh} \sim N$$

As we mentioned before, the horizon radius r_h has not changed, so formally, the entropy of the black hole has not changed either. Therefore, the proportional relation between the number of gravitons N and the entropy S is a robust result. On the contrary, it is assumed by hand that Hawking temperature does not change. The lapse function of the metric has changed with respect to the Schwarzschild metric, and with the knowledge built in chapters 2, and 3 of this thesis, it is expected that the Hawking temperature has some correction. Later we will return to this point. We have arrived at the same result of (4.3) with a more geometrical construction than the black hole N-Portrait proposal. The Hawking temperature T_h, the BH entropy S_{bh}, and the BH energy M can be expressed in terms of only one number, the number of gravitons N.

Finally, in the original interpretation of (55), (56), the scalar field μ given by equation (4.11) plays the role of the chemical potential of the condensate. In this way, μ would be a general feature for any black hole and would only be defined in its interior. This interpretation is difficult to sustain when one analyzes it in more detail. We will discuss this point and other possible problems with this model in the next subsection.

4.1.3 The Enhanced Geometrical BEC-BH Model

Expressing the BEC-Schwarzschild solution in the Advanced Eddington-Finkelstein coordinates [EFC], one has

$$ds^2_{BEC} = -\frac{1}{1-B}\left(1 - \frac{2M}{r}\right) du^2 + \frac{2}{1-B} dudr + r^2 d\Omega^2 \tag{4.15}$$

Where the coordinate transformations from equation (2.12) has been used. We notice that we can use the same transformation because the factor $\frac{1}{1-B}$ is shared by g^{BEC}_{tt} and g^{BEC}_{rr} in the line element (4.8). According to the setting described in the previous subsection, B is different from zero in the interior of the black hole only. The standard Schwarzschild solution still describes the outside metric. This setting introduces a discontinuity in the metric precisely at the horizon. From the exterior solution described in the EFC by equation (2.13) we have $g^{out}_{rr} = 2$ at the horizon, but from the interior solution in EFC given by (4.15), one has $g^{BEC}_{rr} = \frac{2}{1-B}$, then $g_{rr}(r_h)^{out} \neq g_{rr}(r_h)^{BEC}$. This result is not acceptable, according to what we have stated in section 2.2. To avoid this problem, we will accept that $B \neq 0$ for the whole spacetime. Therefore, the BEC-Schwarzschild solution is not asymptotically flat. In the context of Loop Quantum Gravity, a quantum correction of the Schwarzschild BH has been proposed (58). This proposal has a similar problem with asymptotically flat behavior (59). Apparently, quantum corrections cannot live just inside the black hole.

Taking $r >>> 2M$ in the BEC-Schwarzschild line element (4.8), one arrives at

$$\Rightarrow ds^2_{BEC} = -\frac{1}{(1-B)} dt^2 + \frac{1}{(1-B)} dr^2 + r^2 d\Omega^2$$

Defining $T = \frac{t}{\sqrt{1-B}}$ and $R = \frac{r}{\sqrt{1-B}}$, one has that

$$\Rightarrow ds^2_{BEC} = -dT^2 + dR^2 + (1-B)R^2 d\Omega^2$$

We could argue that if we ignore the Ω sector, this line element can be considered asymptotically flat [$ds^2_{BEC} = -dT^2 + dR^2$]. In any case, with the Horizon Thermodynamics approach, we will still be able to compute the thermal behavior for the present model. Later, we will discuss that the constant B has an extremely small value so that it would be naturally suppressed in the real universe. Finally, we notice from equation (4.8) that the radial geodesic and the circular geodesic are not affected by B. The constant B plays a role only in geodesics that mix a circular and radial motion, providing a negligible correction.

Accepting that B is defined in the whole spacetime, a natural question is, does the effective energy-momentum tensor used by the geometrical BEC-BH Model satisfy the energy conditions? At the quantum level, the classical energy conditions can be violated, so we will study these conditions here to see what happens in the geometrical BEC-BH model.

Using equation (D.12) from Appendix D combined with equation (4.13), we obtain the local energy density and local pressure for the graviton condensate

$$\rho = \frac{B}{8\pi r^2}, \quad P_r = -\frac{B}{8\pi r^2} \tag{4.16}$$

From the beginning, we assume $B \geq 0$, so the local energy density is positive. Additionally, we have the following relation between the energy density and the radial pressure $P_r = -\rho$. Also, the tangential pressures are null, and the radial pressure is negative.

Concerning the energy conditions described in Appendix D, the effective energy-momentum tensor satisfies the null, the strong, and the dominant energy conditions. More explicitly, we have the following conditions

$$\rho \geq 0, \quad \rho + P_1 = 0, \quad \rho - |P_1| = 0 \tag{4.17}$$

Therefore, we saturate the null and strong energy condition [go to Appendix D to see the details]. From a classical perspective, the BEC-Schwarzschild BH is well-behaved with regards to its effective energy-momentum tensor. The graviton condensate has a quantum origin, it could perfectly have happened that some energy condition was violated, but it was not the case.

In the original interpretation for the geometrical BEC-BH model, the scalar field $\mu(r)$ given by equation (4.11) plays the role of the chemical potential. However, $\mu(r)$ does not have units of energy as any chemical potential. Besides, from equation (4.14), we know that number of particles satisfies $N \sim M^2$. Nevertheless, according to chapters 2 and 3, if there was a chemical potential, this should contribute to the Smarr relation as $2\mu_{ch}N$. This relation implies that the chemical potential, if any, should scale as $\mu_{ch} \sim M^{-1}$. This result can be seen from $2\mu_{ch}N \sim M$, where we are using M as the natural size of the system because we are in the BEC-Schwarzschild BH [$r_h = 2M$]. In conclusion, the scalar field $\mu(r)$ of equation (4.11) is not the black hole's chemical potential.

This last conclusion seems daunting. However, a fundamental idea of the Dvali and Gomez proposal is that the black hole is at the critical point of the quantum phase transition. Furthermore, according to equation (4.1), gravitons act as if they were free quasi-particles. Consequently, we can consider the graviton condensate as if they were an ideal quantum gas where the critical point for a quantum phase transition is defined when $\mu_{ch}^{critical} = 0$ for a BEC [See the textbook about BEC (57) to remind this condition]. Therefore, we do not need a chemical potential for the BEC; what we need is a pressure term associated with the graviton condensate. Thus, we would have a BEC with well-defined P, V, S, T_h, E, and N at the critical point of the quantum phase transition.

4.2 Thermodynamics of the Geometrical BEC-BH model

In this section, we will obtain the thermodynamics of the geometrical BEC-BH model using the HT approach. We will obtain a small correction to the Hawking temperature and negative pressure for the graviton condensate.

We have for our model the following line element

$$ds^2_{BEC} = -\frac{1}{(1-B)}\left(1-\frac{2M}{r}\right)dt^2 + \frac{1}{(1-B)\left(1-\frac{2M}{r}\right)}dr^2 + r^2 d\Omega^2$$

We are in the case where $f \neq h$ for the line element given by (A.1). So, we must modify the mathematical construction for the first law of HT made in section 3.2. Using the radial component of the Einstein equation, we arrive at

$$G^r{}_r = 8\pi T^r{}_r \Rightarrow \frac{f(h-1)+rhf'}{r^2 f} = 8\pi T^r{}_r$$

Where we have used equation (A.7) for $G^r{}_r$. Reorganizing the terms of this expression, we obtain

$$\Rightarrow h - 1 + \frac{h}{f} r f' = 8\pi T^r{}_r r^2$$

We must notice that $\frac{h(r)}{f(r)} = (1-B)^2$. So, it is well-defined to evaluate this fraction at $r = r_+$. Taking $T^r{}_r|_{r_+} \equiv P$, and recalling that $f(r_+) = h(r_+) = 0$, we arrive at

$$\Rightarrow \frac{(1-B)^2}{2} r_+ f'(r_+) - \frac{1}{2} = 4\pi P r_+^2$$

Finally, multiplying the whole equation with dr_+, and reorganizing the differential in a suggesting way, we have the following expression

$$\Rightarrow (1-B)^2 \frac{f'(r_+)}{4\pi} d(\pi r_+^2) - d(r_+/2) = P d\left(\frac{4}{3}\pi r_+^3\right)$$

In this way, we recover the first law of HT, which is given by

$$dU = T_h dS - P dV, \quad S \equiv \pi\frac{A}{4}, \quad V \equiv \frac{4\pi r_+^3}{3}$$

Where the internal energy and the Hawking temperature are given by

$$U \equiv \frac{r_+}{2} \Rightarrow U = M \tag{4.18}$$

$$T_h \equiv \frac{(1-B)^2 f'(r_+)}{2\pi} \Rightarrow T_h = \frac{(1-B)}{8\pi M} \tag{4.19}$$

We use the equation (4.13) to evaluate the pressure as the HT approach stated to do, which is given by

$$P \equiv T^r{}_r|_{r_+} \Rightarrow P = -\frac{B}{32\pi M^2} \tag{4.20}$$

We also notice the following interesting relation

$$P = -\frac{B}{3}\frac{M}{V} \tag{4.21}$$

This equation looks like the relation between pressure and energy density in the case of electromagnetic radiation (3.31). Finally, using equation (4.14) we can express each thermodynamic quantity in terms of N

$$S \sim N, \quad M \sim \sqrt{N} \quad T_h \sim \frac{1}{\sqrt{N}} \tag{4.22}$$

$$V \sim N^{3/2}, \quad P \sim \frac{1}{N} \tag{4.23}$$

In these expressions, we have ignored the proportional constants because they are irrelevant to the discussion. These relations are entirely satisfactory from the point of view of the N-Portrait proposal. Therefore, the black hole thermodynamics can be parameterized by just one number: the number of gravitons of the BEC given by N. We have successfully associated a volume and a pressure with the graviton condensate. We can see that P given by equation (4.20) is negative. Then, we could consider this negative pressure as a tension instead of pressure. Perhaps, this negative pressure does not allow the matter to fall inside of the BH. We remind that the BEC-Schwarzschild black hole is a vacuum solution that considers a quantum fluctuation given by $h_{\mu\nu}$; it does not contain any type of matter inside of it.

4.3 Graviton Condensate Interpretation

In this section, we will discuss the thermodynamic interpretation for the graviton condensate. We will focus on its pressure given by equation (4.20), its Hawking temperature (4.19), and its internal energy (4.18). In the pertinent literature, we have found some support for our results.

4.3.1 The Pressure of Graviton Condensate

It may be impressive to associate a negative pressure with a black hole without matter content. However, several authors have reached similar conclusions in alternative models to traditional black holes. In 2000, Chaplin and Colleagues argued that a black hole would be a quantum phase transition of the vacuum spacetime. According to them, the black holes would have a negative pressure similar to Bose fluid at the critical point (60). They explain, "A balloon with surface tension T filled with gas at pressure P will acquire a radius r satisfying $T = \frac{rP}{2}$. Similarly the black hole with local pressure $P = \frac{-3c^8}{32^4 M^2}$ in proper coordinates inside must have surface tension $T = \frac{-3c^2}{32^2 M}$. This tension is generated by the spacetime itself as it undergoes the transition between its two phases and thus need not be constrained by the properties of any familiar kinds of matter" (60). The pressure term obtained by them is proportional to our negative pressure, but not equal. To introduce a pressure term, they appropriately redefined the cosmological constant, which is only defined inside the black hole in its model.

Another proposal that requires a negative radial pressure is the "gravastars" model developed by Mazur and Mottola (61). The motivation of this idea is to eliminate the central singularity changing the interior black hole solution by another solution similar to de Sitter spacetime. It is not precisely the dS spacetime because the gravastars need an anisotropic pressure to be stable (62). In a recent paper, Brustein and colleagues have argued that in order to avoid the gravitational collapse, a considerable radial negative pressure is necessary (63). In the same article, they proposed a black hole model made of closed interacting strings with state equation $\rho = -P_r$. They called this model the "collapse polymer model" because, according to them, some polymers can have negative pressures, when they are under attractive interactions (64). There are also other recent proposals with negative pressure associated with black holes, such as (65) and (66).

We can conclude that a common characteristic for many alternative models of black holes is the presence of a negative pressure term. Most of them require negative pressure to avoid the central singularity. Other models relate to the negative pressure with some quantum phase transition. However, the BEC-Schwarzschild solution, which describes the graviton condensate, does not avoid the central singularity. Its original motivation never was to solve this problem. After researching the relevant literature, it does not seem so strange to associate negative pressure with the black hole.

4.3.2 The Hawking Temperature of Graviton Condensate

Professor Ashtekar and coworkers have obtained a quantum correction for the Schwarzschild black hole in the context of Loop Quantum Gravity, which affects both its interior and exterior (58), (59). The line element associated with this quantum correction is quite complicated; we will not state it here. The more significant point for our purpose is that they have achieved a quantum correction to the Hawking temperature, which is given by

$$T_h = \frac{1}{8\pi M}\frac{1}{1+e_M}, \quad e_M = \frac{1}{256}\left(\frac{\gamma\Delta^{1/2}}{\sqrt{2\pi}M}\right)^{8/3} \tag{4.24}$$

Where $\Delta \approx 5,17L_p^2$ is the quantum area gap that comes from LQG, and $\gamma \approx 0.2375$ is the Barbero-Immirzi parameter of LQG. Therefore, e_M is only defined in terms of the mass M. From the previous expression, we can make the following Taylor expansion

$$T_h = \frac{1}{8\pi M}\frac{1}{1+e_M} \Rightarrow T_h \approx \frac{(1-e_m)}{8\pi M} \tag{4.25}$$

This expansion is possible because e_M is an extremely small value. For a solar mass black hole, $e_M \approx 10^{-106}$ (59). On the other hand, our correction to the Hawking temperature is

$$T_h^{(BEC)} \approx \frac{(1-B)}{8\pi M}$$

Both corrections of the Hawking temperature have the same mathematical structure. However, they are obtained in a very different way. Even so, we can estimate the possible value of B assuming that $B \approx e_M$. Therefore, for a solar mass black hole, B would be of order $B \approx 10^{-106}$. From this estimation, we can conclude that the parameter B related to the graviton condensate would be naturally suppressed in our universe. Then, B would affect in a negligible way any classical test of GR.

4.3.3 The Internal Energy of Graviton Condensate

In the discussion of section 3.2, we stated that the Misner-Sharp mass is equal to the Komar mass only for the case of Schwarzschild black hole. However, for the BEC-Schwarzschild black hole, we obtained $U_{BEC} = M$. From the point of view of HT, this result looks like a peculiar one. To clarify this point, we will compute the Komar mass for our geometrical BEC-BH model.

We start making the following coordinate transformation $t = \frac{T}{\sqrt{1-B}}$ in the line element of BEC-Schwarzschild black hole (4.8). Then, we obtain

$$ds^2_{BEC} = -(1-B)\left(1-\frac{2M}{r}\right)dT^2 + \frac{1}{(1-B)\left(1-\frac{2M}{r}\right)}dr^2 + r^2 d\Omega^2 \tag{4.26}$$

With this form of the metric, we can use our previous construction in section 2.3 to obtain the Komar mass. The lapse function is

$$f(r) = (1-B)\left(1-\frac{2M}{r}\right) \tag{4.27}$$

Using the Komar mass formula (2.25), we obtain that

$$E_k = \frac{R}{2}f'(R) \Rightarrow E_k = M(1-B) \tag{4.28}$$

We have obtained a finite result, even though our metric is not asymptotically flat. The Komar mass for the graviton condensate is different from the Misner-Sharp mass $U = M$, such as we stated in section 3.2. Both energies satisfy the following relation

$$U = E_k + BM \tag{4.29}$$

Therefore, the internal energy contains the Komar mass plus the additional term aM. This result is quite intriguing in the HT approach. What does the extra factor aM mean in equation (4.29)? Inspired by the discussion of the internal energy U in the case of AdS Reissner-Nordstrom BH made in chapter 3, we propose to compute the following integral

$$E^* \equiv \int_0^{r_h} \rho_{local} dV, \quad \rho_{local} = \frac{B}{8\pi r^2} \tag{4.30}$$

Where ρ_{local} is the local energy density of equation (4.16), and we use the differential "thermodynamic" volume $dV = r^2 sin(\theta) dr d\theta d\phi$. We propose to compute only the interior of the black hole. Thus, we obtain

$$E^* = \frac{a}{2} r_h \Rightarrow E^* = BM \tag{4.31}$$

We obtain the right factor for the energy BM, which appears in equation (4.29). Although we are using an effective energy-momentum tensor, we still obtained that the internal energy is given by M. However, U can be split as $U = E_k + BM$ where E_k is the Komar mass and the term BM that could be interpreted as energy associated with the quantum fluctuation $h_{\nu\mu}$.

4.4 The Formal Equivalence with Letelier Spacetime

The Letelier spacetime is described in Appendix F in conjunction with a study of their thermodynamic behavior. We will refer here only to its final results that in Appendix F are worked in detail. This solution describes a cloud of string surrounding the Schwarzschild black hole (67). The line element is given by equation (F.3), which is

$$ds^2 = -\left(1 - a - \frac{2m}{R}\right) dT^2 + \frac{1}{\left(1 - a - \frac{2m}{R}\right)} dR^2 + R^2 d\Omega^2 \tag{4.32}$$

Here a is an adimensional parameter related to the energy density of the cloud of string, hence, $a > 0$. From equation (F.2), we have the local energy density

$$\rho = -T^t{}_t = \frac{a}{r^2} \tag{4.33}$$

Where the negative sign comes from the coordinate transformations to the local frame described in the second part of Appendix A. Supposedly, m represents the mass of the black hole. To arrive at this conclusion, one takes $a = 0$ and demands to recover the Schwarzschild solution. This conclusion cannot be obtained using the asymptotically flat limit, because a does not disappear in this limit. Therefore, the Letelier spacetime is not asymptotically flat, which introduces an ambiguity to define m as the mass of the BH. Then, we have the right to define $m = M(1 - a)$ and obtain that

$$ds^2 = -\left(1 - a - \frac{2M(1-a)}{R}\right) dT^2 + \frac{1}{\left(1 - a - \frac{2M(1-a)}{R}\right)} dR^2 + R^2 d\Omega^2$$

In this expression, we can still recover the Schwarzschild solution demanding that $a = 0$, where obviously $m = M$. Factorizing the term $(1 - a)$ in this line element, and introducing the following coordinate transformations $R = r$ and $T = \frac{t}{\sqrt{1-a}}$, we obtain

$$ds^2 = -\frac{1}{(1-a)} \left(1 - \frac{2M}{r}\right) dt^2 + \frac{1}{(1-a)\left(1 - \frac{2M}{r}\right)} dr^2 + r^2 d\Omega^2$$

This line element is formally the same as that of the BEC-Schwarzschild BH [See equation (4.8) with $a = B$]. Is this a mere coincidence? In the Letelier spacetime, the energy-momentum tensor is an effective one due to a cloud of string with their respective worksheet. In the graviton condensate, the energy-momentum tensor is built from the quantum fluctuation of the background metric given by $h_{\mu\nu}$. Besides, the Letelier solution comes from of a Nambu-Goto action (F.1). In the graviton condensate, we use the action given by (4.5). They are different actions, even more important: they are entirely different at a conceptual level. Could it be possible that quantum fluctuations of metric are due to a cloud of string? We will leave this possibility for later works.

From Appendix F, we know the thermodynamic quantities for this black hole. The Hawking temperature and BH entropy are given by equation (F.2) and (F.5), respectively. Then, we have

$$S = \frac{4\pi m^2}{(1-a)^2}, \quad T_h = \frac{(1-a)^2}{8\pi m} \tag{4.34}$$

From the HT approach, we also have the volume and pressure (F.11) for the Letelier black hole, which are

$$V = \frac{32\pi m^3}{3(1-a)^3}, \quad P = \frac{-a(1-a)^2}{32\pi m^2} \tag{4.35}$$

Finally, the Komar mass from equation (F.7) and the Misner-Mass sharp from equation (F.10) are

$$E_k = m, \quad U = \frac{m}{(1-a)} \tag{4.36}$$

We notice that in the standard approach to BH thermodynamics, we can write $S = S(E_k)$, where E_k is the Komar mass, because $dE_k = T_h dS$. Therefore

$$\frac{\partial S}{\partial E_k} \equiv \frac{1}{T_h} \tag{4.37}$$

$$\Rightarrow \frac{\partial S}{\partial E_k} = \frac{8\pi m}{(1-a)^2} \Rightarrow T_h = \frac{(1-a)^2}{8\pi m}$$

We have recovered, in the usual thermodynamic way, the temperature of the Letelier BH. However, in the HT approach, we must be more careful with this procedure. In this approach, we have $dU = T_h dS - PdV$, then, we need to write the entropy in terms of U and V, i.e., $S = S(U, V)$, and then, take the partial derivative with respect to U to obtain the temperature. The most recommended way is to use the Euler-Smarr relation of the HT approach, which is given by

$$U = 2T_h S - 3PV$$

Then, we use the generalized Euler theorem (2.31), and thus we get the partial derivatives correctly, just as we have done in Appendix F.

Finally, redefining as before $m = M(1-a)$ and $a = B$ we recover the thermodynamic quantities for the graviton condensate of our geometrical BEC-BH model, which are

$$S = 4\pi M^2, \quad T_h = \frac{1-B}{8\pi M}$$

$$U = M, \quad P = -\frac{B}{32\pi M^2}, \quad V = \frac{32\pi M^3}{3}$$

We also recover the Komar mass for our model, which is given by

$$E_k = M(1 - B)$$

These quantities are the same that we have obtained in section 4.2 and subsection 4.3.3. These results are expected because the Letelier black hole and the BEC-Schwarzschild black hole have formally the same line element. However, these solutions are quite different at a conceptual level. In our model, there is no ambiguity in the definition of the mass of the black hole because the background metric is, by definition, the Schwarzschild black hole. We cannot redefine the mass as we want for the graviton condensate. In future works, we will exploit this formal equivalence more between these black holes to extend our geometrical BEC-BH model. Perhaps, it could be possible to explain the quantum fluctuation $h_{\mu\nu}$ related to the graviton condensate in terms of a quantized cloud of strings in the spacetime. More intriguing yet, it could be possible to obtain the black hole entropy with a quantum computation instead of a thermodynamic way assumed in the Bekenstein-Hawking relation (2.36).

Chapter 5

Thermodynamics of the Kiselev Black Hole

In this chapter, we will study a black hole that can parameterize several well-known spherically symmetric BHs such as dS/AdS/flat Schwarzschild BH, dS/AdS/flat Reissner-Nordstrom BH, and the dS/AdS/flat Letelier BH. This solution is called the Kiselev black hole. Besides, we will study its thermodynamics from the standard perspective described in chapter 2. Finally, we will show a new solution called the BEC-Kiselev BH, extending the analysis carried out in sections 4.2, 4.3 so that the graviton condensate now includes different matter contents.

5.1 Kiselev Black Hole

The multi-components Kiselev BH has the following lapse function

$$f(r) = 1 - \frac{2M}{r} - \sum_i \frac{C_i}{r^{3\omega_i+1}} \tag{5.1}$$

The energy-momentum tensor which generates this solution is given by

$$T^t{}_t = T^r{}_r = \sum_i \rho_i, \quad T^\theta{}_\theta = T^\phi{}_\phi = -\frac{1}{2}\sum_i \rho_i(3\omega_i + 1), \quad \rho_i \equiv \frac{3C_i\omega_i}{8\pi r^{3(\omega_i+1)}} \tag{5.2}$$

Kiselev obtained this solution in 2002 (70). We will call the parameter ω_i the state parameter, and C_i the Kiselev charge. According to Kiselev, when the only nonzero state parameter is taken as $\omega_1 = -2/3$, this solution represents the Schwarzschild black hole surrounded by the quintessence. For this case, the associated lapse function and energy-momentum tensor are

$$\Rightarrow f(r) = 1 - \frac{2M}{2} - C_{(quint)}r$$

$$\Rightarrow T^t{}_t = T^r{}_r = 2T^\phi{}_\phi = 2T^\theta{}_\theta = -\frac{2C_{(quint)}}{8\pi r}$$

The Kiselev BH is a very famous toy model with more than 250 cites. Virtually all publication on this model has preserved the original interpretation given by Kiselev. In 2019 Matt Visser pointed out that this interpretation is inadequate (71). The quintessence is a scalar field that has associated a perfect fluid type energy-momentum tensor (9). However, we can see from equation (5.2) that $T^r{}_r \neq T^\theta{}_\theta$ which implies anisotropic pressure. Nevertheless, we know that a perfect fluid has isotropic pressure. Therefore, for a conservative perspective, the Kiselev BH cannot be related to a quintessence fluid.

If we take $\omega_1 = -\frac{1}{3}$, $\omega_2 = \frac{1}{3}$ and $\omega_3 = -1$ as nonzero state parameters in equation (5.1), we obtain the following lapse function

$$\Rightarrow f(r) = 1 - \frac{2M}{r} - C_{[-1/3]} - \frac{C_{[1/3]}}{r^2} - C_{[-1]}r^2$$

We can define $C_{[-1/3]} \equiv a$, $C_{[1/3]} \equiv -Q^2$, and $C_{[-1]} \equiv \frac{\Lambda}{3}$. Then, we obtain

$$\Rightarrow f(r) = 1 - a - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{\Lambda}{3}r^2$$

This result is the lapse function for the dS/AdS Letelier-Reissner-Nordstrom BH. Marvelous, the Kiselev BH can parameterize the most famous black holes with static spherical symmetry. Of course, this idea works with the energy-momentum tensor too. Using the aforementioned values for ω_i and definitions for C_i in equation (5.2), we obtain the following expressions

$$\Rightarrow T^{t[Let]}_{\ \ t} = T^{r[Let]}_{\ \ r} = -\frac{a}{8\pi r^2}, \quad T^{\theta[Let]}_{\ \ \theta} = T^{\phi[Let]}_{\ \ \phi} = 0$$

$$\Rightarrow T^{t[RN]}_{\ \ t} = T^{r[RN]}_{\ \ r} = -T^{\theta[RN]}_{\ \ \theta} = -T^{\phi[RN]}_{\ \ \phi} = \frac{Q^2}{8\pi r^4}$$

$$\Rightarrow T^{t[\Lambda]}_{\ \ t} = T^{r[\Lambda]}_{\ \ r} = T^{\theta[\Lambda]}_{\ \ \theta} = T^{\phi[\Lambda]}_{\ \ \phi} = -\frac{\Lambda}{8\pi}$$

Where we have written each component of the energy-momentum separately. Taking into account the sum of equation (5.2), we have actually that

$$\Rightarrow T^t{}_t = T^r{}_r = -\frac{a}{8\pi r^2} + \frac{Q^2}{8\pi r^4} - \frac{\Lambda}{8\pi}$$

$$\Rightarrow T^t{}_t = T^r{}_r = -\frac{Q^2}{8\pi r^4} - \frac{\Lambda}{8\pi}$$

We are not going to debate whether or not the Kiselev metric can describe the quintessence. However, we are going to exploit the ability of this metric to parameterize other black holes. In any case, the Kiselev's original interpretation has limited the physical insight in some publications. Several authors have obtained the Kiselev black hole's thermodynamics, but only under the original interpretation given by Kiselev. These authors use only the state parameter $\omega = -\frac{2}{3}$ because, for this parameter, the lapse function has roots that can be computed analytically. A sample of these publications are (72), (73), (74). Other authors have obtained the Kiselev thermodynamics for one arbitrary state parameter ω (75). However, they did not connect the fact that the Kiselev BH can parameterize other BHs. In section 5.2, we will show how to obtain several black holes thermodynamics from the Kiselev BH thermodynamics in one go.

5.1.1 Null Energy Condition for the Kiselev BH

We will study the null energy condition [NEC] only for the single-component Kiselev BH for simplicity. When the state parameter is taken as $\omega \in (-1/3, \infty)$, the Kiselev black hole is asymptotically flat. On the contrary, when we take ω outside the range mentioned above, we expected that the Komar mass is infinity just as the case of AdS Reissner-Nordstrom described in section 3.1. We will use the NEC to limit the range of ω. We only use this condition because this is an essential hypothesis of the Hawking area theorem.

Using equation (D.12) from Appendix D combined with equation (5.2), we obtain the local energy density and local pressures

$$\epsilon = -P_r = -\frac{3C\omega}{8\pi r^{3(\omega+1)}}, \quad P_\phi = P_\theta = -\frac{3C\omega(3\omega+1)}{16\pi r^{3(\omega+1)}} \tag{5.3}$$

The null energy condition (D.7) requires that

$$\epsilon + P_i \geq 0$$

We notice that $\epsilon + P_r = 0$, so this expression does not give us any condition. Therefore, we have only one equation from $\epsilon + P_\phi \geq 0$, which is

$$\Rightarrow -\frac{3C\omega}{8\pi r^{3(\omega+1)}} - \frac{3C\omega(3\omega+1)}{16\pi r^{3(\omega+1)}} \geq 0$$

$$\Rightarrow C\omega(\omega+1) \leq 0 \tag{5.4}$$

For instance, for $\omega = 1/3$, the constant C must be negative to satisfy the inequality (5.4). However, for $\omega = 1/3$, we recover the Reissner-Nordstrom BH with the constant $C = -Q^2$, so this BH satisfies the null energy condition. In the same way, for the Letelier BH, we have $\omega = -1/3$ and $C = a$, which satisfies the null energy condition too. Finally, the AdS/dS saturates this inequality because its state parameter is $\omega = -1$. This analysis illustrates how useful the Kiselev BH is to study different BH solutions in a unified way. More generally, if $C > 0$, the state parameter must belong to $\omega \in [-1, 0]$ to satisfy the NEC. In the case of $C < 0$, we need $\omega \in (\infty - 1, -1] \cup [0, \infty)$ to satisfy the NEC condition.

5.2 Thermodynamics of the Kiselev BH

We start recalling the Komar mass E_k from equation (2.25), which is

$$E_k = \frac{R^2}{2} f'(R)$$

Using the lapse function (5.1) for the Kiselev BH without taking the limit $R \to \infty$ immediately, we obtain

$$E_k = M + \sum_i \frac{C_i}{2}(3\omega_i + 1) R^{-3\omega_i} \tag{5.5}$$

We also recall the general version of the Smarr relation (3.3) used in chapter 3, which is given by

$$E_k = 2T_h S - 2\int_{\Sigma_{BH}} d^3\Sigma_t (T^t{}_t - \frac{1}{2}T)$$

Where we must integrate between the outermost horizon r_+ and infinity. We again perform the calculation without taking the limit $R \to \infty$ immediately. Then, we have

$$\Rightarrow E_k = 2T_h S - 8\pi \int_{r_+}^{R} d^3\Sigma_t (T^t{}_t - \frac{1}{2}T)$$

From the energy-momentum tensor (5.2), we compute its trace, which is

$$\Rightarrow T = 3\sum_i \rho_i (1+\omega_i)$$

Therefore, the expression inside of the integral is given by

$$\Rightarrow T^t{}_t - \frac{1}{2}T = -\frac{1}{2}\sum_i \rho_i(1+3\omega_i)$$

This last expression vanishes for $\omega_i^* = -\frac{1}{3}$. Then, computing the integral, we obtain

$$\Rightarrow E_k = 2T_h S + \sum_i \frac{C_i}{2}(3\omega_i+1)\left(R^{-3\omega_i} - r_+^{-3\omega_i}\right)$$

In this expression, we replace the Komar mass given by equation (5.5). Then, we obtain the following equality

$$\Rightarrow M + \sum_i \frac{C_i}{2}(3\omega_i+1)R^{-3\omega_i} = 2T_h S + \sum_i \frac{C_i}{2}(3\omega_i+1)\left(R^{-3\omega_i} - r_+^{-3\omega_i}\right)$$

Finally, the Smarr relation for the Kiselev black hole is

$$M = 2T_h S - \sum_i \frac{C_i}{2}(3\omega_i+1) r_+^{-3\omega_i} \tag{5.6}$$

This result does not depend on R and is finite regardless of the state parameter ω_i. In the summatory, the term associated with $\omega_i^* = -\frac{1}{3}$ vanishes. Therefore, this result includes the cases for ω_i when Kiselev black hole is not asymptotically flat. We can compute the Hawking temperature using equation (2.40), which results

$$T_h = \frac{f'(r_+)}{4\pi} \Rightarrow T_h = \frac{1}{4\pi}\left(\frac{2M}{r_+^2} + \sum_i \frac{(3\omega_i+1)C_i}{r_+^{3\omega_i+2}}\right) \tag{5.7}$$

We know that entropy scales with the outermost horizon radius as $S \sim r_+^2$. From the lapse function (5.1) we read the scaling relation between C_i and r_+ that is $C_i \sim r_+^{3\omega_i+1}$. Then, using the generalized Euler theorem (2.31), one has that

$$M = 2\frac{\partial M}{\partial S}S + \sum_i (3\omega_i+1)\frac{\partial M}{\partial C_i}C_i \tag{5.8}$$

Comparing equations (5.6) and (5.8), we obtain the partial derivatives

$$\frac{\partial M}{\partial S} = T_h, \quad \frac{\partial M}{\partial C_i} = -\frac{1}{2}r_+^{-3\omega_i} \tag{5.9}$$

The first equality is the expected result, and the second one is the interesting part related to the matter content. We must remark the second equation of (5.9) is not valid for $\omega_i^* = -\frac{1}{3}$ because the second part of equation (5.8) is zero for ω_i^*. Then, the first law of black holes thermodynamics is

$$dM = T_h dS + \sum_i \left(-\frac{1}{2} r_+^{-3\omega_i} \right) dC_i$$

Let be $C_i = C_i(\lambda_i)$, i.e, C is function on some arbitrary parameter λ_i. Then, from the chain rule we have $dC_i = \frac{\partial C_i}{\partial \lambda_i} d\lambda_i$. Finally, the first law of thermodynamics for multi-components Kiselev BH is

$$dM = T_h dS + \sum_i \theta_i d\lambda_i, \quad \theta_i \equiv -\frac{1}{2} r_+^{-3\omega_i} \frac{\partial C_i}{\partial \lambda_i} \tag{5.10}$$

From this equation, we can recover the first law of thermodynamics of several well-known black holes. Even more importantly, we have made a general mathematical construction where each parameter that appears in the lapse function for a black hole can be considered a thermodynamic variable. In fact, we can demand some physical behavior to some fixed parameters (ω_i, C_i), and then rederive a lapse function using equation (5.1). That is to say, reverting the mathematical construction. This result is quite striking in itself.

As we have mentioned before, this solution is even valid for values of ω_i that are not asymptotically flat. Then, we could use some values of ω_i to have a "boundary" at infinity just as the AdS black holes. Besides, we know that AdS BHs are used like "boundaries" in the AdS/CFT correspondence (3). Therefore, it could be possible to explore the Kiselev/CFT correspondence using this section's results. The first attempt to establish this correspond is made in (76). We will leave these speculations about this possible correspondence for future work.

In order to illustrate our result, we take $\omega_1 = \frac{1}{3}$ and $\omega_2 = -1$ as nonzero state parameters. Then, we define as before $C_{[1/3]} \equiv -Q^2$ and $C_{[-1]} \equiv \frac{\Lambda}{3}$. Thus, using the definition of θ_i given by equation (5.10) we obtain that

$$\theta_{[1/3]} = -\frac{1}{2} r_+^{-1} \frac{\partial C_i}{\partial \lambda_i} \Rightarrow \theta_{[1/3]} = -\frac{1}{2} r_+^{-1}(-2Q) \Rightarrow \theta_{[1/3]} = \frac{Q}{r_+}$$

$$\theta_{[-1]} = -\frac{1}{2} r_+^{3} \frac{\partial C_i}{\partial \lambda_i} \Rightarrow \theta_{[1/3]} = -\frac{1}{2} r_+^{3} \left(\frac{1}{3} \right) \Rightarrow \theta_{[1/3]} = -\frac{r_+^3}{6}$$

Here we have taken for $C_{[1/3]}$ that $\lambda_{[1/3]} = Q$, and for $C_{[-1]}$ that $\lambda_{[-1]} = \Lambda$ in order to take the partial derivatives. Inserting these results in equation (5.10), we arrive at

$$\Rightarrow dM = T_h dS + \frac{Q}{r_+} dQ - \frac{r_+^3}{6} d\Lambda$$

$$\Rightarrow dM = T_h dS + \frac{Q}{r_+} dQ + \frac{4\pi r_+^3}{3} d\left(\frac{-\Lambda}{8\pi} \right)$$

$$\Rightarrow dM = T_h dS + \phi_Q dQ + P_\Lambda dV$$

Where we have recognized the electrical potential $\phi_Q = \frac{Q}{r_+}$, and the pressure related to the cosmological constant $P_\Lambda = \frac{-\Lambda}{8\pi}$. This equation is the first law of AdS Black Hole Thermodynamics described in section 3.1 [equation (3.7)]. Finally, we can compute the Hawking temperature using (5.7) in order to complete the illustration. So, we have

$$\Rightarrow T_h = \frac{1}{4\pi}\left(\frac{2M}{r_+^2} - \frac{2Q^2}{r_+^3} - \frac{2\Lambda}{3} r_+\right)$$

$$\Rightarrow T_h = \frac{1}{2\pi r_+^2}\left(M - \frac{Q^2}{r_+} - \frac{\Lambda}{3} r_+^3\right)$$

This result is precisely the Hawking temperature obtained in equation (3.8). As we promised, the Kiselev thermodynamics includes the other black holes' thermodynamics. It is even more. We can have a more general law of black hole thermodynamics as follows

$$dM = T_h dS + \phi_Q dQ + P_\Lambda dV + \theta d\lambda \tag{5.11}$$

Here the function θ and the parameter λ are defined by equation (5.10). Besides, the Hawking temperature for the Kiselev-AdS-Reissner-Nordstrom black hole can be obtained using equation (5.7). In this way, we have extended the usual thermodynamics of black holes. In future works, we will compute cycles from this extended thermodynamics and extract more consequences.

5.3 BEC-Kiselev Black Hole

In this last section, we will generalize the discussion made in chapter 4. This time, the graviton condensate will include matter. To do this, we extend the action given in (4.5) as follows

$$S_{BEC} = -\frac{1}{8}\int d^4x\sqrt{-\tilde{g}}(\nu(x)h^2 + \mu(x)h_{\alpha\beta}h^{\alpha\beta}) \tag{5.12}$$

In this action, we have considered the additional term $\nu(x)h^2$, which allows us to have BEC-solution with matter content. As in subsection 4.1.2 the metric is split $g_{\mu\nu} = \tilde{g}_{\mu\nu} + h_{\mu\nu}$. The indices of all tensor are raised and lowered with the full metric $g_{\mu\nu}$. From this action principle, we obtain the following equation of motion

$$G_{\alpha\beta}(\tilde{g} + h) = \Sigma\left(\frac{1}{2}\nu(x)h^\sigma{}_\sigma\tilde{g}_{\alpha\beta} + \mu(x)(h_{\alpha\beta} - h_{\alpha\sigma}h^\sigma{}_\beta)\right) + T^{matter}_{\alpha\beta} \tag{5.13}$$

Where we have defined $\Sigma \equiv \sqrt{\frac{\tilde{g}}{g}}$ with $\sqrt{-\tilde{g}}$ the determinant of background metric and $\sqrt{-g}$ the determinant of the full metric. The details of the variation of the action are given in Appendix E. The term associated with $\nu(x)$ allows us to have a closed algebraic system to solve the Einstein equation.

Using as "matter" energy-momentum tensor $T^{matter}_{\alpha\beta}$ the equation (5.2), we have found the following line element

$$ds^2_{BEC} = -\frac{1}{(1-B)}\left(1 - \frac{2M}{r} - \frac{C}{r^{3\omega+1}}\right)dt^2 + \frac{1}{(1-B)\left(1 - \frac{2M}{r} - \frac{C}{r^{3\omega+1}}\right)}dr^2 + r^2 d\Omega^2 \tag{5.14}$$

An expression that can be written as

$$g_{\alpha\beta} = diag\left(\frac{1}{(1-B)}\tilde{g}_{tt}, \frac{1}{(1-B)}\tilde{g}_{rr}, \tilde{g}_{\phi\phi}, \tilde{g}_{\theta\theta}\right) \tag{5.15}$$

This equation has the same mathematical structure as (4.9). With this structure we always obtain that $G^t{}_t = G^t{}_t$ and $G^\phi{}_\phi = G^\theta{}_\theta$. Then, we have two effective equations to solve, and this is why we need $\mu(r)$ and $\nu(r)$ to close an algebraic system. These scalar fields are given by

$$\nu(r) = \frac{3}{4(1-B)}\frac{C\omega}{r^{3(\omega+1)}}(3\omega+1) \tag{5.16}$$

$$\mu(r) = \frac{3}{2}\frac{[B(3\omega+1)-3(\omega+1)]C\omega}{(1-B)^2 r^{3+3\omega}} - \frac{1}{(1-B)^2 r^2} \tag{5.17}$$

We do not have any particular interpretation for μ and ν, but they play a fundamental role in obtaining this solution. In this way, we have succeeded to generalize the BEC-Schwarzschild line element (4.8), which describes a graviton condensate without matter content. The line element (5.14) would describe a graviton condensate that is surrounded with different matter contents because, as we have discussed throughout this chapter, the Kiselev black hole can parameterize other black holes which have matter content. This result is the natural extension of the metrics obtained in (56).

We can compute the graviton number recalling from subsection 4.1.2 that

$$N = \int_0^{r_+} \rho dV, \quad \rho = \frac{1}{2r_+}h_{\alpha\beta}h^{\alpha\beta}$$

Therefore, we obtain that the number of gravitons is given by

$$N = \frac{4\pi}{3}(r_+)^2 B^2 \Rightarrow N \sim S \tag{5.18}$$

As before, the number of gravitons is proportional to the entropy. We remark that in the graviton condensate without matter, the horizon radius was $r = 2M$. In this solution, r_+ is not expressible analytically, it must satisfied that $1 - \frac{2M}{r_+} - \frac{C}{r_+^{3\omega+1}} = 0$. We must not forget that r_+ denotes the outermost horizon radius.

We will only give the radial mixed component of the Einstein tensor. The other components are unnecessary for our purposes. Then, we have

$$G^r{}_r = -\frac{B}{r^2} + \frac{3C\omega(1-B)}{r^{3(\omega+1)}} \tag{5.19}$$

Therefore, the effective radial mixed component of the energy-momentum tensor is

$$T^r{}_r = -\frac{B}{8\pi r^2} + \frac{3C\omega(1-B)}{8\pi r^{3(\omega+1)}} \tag{5.20}$$

Evaluating at the horizon, we obtain the thermodynamic pressure used in HT approach, which is

$$P_{tot} = -\frac{B}{8\pi r_+^2} + \frac{3C\omega(1-B)}{8\pi r_+^{3(\omega+1)}} \tag{5.21}$$

We split the total pressure as $P_{tot} \equiv P_{vac} + P_{matt}$. Then, we have

$$P_{vac} \equiv -\frac{B}{8\pi r_+^2}, \quad P_{matt} \equiv \frac{3C\omega(1-B)}{8\pi r_+^{3(\omega+1)}} \tag{5.22}$$

The first term is the pressure related to the vacuum solution, which we denote P_{vac}, and the second term is the pressure related to the matter content P_{matt}. If we take $\omega = 0$, we recover the thermodynamic pressure for graviton condensate without matter (4.20). Using the equation (2.40), we obtain the Hawking temperature

$$T_h = \frac{(1-B)}{4\pi}\left(\frac{2M}{r_+^2} + \frac{(3\omega+1)C}{r_+^{3\omega+2}}\right) \tag{5.23}$$

Therefore, the first law and the Euler-Smarr relation of HT are

$$dU = T_h dS - (P_{vac} + P_{mat})dV \tag{5.24}$$

$$U = 2T_h S - 3(P_{vac} + P_{mat})V \tag{5.25}$$

The entropy and volume have the usual definition: $S = \pi r_+^2$ and $V = \frac{4\pi r_+^3}{3}$. Besides, the internal energy is still $U = \frac{r_+}{2}$. Now we would have a graviton condensate surrounded by matter. In this way, we were able to extend the results obtained in section 4.2. We must emphasize that we cannot use the number of gravitons N to express each thermodynamic quantity due to the presence of matter, as we did in chapter 4. In this sense, the N-portrait proposal of Dvali and Gomez only works for the vacuum solution.

5.3.1 BEC-Reissner-Nordstrom BH and BEC-AdS Schwarzschild BH

We will illustrate our result for the cases of two well-known black holes. We obtain the BEC-Reissner-Nordstrom BH taking $\omega = \frac{1}{3}$ and $C_{[1/3]} = -Q^2$. From equation (5.14), we obtain the line element

$$ds^2_{BEC} = -\frac{1}{(1-B)}\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)dt^2 + \frac{1}{(1-B)\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)}dr^2 + r^2 d\Omega^2 \tag{5.26}$$

Using the equation (5.23), we obtain the Hawking temperature

$$T_h = \frac{(1-B)}{2\pi r_+^2}\left(M - \frac{Q^2}{r_+}\right)$$

We know that for BEC-Reissner-Nordstrom black hole, the outermost horizon is located at $r_+ = M + \sqrt{M^2 - Q^2}$. Therefore, another expression for the temperature is

$$\Rightarrow T_h = \frac{(1-B)\sqrt{M^2 - Q^2}}{2\pi(M + \sqrt{M^2 - Q^2})^2} \tag{5.27}$$

This temperature is almost the same that (2.41) from chapter 2 corrected by a factor $(1-B)$ in a similar line that we obtained in section 4.2. Using the second term of equation (5.22), we obtain the electrical pressure

$$P_{matter} = -\frac{Q(1-B)}{8\pi r_+^4} \tag{5.28}$$

This pressure has almost the same form that the electrical pressure obtained in equation (3.27) in chapter 3. P_{matter} also includes the correction factor $(1-B)$. The volume V, the entropy S, and internal energy U are functions of r_+ only; hence, they do not change with respect to the standard Reissner-Nordstrom BH. In summary, the temperature (5.27) and the pressure (5.28) are for a graviton condensate surrounded by an electric field.

In a similar line, we obtain the BEC-AdS Schwarzschild black hole taking $\omega = -1$ and $C_{[-1]} = \frac{\Lambda}{3}$, which has the following line element

$$ds^2_{BEC} = -\frac{1}{(1-B)}\left(1 - \frac{2M}{r} - \frac{\Lambda}{3}r^2\right)dt^2 + \frac{1}{(1-B)\left(1 - \frac{2M}{r} - \frac{\Lambda}{3}r^2\right)}dr^2 + r^2 d\Omega^2 \tag{5.29}$$

The Hawking temperature and pressure are respectively

$$T_h = \frac{(1-B)}{2\pi r_+^2}\left(M - \frac{\Lambda}{3}r_+^3\right), \quad P_{matter} = -\frac{(1-B)\Lambda}{8\pi} \tag{5.30}$$

Except for the correction $(1-B)$, these quantities are precisely the same obtained in chapter 3, i.e., equations (3.13) and (3.27) respectively. This graviton condensate would be surrounded by a cosmological fluid.

Once again, in this chapter, we have illustrated the powerful mathematical tool that is Kiselev's solution. We can draw general conclusions from various black holes only by using this solution. We have been able to study from energy conditions to the proposal of graviton condensate with different types of matter in one go, thanks to the Kiselev black hole's ability to parameterize other black holes.

Chapter 6

Conclusions

The N-Portrait proposal asserts that black holes physics can be understood in terms of a graviton condensate at the critical point of a quantum phase transition. Besides, it states that each thermodynamic quantity of the graviton condensate can be expressed in terms of the number of gravitons N (52). The N-Portrait analysis made by its authors is more qualitative than quantitative, further, of being not geometrical at all. On the other hand, the geometrical BEC-BH model is a more geometric proposal to the graviton condensate in a more quantitative setting, which we have presented in subsection 4.1.2. If a black hole is a graviton condensate at the critical point $[\mu_{chem} = 0]$, it must have well-established thermodynamic variables. In this thesis, we have calculated a negative pressure for the graviton condensate in conjunction with a correction to the Hawking temperature for a black hole of mass M. These results are

$$T_h = \frac{(1-B)}{8\pi M}, \quad P = -\frac{B}{32\pi M^2}$$

Where B is a small constant associated with the quantum fluctuation $h_{\mu\nu}$ (4.10). In the spirit of the N-Portrait proposal, we have been able to express each thermodynamic variable in terms of the number of gravitons N as follows

$$S \sim N, \quad M \sim \sqrt{N} \quad T_h \sim \frac{1}{\sqrt{N}}, \quad V \sim N^{3/2}, \quad P \sim \frac{1}{N}$$

We have successfully defined the thermodynamic variables of the graviton condensate. We recall that these quantities, under the N-Portrait proposal, are only defined at the critical point where the chemical potential is zero, as is the case with an ideal quantum gas. This fact is possible because the coupling between gravitons is extremely small (52). Finally, we have also established a formal equivalence between the Letelier black hole with the BEC-Schwarzchild line element, which describes the graviton condensate. We hope in future works to be able to exploit this equivalence more and perhaps be able to calculate the entropy of the black hole using the cloud of strings associated with Letelier's solution with mathematical tools from Loop Quantum Gravity or String Theory.

In the second part of this thesis, we have shown how the Kiselev black hole can be used to parameterize other well-known spherically symmetric black holes. Under this premise, we have obtained the Smarr relation and the first law of thermodynamics for an arbitrary state parameter ω_i. These results are

$$M = 2T_h S - \sum_i \frac{C_i}{2}(3\omega_i + 1) r_+^{-3\omega_i}$$

$$dM = T_h dS + \sum_i \theta_i d\lambda_i, \quad \theta_i \equiv -\frac{1}{2} r_+^{-3\omega_i} \frac{\partial C_i}{\partial \lambda_i}$$

With these equations, appropriately choosing the state parameter ω_i and the Kiselev charge C_i, we recover the thermodynamics of the AdS Reissner-Nordstrom black hole. Furthermore, we extended the standard thermodynamics approach by introducing an arbitrary pair (ω_i, C_i) in the first law of the black hole thermodynamics [See equation (5.11)]. In future works, we hope to study thermodynamic cycles with these equations and speculate if it is possible to establish a Kiselev/CFT correspondence.

The third result of this thesis was to extend the geometrical BEC-BH model in order to include different types of matter. We have once again used the Kiselev black hole and its ability to parameterize other solutions. We have obtained the following new solution called the BEC-Kiselev black hole

$$ds^2_{BEC} = -\frac{1}{(1-B)}\left(1 - \frac{2M}{r} - \frac{C}{r^{3\omega+1}}\right)dt^2 + \frac{1}{(1-B)\left(1 - \frac{2M}{r} - \frac{C}{r^{3\omega+1}}\right)}dr^2 + r^2 d\Omega^2$$

By choosing the pair (ω_i, C_i) appropriately, we can obtain BEC-Reissner-Nordstrom black hole and BEC-AdS Schwarzschild black hole [See equations (5.26) and (5.29) respectively]. Finally, the temperature and pressure associated with this solution are

$$T_h = \frac{(1-B)}{4\pi}\left(\frac{2M}{r_+^2} + \frac{(3\omega+1)C}{r_+^{3\omega+2}}\right)$$

$$P_{tot} = P_{vac} + P_{matt}, \quad P_{vac} \equiv -\frac{B}{8\pi r_+^2}, \quad P_{matt} \equiv \frac{3C\omega(1-B)}{8\pi r_+^{3(\omega+1)}}$$

In this way, we can describe a graviton condensate with different matter contents, as we have done in subsection 5.3.1. When we have matter contents, we cannot express all the thermodynamic quantities in terms of the number of gravitons N. However, this number N continues to be proportional to the entropy of the black hole [See equation (5.18)].

Appendix A

Connection, Einstein Tensor, and Local Frame

We write the line element for static spherical symmetry solutions as follows

$$ds^2 = -f(r)dt^2 + \frac{1}{h(r)}dr^2 + r^2 d\Omega^2 \tag{A.1}$$

Here $d\Omega \equiv d\theta^2 + \sin^2(\theta)d\phi^2$ is the unit 2-sphere. The nonzero components of the connection are given by

$$\Gamma^r_{tt} = \frac{1}{2}h(r)f'(r), \quad \Gamma^t_{tr} = \frac{1}{2}\frac{f'(r)}{f(r)}, \tag{A.2}$$

$$\Gamma^r_{rr} = -\frac{1}{2}\frac{h'(r)}{h(r)}, \quad \Gamma^\theta_{r\theta} = \Gamma^\phi_{r\phi} = \frac{1}{r}, \tag{A.3}$$

$$\Gamma^r_{\theta\theta} = -h(r)r, \quad \Gamma^\phi_{\theta\phi} = \frac{\cos(\theta)}{\sin(\theta)} \tag{A.4}$$

$$\Gamma^r_{\phi\phi} = -h(r)r\sin^2(\theta), \quad \Gamma^\theta_{\phi\phi} = -\cos(\theta)\sin(\theta) \tag{A.5}$$

The nonzero mixed components of the Einstein tensor are

$$G^t{}_t = \frac{h - 1 + rh'}{r^2} \tag{A.6}$$

$$G^r{}_r = \frac{f(h-1) + rhf'}{r^2 f} \tag{A.7}$$

$$G^\theta{}_\theta = G^\phi{}_\phi = \frac{2hff'' + h'f'fr - h(f')^2 r + 2h'f^2 + 2hh'f}{4r^2} \tag{A.8}$$

Where we used as notation $f \equiv f(r)$, $h \equiv h(r)$, $f' \equiv \frac{df}{dr}$ and $h' \equiv \frac{dh}{dr}$.

A.1 Local Frame

For any arbitrary metric one can introduce a set of orthonormal units vectors, which is called the orthonormal basis, and fulfills

$$\vec{e}_{\hat{\mu}} \cdot \vec{e}_{\hat{\nu}} = \eta_{\hat{\mu}\hat{\nu}} \tag{A.9}$$

The local components of an arbitrary basis $\vec{e}_\mu$ is given by

$$\vec{e}_\mu = e^{\hat{\mu}}{}_\mu \vec{e}_{\hat{\mu}} \tag{A.10}$$

Then, the metric can be written as

$$g_{\mu\nu} = e^{\hat{\mu}}{}_\mu e^{\hat{\nu}}{}_\nu \eta_{\hat{\mu}\hat{\nu}} \tag{A.11}$$

The hat indices are with respect to the orthonormal basis. The no hat indices are the global indices of the spacetime. To see more details go to sections 4.6-4.8 of (9).

Doing the transformation of g_{tt} to the local frame in a diagonal metric, one has that

$$g_{tt} = e^{\hat{\mu}}{}_t e^{\hat{\nu}}{}_t \Rightarrow g_{tt} = -\left(e^{\hat{t}}{}_t\right)^2 \Rightarrow e^{\hat{t}}{}_t = \sqrt{-g_{tt}}$$

Doing the same for each component of the diagonal metric, we obtain

$$e^{\hat{t}}{}_t = \sqrt{-g_{tt}}, \quad e^{\hat{r}}{}_r = \sqrt{g_{rr}}, \quad e^{\hat{\phi}}{}_\phi = \sqrt{g_{\phi\phi}}, \quad e^{\hat{\theta}}{}_\theta = \sqrt{g_{\theta\theta}} \tag{A.12}$$

We can use these coordinate transformations to obtain the local components of the Einstein tensor using that

$$G^{\hat{\mu}\hat{\nu}} = e^{\hat{\mu}}{}_\mu e^{\hat{\nu}}{}_\nu G^{\mu\nu} \tag{A.13}$$

From this, we obtain for the case of diagonal metric that

$$G^{\hat{t}\hat{t}} = -G^t{}_t, \quad G^{\hat{r}\hat{r}} = G^r{}_r, \quad G^{\hat{\phi}\hat{\phi}} = G^\phi{}_\phi, \quad G^{\hat{\theta}\hat{\theta}} = G^\theta{}_\theta \tag{A.14}$$

Using the Einstein equation (2.6), we obtain the local components of the energy-momentum tensor

$$T^{\hat{t}\hat{t}} = -\frac{G^t{}_t}{8\pi}, \quad T^{\hat{r}\hat{r}} = \frac{G^r{}_r}{8\pi}, \quad T^{\hat{\phi}\hat{\phi}} = \frac{G^\phi{}_\phi}{8\pi}, \quad T^{\hat{\theta}\hat{\theta}} = \frac{G^\theta{}_\theta}{8\pi} \tag{A.15}$$

These results are valid for the line element (A.1), which is diagonal. Notice the negative sign between the time-time local component of the energy-momentum tensor and the time-time mixed component of the Einstein tensor. This negative sign will be necessary for interpreting the energy conditions, which will be given in Appendix D.

Appendix B

Rindler Frame

Let (T, X, Y, Z) be an inertial coordinate system in flat spacetime. The Minkowski line element is

$$ds^2 = -dT^2 + dX^2 + dY^2 + dZ^2 \tag{B.1}$$

The coordinate transformations between the inertial system and the system of the uniformly accelerated observer are given by

$$X = a^{-1}e^{ax}\cosh(at), \quad T = a^{-1}e^{ax}\sinh(at), \quad Y = y, \quad Z = z \tag{B.2}$$

$$\Rightarrow dX^2 - dT^2 = e^{2ax}(dx^2 - dt^2)$$

Where the uniform acceleration of the Rindler observer is a. Besides, we notice that these coordinate transformations only described one-quarter of the total Minkowski spacetime because of $\cosh(at) \geq 1$ $(\forall t)$. Therefore, the coordinates (x, t) describe only the region $X > |T|$. See the diagram B.1. Then, the line element becomes

$$ds^2 = e^{2ax}(-dt^2 + dx^2) + dL^2 \tag{B.3}$$

We have defined the transversal part of this line element as $dL^2 \equiv dy^2 + dx^2$. We also notice that the factor e^{2ax} appears because the coordinate transformations used are nonlinear. We can do a new coordinate transformation for x to obtain an alternative line element for the Rindler frame. Then, we have

$$e^{ax} = 1 + a\bar{x} \Rightarrow dx = -e^{ax}\bar{x}$$

The line element gets the new form

$$ds^2 = -\left(1 + a\bar{x}\right)^2 dt^2 + d\bar{x}^2 + dL^2 \tag{B.4}$$

This form of Rindler frame is fundamental at a conceptual level to understand GR. We remind the line element of the weak gravitational approximation, which is

$$ds^2 = -\left(1 + 2\phi\right) dt^2 + d\bar{x}^2 + dy^2 + dz^2$$

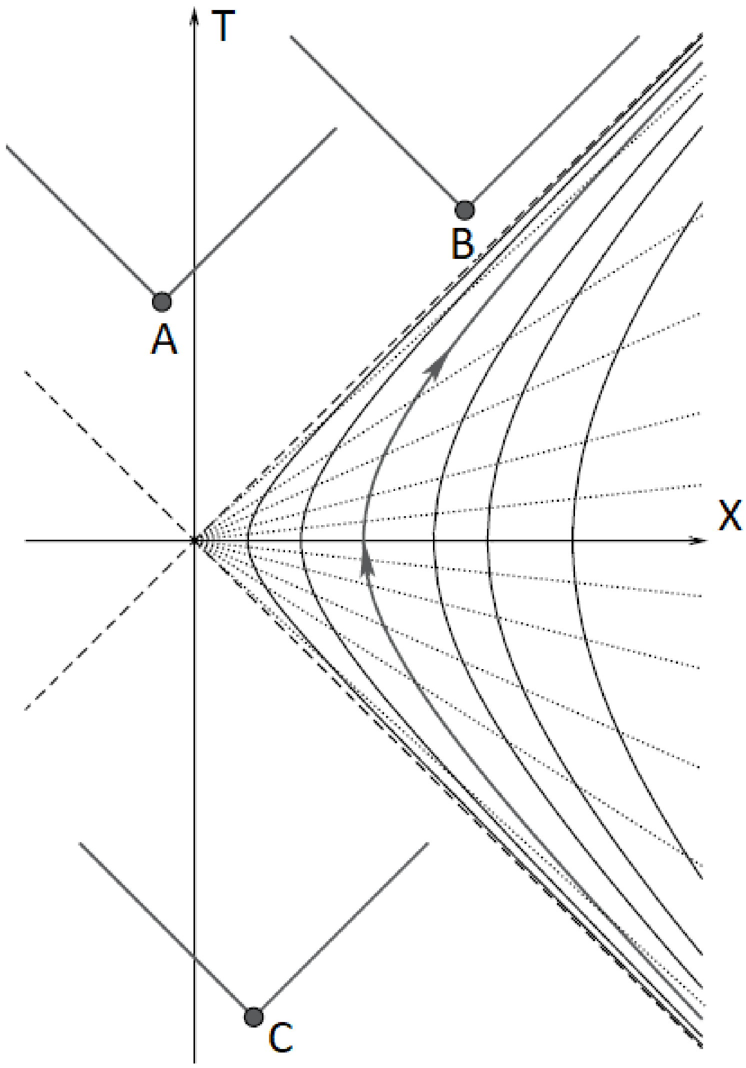

Figure B.1: This diagram shows the region that is described by the coordinate transformations given by (B.2). We notice that the events A, B cannot reach with a signal to an accelerated observer. On the other hand, the event C is not reachable by the accelerated observer. In this way, the lines $X = T$ and $X = -T$ act as a one-way membrane, according to the accelerated observer. We call $X = T$ the future Rindler horizon and $X = -T$ the past Rindler horizon. This diagram is similar to the Kruskal diagram 2.1 showed in section 2.2. [Image extracted from (8)]

The scalar field ϕ is the Newtonian potential. Doing a Taylor expansion in (B.4) $(1 + a\bar{x})^2 \approx 1 + 2a\bar{x}$ for $a\bar{x} << 1$ [In the SI units $\frac{a\bar{x}}{c^2} << 1$]. From this, we recognize that $\phi = a\bar{x}$. Therefore, we arrive at the conclusion that the gravitational field are locally indistinguishable from accelerated frames. Unintentionally we have reached the equivalence principle.

To establish a direct relation between the near-horizon metric of a BH and the Rindler frame for the Minkowski spacetime, we perform the following transformation

$$1 + a\bar{x} = \sqrt{2al} \Rightarrow d\bar{x} = \frac{1}{\sqrt{2al}}dl$$

In this way, we have another alternative form of the Rindler line element

$$ds^2 = -2al dt^2 + \frac{1}{2al} dl^2 + dL^2 \tag{B.5}$$

This line element has the same mathematical structure in the sector $t - l$ that the near-horizon metric of a BH given by equation (2.20). Finally, to reach other useful form related to the Euclidean trick for the Rindler frame, we do the next transformation

$$r = \sqrt{\frac{2l}{a}} \Rightarrow dr = \frac{1}{\sqrt{2al}} dl$$

The line element becomes

$$ds^2 = -(ar)^2 dt^2 + dr^2 + dL^2 \tag{B.6}$$

Just the time part of the metric contains "a special term" different from the standard Minkowski line element. This form is pretty useful to apply the Euclidean trick. The Rindler frame is a powerful tool to develop insight into the nature of spacetime, in a general setting, and the spacetime near the black hole.

Appendix C

Killing Vectors and Surface Gravity

The generally covariant way to determine the symmetries of the metric is through the Lie Derivative

$$\pounds_{\xi} g_{\mu\nu} = \nabla_{\mu}\xi_{\nu} + \nabla_{\nu}\xi_{\mu} = 0 \tag{C.1}$$

Any vector that satisfies this equation is called a Killing vector. If the metric tensor is independent of a particular coordinate, for instance, x^t in a given coordinate system, then $\pounds_{\xi} g_{\mu\nu} = 0$ where the killing vector is simple $\xi^{\mu}_{(t)} = \delta^{\mu}_{t}$. To see this result, we take the covariant derivative, so that

$$\nabla_{\mu}\xi_{\alpha} = \partial_{\mu}\xi_{\alpha} - \Gamma^{\nu}_{\mu\alpha}\xi_{\nu} \Rightarrow \nabla_{\mu}\xi_{\alpha(t)} = -g_{\alpha\beta}\Gamma^{\beta}_{\mu t} = -\Gamma_{\alpha\mu t}$$

Therefore, in this particular case, the Killing equation is

$$\nabla_{\mu}\xi_{\nu(t)} + \nabla_{\nu}\xi_{\mu(t)} = -\left(\Gamma_{\mu\nu t} + \Gamma_{\nu\mu t}\right)$$

We recall that $\partial_{\gamma} g_{\mu\nu} = \Gamma_{\mu\nu\gamma} + \Gamma_{\nu\nu\gamma}$, see, e.g, (2). From this, we obtain

$$\nabla_{\mu}\xi_{\nu(t)} + \nabla_{\nu}\xi_{\mu(t)} = -\partial_t g_{\mu\nu} \Rightarrow \partial_t g_{\mu\nu} = 0$$

This result was calculated in a given coordinate system. However, the Killing equation (C.1) is covariant. Therefore, we know the spacetime itself has a time Killing vector, which satisfies the Killing equation, even when $\partial_t g_{\mu\nu} = 0$ is not satisfied in some other coordinate system. In spherical symmetry, a static spacetime is one that has a time Killing vector or equivalently a spacetime where at least there is one coordinate system where the metric does not depend on the time coordinate.

The equation (C.1) implies immediately that Killing vector fields have zero divergence

$$\nabla_{\beta}\xi^{\beta} = 0 \tag{C.2}$$

Another pretty useful property is

$$\nabla_{\alpha}\nabla^{\alpha}\xi^{\beta} = -R^{\beta}{}_{\mu}\xi^{\mu} \tag{C.3}$$

To prove this equation, we remember that one of the possible definitions of the Riemann tensor is

$$[\nabla_{\alpha}, \nabla_{\beta}]v_{\gamma} = R^{\mu}{}_{\gamma\beta\alpha}v_{\mu}$$

Where v_γ is an arbitrary vector. So, taking $v^\gamma = \xi^\gamma$, we have

$$[\nabla_\alpha, \nabla_\beta]\xi^\gamma = R^{\mu\gamma}{}_{\beta\alpha}\xi_\mu \Rightarrow -\nabla_\alpha\nabla^\alpha\xi_\beta = R^{\mu\gamma}{}_{\beta\gamma}\xi_\mu \Rightarrow \nabla_\alpha\nabla^\alpha\xi^\beta = -R^\beta{}_\mu\xi^\mu$$

In the first equality, we contracted the indices α, γ, and we used equations (C.1) and (C.2) to compute the commutator. In the second equality, we use the definition of the Ricci tensor (2.3), and we raised the index β using the metric.

Now we will focus only on static spherically symmetric spacetimes where we have for the line element given by (A.1) that $\xi^\mu_{(t)} = \delta^\mu_t$. Therefore, the norm of the time Killing vector is

$$\xi^\mu_{(t)}\xi_{\mu(t)} = g_{tt} = -f(r) \tag{C.4}$$

On the horizon, the norm of the time Killing vector vanishes. Outside the horizon, the norm is negative, and inside is positive. These signs have an invariant meaning by construction.

C.1 Surface Gravity

We define the surface gravity κ, on the horizon, following two equivalent ways

$$\xi^\mu\nabla_\mu\xi^\alpha \equiv \kappa\xi^\alpha \tag{C.5}$$

$$\kappa^2 \equiv -\frac{1}{2}\nabla^\mu\xi^\nu\nabla_\mu\xi_\nu \tag{C.6}$$

It is not difficult to demonstrate this equivalence, but it is a long calculation. We omit it. We will compute the surface gravity for the line element given by (A.1). Then, we have

$$\nabla_\mu\xi^\alpha = \partial_\mu\xi^\alpha + \Gamma^\alpha_{\mu\nu}\xi^\nu \Rightarrow \nabla_\mu\xi^\alpha_{(t)} = \Gamma^\alpha_{\mu t}$$

Using the components of the affine connection given in equation (A.2) in equation (C.6), we obtain

$$\kappa^2 = -\frac{1}{2}g^{\mu\nu}g_{\alpha\beta}\Gamma^\alpha_{\mu t}\Gamma^\beta_{\nu t} \Rightarrow \kappa^2 = -\frac{1}{2}\left(g^{tt}g_{rr}\left(\Gamma^r_{tt}\right)^2 + g^{rr}g_{tt}\left(\Gamma^t_{rt}\right)^2\right)$$

Taking particularly $f = h$ in the line element (A.1), one obtains

$$\kappa = \frac{1}{2}f'(r_+) \tag{C.7}$$

The derivative of the lapse function is evaluated at the outermost horizon because the other possible interior horizons are causally disconnected from the outside spacetime. The gravity surface is physically related to the force on a massless (unphysical) string at infinity, see, for instance, (4). Another interpretation of κ is the acceleration of a static particle near the horizon as measured at spatial infinity (5). The basis for these interpretations is that a static observer is not in free fall, so roughly speaking, a static observer needs a motor to keep her static position.

Appendix D

Energy Conditions

This appendix will be based on the textbook called A relativist's toolkit written by Poisson (6). Energy conditions are essential for two central purposes. First, to show some powerful theorems in GR, such as the Hawking area theorem, which requires the null energy condition. Second, the geometrical side of the Einstein equation $G_{\mu\nu}$ can be computed for an arbitrary metric $g_{\mu\nu}$ as a mathematical exercise regardless of any physical content. Hence, we can propose any metric, then compute $G_{\mu\nu}$ and equate with $G_{\mu\nu} = 8\pi T_{\mu\nu}$. Thus we can define an energy-momentum tensor with this procedure. Doing this, no one can assure that the energy-momentum tensor found is well-founded in physical terms. The energy conditions give us some physical criteria to discard nonphysical $T_{\mu\nu}$. Here, we are going to give only the classical energy conditions, no quantum ones. The latter are currently hot topics of research.

To start, we expand the energy-momentum tensor in the local frame as follows

$$T^{\hat{\alpha}\hat{\beta}} = \rho e^{\hat{\alpha}}{}_{t} e^{\hat{\beta}}{}_{t} + P_r e^{\hat{\alpha}}{}_{r} e^{\hat{\beta}}{}_{r} + P_\phi e^{\hat{\alpha}}{}_{\phi} e^{\hat{\beta}}{}_{\phi} + P_\theta e^{\hat{\alpha}}{}_{\theta} e^{\hat{\beta}}{}_{\theta} \tag{D.1}$$

We will require a future-directed timelike vector v^α, which can be decomposed as

$$v^\alpha = \gamma \left(e^{\hat{\alpha}}{}_{t} + a e^{\hat{\alpha}}{}_{r} + b e^{\hat{\alpha}}{}_{\phi} + c e^{\hat{\alpha}}{}_{\theta} \right), \quad \gamma = (\sqrt{1 - a^2 - b^2 - c^2})^{-1} \tag{D.2}$$

Here a, b, c are arbitrary functions. Also, we will require a future-directed null like vector k^α, which can be decomposed as

$$k^\alpha = \left(e^{\hat{\alpha}}{}_{t} + f e^{\hat{\alpha}}{}_{r} + g e^{\hat{\alpha}}{}_{\phi} + h e^{\hat{\alpha}}{}_{\theta} \right), \quad 1 = f^2 + g^2 + h^2 \tag{D.3}$$

We recall that the normalization of a null vector is always arbitrary.

Weak Energy Condition:

An observer with four-velocity v^α will measure the energy density as $T_{\mu\nu} v^\mu v^\nu$. Therefore, we demand

$$T_{\mu\nu} v^\mu v^\nu \geq 0 \tag{D.4}$$

From the local energy-momentum tensor (D.1) and equation (D.2), we obtain from the previous condition that

$$\rho + a^2 P_r + b^2 P_\phi + c^2 P_\theta \geq 0$$

Since a, b, c are arbitrary functions, we set $a = b = c = 0$, so we get $\rho \geq 0$. Now, we only demand $b = c = 0$, then $\rho + a^2 P_r \geq 0$, but from the normalization of v^α (D.2) we know that $1 > a^2$, therefore, $\rho + P_r > 0$. We can do the same for the other pressures to obtain similar inequalities. To sum up, we obtain the following two conditions

$$\rho \geq 0, \quad \rho + P_i > 0 \tag{D.5}$$

Null Energy Condition:

The null energy condition is pretty similar to the weak one, except we use k^α instead of v^α. Then, we demand the condition

$$T_{\mu\nu} k^\mu k^\nu \geq 0 \tag{D.6}$$

From the local energy-momentum tensor (D.1) and equation (D.3), one obtains

$$\rho + f^2 P_r + g^2 P_\phi + h^2 P_\theta \geq 0$$

We cannot demand simultaneously $f = g = h = 0$ due to the normalization condition over k^α (D.3). Choosing $g = h = 0$ implies $a = 1$, so we have $\rho + P_r \geq 0$. The other pressures hold similar inequalities. To sum up, we have for the null energy condition that

$$\rho + P_i \geq 0 \tag{D.7}$$

The weak energy condition implies the null energy condition.

Strong Energy Condition:

For the strong energy condition, we demand that the energy-momentum tensor satisfies

$$\left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T\right) v^\mu v^\nu \geq 0 \tag{D.8}$$

This condition is actually a condition for the Ricci tensor. We see this from the alternative Einstein equation (2.9), so $R_{\mu\nu} v^\mu v^\nu \geq 0$. In this sense, this is a mathematical condition over the Ricci tensor, not a physical one. With the same philosophy of the weak and null energy conditions, it is possible to show that the strong energy condition implies the following restrictions

$$\rho + P_r + P_\phi + P_\theta \geq 0, \quad \rho + P_i \geq 0 \tag{D.9}$$

Where i can be (r, ϕ, θ). The strong energy condition does not imply the weak one.

Dominant Energy Condition :

The physical insight tells us that the matter should flow along timelike or null like world lines. Mathematically, we impose that

$$-T^\alpha_\beta v^\beta \geq 0 \tag{D.10}$$

This quantity must be future-directed and represents the density momentum of matter as measured by a local observer with four-velocity v^β. Doing the same procedure of weak and null energy condition, one can prove

$$\rho \geq 0, \quad \rho \geq |P_i| \tag{D.11}$$

Where the index i can be taken from (r, ϕ, θ).

The weak, null, and dominant energy conditions are based on the physical requirements of the local behavior of matter. In comparison, the strong energy condition is based on a mathematical need for specific proofs of GR. These conditions hold classically; in the quantum realm, they can be violated. The most notable example of this is the Hawking radiation, which decreases the black hole area. Here, the null energy condition, which is a hypothesis of the area theorem is violated at the quantum level. Then, one must demand that the total entropy, the sum of S_{bh} and S_{matter}, does not decrease, so there is not a problem with the decreasing of the BH entropy while the entropy of surrounding increases enough to hold the generalized second law of BH thermodynamics (17).

Finally, we use from Appendix A equation (A.15) combining with (D.1) to obtain that

$$\rho = -\frac{G^t{}_t}{8\pi}, \quad P_r = \frac{G^r{}_r}{8\pi}, \quad P_\phi = \frac{G^\phi{}_\phi}{8\pi}, \quad P_\theta = \frac{G^\theta{}_\theta}{8\pi} \tag{D.12}$$

Thanks to this result, it is straightforward to study the energy conditions for a diagonal metric.

Appendix E

BEC Energy-Momentum Tensor

We propose the following action principle to obtain the energy-momentum tensor associated with the BEC solutions

$$S_{BEC} = -\frac{1}{8}\int d^4x\sqrt{-\tilde{g}}h_{\alpha\beta}h_{\lambda\sigma}U^{\alpha\beta\lambda\sigma} \tag{E.1}$$

Where we have separated the full metric $g_{\alpha\beta}$ in the following way: $g_{\alpha\beta} = \tilde{g}_{\alpha\beta} + h_{\alpha\beta}$. Here $\tilde{g}_{\alpha\beta}$ is the background metric and $h_{\alpha\beta}$ is a correction to background metric which is coupled with the BEC tensor $U^{\alpha\beta\lambda\sigma}$. We assume that this tensor can only depend on the full metric, not in any power of derivative of $g_{\alpha\beta}$. With this assumption the more general way for $U^{\alpha\beta\lambda\sigma}$ is

$$U^{\alpha\beta\lambda\sigma} = \nu(x)g^{\alpha\beta}g^{\lambda\sigma} + \mu(x)[g^{\alpha\lambda}g^{\beta\sigma} + g^{\alpha\sigma}g^{\beta\lambda}] \tag{E.2}$$

Here $\mu(x)$ and $\nu(x)$ are scalar functions of spacetime. The action takes the following form

$$S_{BEC} = -\frac{1}{8}\int d^4x\sqrt{-\tilde{g}}\left(\nu(x)g^{\alpha\beta}g^{\lambda\sigma}h_{\alpha\beta}h_{\lambda\sigma} + \mu(x)h_{\alpha\beta}h_{\lambda\sigma}[g^{\alpha\lambda}g^{\beta\sigma} + g^{\alpha\sigma}g^{\beta\lambda}]\right)$$

We notice that $dx^4\sqrt{-\tilde{g}}$ is the measure of the background metric. Also, we assume that the background metric is fixed $\delta\tilde{g}_{\alpha\beta} = 0$, which implies that $\delta g_{\alpha\beta} = \delta h_{\alpha\beta}$. We remark that the indices of all tensors are raised and lowered with $g_{\alpha\beta}$. Particularly, we have $h^{\alpha}{}_{\beta} = g^{\alpha\omega}h_{\omega\beta}$ and $h^{\alpha\beta} = g^{\alpha\omega}g^{\beta\lambda}h_{\omega\lambda}$.

For simplicity, we will split the total Lagrangian density as follows

$$\mathcal{L}_1 = -\frac{1}{8}\nu(x)g^{\alpha\beta}g^{\lambda\sigma}h_{\alpha\beta}h_{\lambda\sigma} \tag{E.3}$$

$$\mathcal{L}_2 = -\frac{1}{8}\mu(x)h_{\alpha\beta}h_{\lambda\sigma}[g^{\alpha\lambda}g^{\beta\sigma} + g^{\alpha\sigma}g^{\beta\lambda}] \tag{E.4}$$

Although it is true that the action proposed conceptually is a quantum correction, we can still consider it as an effective matter action and use equation (2.8) to obtain an effective energy-momentum tensor. Then, we recall its definition

$$T_{\mu\nu} = \frac{-2}{\sqrt{-g}}\frac{\delta(\sqrt{-\tilde{g}}\mathcal{L})}{\delta h^{\mu\nu}}$$

Where we already have considered that the background metric is fixed. Besides, we recall the following result (2)

$$\delta g^{\mu\nu} = -g^{\mu\gamma} g^{\nu\epsilon} \delta g_{\gamma\epsilon}$$

Calculating the variation of the first Lagrangian density

$$\delta(\sqrt{-\tilde{g}}\mathcal{L}_1) = -\frac{1}{8}\sqrt{-\tilde{g}}\nu(x)((\delta g^{\alpha\beta} g^{\lambda\sigma} + g^{\alpha\beta}\delta g^{\lambda\sigma})h_{\alpha\beta}h_{\lambda\sigma} + g^{\alpha\beta}g^{\lambda\sigma}(\delta h_{\alpha\beta}h_{\lambda\sigma} + h_{\alpha\beta}\delta h_{\lambda\sigma}))$$

$$\Rightarrow \delta(\sqrt{-\tilde{g}}\mathcal{L}_1) = -\frac{1}{4}\sqrt{-\tilde{g}}\nu(x)h^{\sigma}{}_{\sigma}(-h_{\alpha\beta}g^{\alpha\gamma}g^{\beta\epsilon}\delta g_{\gamma\epsilon} + g^{\alpha\beta}\delta h_{\alpha\beta})$$

$$\Rightarrow \delta(\sqrt{-\tilde{g}}\mathcal{L}_1) = -\frac{1}{4}\nu(x)h^{\sigma}{}_{\sigma}(g^{\alpha\beta} - h^{\alpha\beta})\delta h_{\alpha\beta}\sqrt{-\tilde{g}}$$

Where we have used $\delta g_{\mu\nu} = \delta h_{\mu\nu}$ in the last step. Hence, we obtain

$$\Rightarrow T_1^{\alpha\beta} = \frac{1}{2}\sqrt{\frac{-\tilde{g}}{-g}}\nu(x)h^{\sigma}{}_{\sigma}(g^{\alpha\beta} - h^{\alpha\beta})$$

However, we know that $h^{\alpha\beta} = g^{\alpha\omega}g^{\beta\lambda}h_{\omega\lambda}$. Then, we obtain

$$\Rightarrow T^1_{\alpha\beta} = \frac{1}{2}\nu(x)h^{\sigma}{}_{\sigma}\tilde{g}_{\alpha\beta}\Sigma \tag{E.5}$$

Where we have defined $\Sigma \equiv \sqrt{\frac{-\tilde{g}}{-g}}$. For the second Lagrangian density $\mathcal{L}_2$, it is better to do some simplification before

$$\mathcal{L}_2 = -\frac{1}{8}\mu(x)(h_{\alpha\beta}h_{\lambda\sigma}g^{\alpha\lambda}g^{\beta\sigma} + h_{\alpha\beta}h_{\lambda\sigma}g^{\alpha\sigma}g^{\beta\lambda})$$

$$\Rightarrow \mathcal{L}_2 = -\frac{1}{4}\mu(x)h_{\alpha\beta}h_{\lambda\sigma}g^{\alpha\sigma}g^{\beta\lambda}$$

Calculating the variation, one has

$$\delta(\sqrt{-\tilde{g}}\mathcal{L}_2) = -\frac{1}{2}\sqrt{-\tilde{g}}\mu(x)(h_{\lambda\sigma}g^{\alpha\sigma}g^{\beta\lambda}\delta h_{\alpha\beta} + h_{\alpha\beta}h_{\lambda\sigma}g^{\alpha\sigma}\delta g^{\beta\lambda})$$

$$\Rightarrow (\sqrt{-\tilde{g}}\mathcal{L}_2) = -\frac{1}{2}\sqrt{-\tilde{g}}\mu(x)(h^{\alpha\beta} - h^{\alpha}{}_{\sigma}h^{\sigma\beta})\delta h_{\alpha\beta}$$

Therefore, the second contribution to the effective energy-momentum tensor is

$$\Rightarrow T_2^{\alpha\beta} = \Sigma\mu(x)(h^{\alpha\beta} - h^{\alpha}{}_{\sigma}h^{\sigma\beta}) \tag{E.6}$$

The total effective energy-momentum is given by

$$T^{(BEC)}_{\alpha\beta} = \Sigma\left(\frac{1}{2}\nu(x)h^{\sigma}{}_{\sigma}\tilde{g}_{\alpha\beta} + \mu(x)(h_{\alpha\beta} - h_{\alpha\sigma}h^{\sigma}{}_{\beta})\right) \tag{E.7}$$

The Einstein equation becomes

$$G_{\alpha\beta}(\tilde{g} + h) = \Sigma\left(\frac{1}{2}\nu(x)h^{\sigma}{}_{\sigma}\tilde{g}_{\alpha\beta} + \mu(x)(h_{\alpha\beta} - h_{\alpha\sigma}h^{\sigma}{}_{\beta})\right) + T^{matter}_{\alpha\beta} \tag{E.8}$$

Where $T_{\alpha\beta}^{matter}$ represents any other contribution to the energy-momentum tensor aside, which we have calculated. A more compact action to obtain the effective energy-momentum tensor is given by

$$S_{BEC} = -\frac{1}{8}\int d^4x\sqrt{-\tilde{g}}(\nu(x)h^2 + \mu(x)h_{\alpha\beta}h^{\alpha\beta})$$

Where $h \equiv h^{\beta}{}_{\beta}$. This result is obtained after doing all the indices contractions. In this way, the action S_{BEC} contains all possible scalars built with $h_{\alpha\beta}$ up to quadratic power.

Appendix F

Letelier Spacetime

We will compute the spacetime of a black hole surrounded by a cloud of strings based on (68). We start considering a moving infinitesimally thin string that traces out a two-dimensional world sheet Σ, which is parameterized as follows

$$x^{\mu} = x^{\mu}(\lambda^{a}), \quad a = 0, 1$$

Where λ^0 is a time-like parameter and λ^1 a space-like parameter. Assuming the action depends only on λ^0 and λ^1, the Nambu-Goto action is proportional to the area of the worldsheet expanded by the string motion. Therefore, we have

$$S_{NG} = c \int_{\Sigma} \sqrt{-\gamma} d\lambda^0 d\lambda^1 \tag{F.1}$$

Where c is a positive constant related to the string tension, and γ is the determinant of the induced metric, which is given by

$$\gamma_{ab} = g_{\mu\nu} \frac{dx^{\mu}}{d\lambda^{a}} \frac{dx^{\nu}}{d\lambda^{b}}$$

We define the bi-vector $\Sigma^{\mu\nu}$ as follows

$$\Sigma^{\mu\nu} \equiv \epsilon^{ab} \frac{dx^{\mu}}{d\lambda^{a}} \frac{dx^{\nu}}{d\lambda^{b}}$$

Where ϵ^{ab} is the two-dimensional Levi-Civita symbol. Using this definition in the Nambu-Goto action, we obtain

$$S_{NG} = c \int_{\Sigma} \sqrt{-\frac{1}{2} \Sigma_{\mu\nu} \Sigma^{\mu\nu}} d\lambda^0 d\lambda^1$$

Using equation (2.8) we obtain the energy-momentum tensor

$$T^{\mu\nu}_{(string)} = c \frac{\Sigma^{\mu\sigma} \Sigma_{\sigma}^{\ \nu}}{\sqrt{-\gamma}}$$

This result is the energy-momentum tensor for one string. For a cloud of string, we use $T^{\mu\nu} \equiv \rho T^{\mu\nu}_{(string)}$, where ρ is the number density of the string cloud.

We will now consider a static spherically symmetric solution. The non-null components of the bi-vector are $\Sigma^{RT} = -\Sigma^{TR}$, so we have

$$T^T{}_T = T^R{}_R = c\rho\frac{\Sigma^{TR}\Sigma_{TR}}{\sqrt{-\gamma}}$$

Imposing the condition $\nabla_\mu T^{\mu\nu} = 0$, and after long computations [see the details in (68)], one arrives at

$$T^T{}_T = T^R{}_R = -\frac{a}{R^2} \tag{F.2}$$

Here a is an integration constant related to the local energy density. Having an explicit energy-momentum tensor, we can compute the line element for the so-called Letelier spacetime (67)

$$ds^2 = -\left(1 - a - \frac{2m}{R}\right)dT^2 + \frac{1}{\left(1 - a - \frac{2m}{R}\right)}dR^2 + R^2 d\Omega^2 \tag{F.3}$$

According to Letelier's interpretation, this solution represents the Schwarzschild black hole surrounded by a cloud of strings.

F.1 Standard Thermodynamics

From the Letelier line element (F.3), we have the following lapse function

$$f(r) = 1 - a - \frac{2m}{R} \tag{F.4}$$

The event horizon is located at $R_h = \frac{2m}{1-a}$. Using the Bekenstein-Hawking relation (2.36), the BH entropy is

$$S = \pi R_h^2 \Rightarrow S = \pi\frac{4m^2}{(1-a)^2} \tag{F.5}$$

The temperature is given by the Hawking relation (2.40) as follows

$$T_h = \frac{f'(r_h)}{4\pi} \Rightarrow T_h = \frac{(1-a)^2}{8\pi m} \tag{F.6}$$

The Komar energy in static spherical symmetry (2.25) is given by

$$E_k = \frac{R^2}{2}f'(r_h) \Rightarrow E_k = m \tag{F.7}$$

From the global aspect of Black Holes we can compute the Smarr relation (3.3)

$$E_k - 2T_h S = -8\pi\int_{r_h}^{R} dr \cdot r^2\left(T^t{}_t - \frac{T}{2}\right)$$

However, in this spacetime, we have $T = 2T^t{}_t$, so the "matter" integral vanishes. From this, we obtain the Smarr relation for the Letelier BH

$$E_k = 2T_h S \Leftrightarrow m = 2T_h S \tag{F.8}$$

It is easy to prove that multiplying the entropy S (F.5) with temperature T_h (F.6) the Smarr relation is satisfied. The first law of BH thermodynamics is

$$dm = T_h dS \tag{F.9}$$

The internal energy does not change due to the cloud of string. While the Hawking temperature and BH entropy change by a factor proportional to $(1-a)^2$. These results appear in (69).

F.2 Horizon Thermodynamics

We recall the definition of Misner-Sharp mass from equation (3.23)

$$f(R) = 1 - \frac{2}{r}U(R)$$

Reading from the lapse function (F.4), we have

$$\Rightarrow U(R) = m + \frac{aR}{2}$$

Evaluating at the horizon $U \equiv U(R_h)$, we obtain

$$\Rightarrow U = \frac{m}{(1-a)} \tag{F.10}$$

According to the HT approach the pressure and volume (3.26) are given by

$$V \equiv \frac{4\pi R_h^3}{3}, \quad P \equiv T^r{}_r|_{R_h},$$

Using $R_h = \frac{2m}{1-a}$ and equation (F.2), we obtain that

$$\Rightarrow P = \frac{-a(1-a)^2}{32\pi m^2}, \quad V = \frac{32\pi m^3}{3(1-a)^3} \tag{F.11}$$

The Hawking temperature and BH entropy are given by equations (F.6) and (F.5) respectively. We can prove explicitly that the Euler-Smarr relation (3.34) is satisfied. We start with

$$2T_hS - 3PV = 2\left(\frac{(1-a)^2}{8\pi m}\right)\cdot\left(\pi\frac{4m^2}{(1-a)^2}\right) - 3\left(\frac{32\pi m^3}{3(1-a)^3}\right)\cdot\left(\frac{-a(1-a)^2}{32\pi m^2}\right)$$

$$\Rightarrow 2T_hS - 3PV = m + \frac{am}{(1-a)} \Rightarrow 2T_hS - 3PV = \frac{m}{(1-a)}$$

$$\Rightarrow 2T_hS - 3PV = U$$

Where we used the Misner-Sharp mass from equation (F.10). Of course, from this relation, we obtain the first law of HT immediately

$$dU = T_h dS - PdV$$

We notice that the Komar mass is $E_k = m$, while the Misner-Sharp mass is $U = \frac{m}{1-a}$. They are not equal due to the presence of the cloud of strings.

Appendix G

Unruh Effect and Hawking Radiation

In section 2.4, we obtained the Hawking temperature T_h using the Euclidean trick. Here, we will obtain the Hawking radiation using a test scalar field on the Schwarzschild BH background. This computation will be based on the following textbooks (1), (8). We will start with the Unruh effect because it is conceptually simpler and has the same math as Hawking radiation computation. Besides, we will perform the calculation in 1+1 dimensions, and then we will comment on the necessary changes for 3+1 dimensions.

G.1 Unruh Effect

From Appendix B we know that the Minkowski and Rindler line elements are respectively

$$ds^2 = -dT^2 + dX^2, \quad ds^2 = e^{2ax}(-dt^2 + dx^2) \tag{G.1}$$

Where the coordinate transformations that connect both line elements are

$$X(t,x) = a^{-1}e^{ax}\cosh(at), \quad T(t,x) = a^{-1}e^{ax}\sinh(at)$$

We define the lightcone coordinates (U, V) for the Minkowski spacetime and the lightcone coordinates (u, v) for the Rindler spacetime in the following way

$$U \equiv T - X, \quad V \equiv T + X \tag{G.2}$$

$$u \equiv t - x, \quad v \equiv t + x \tag{G.3}$$

In these coordinates the line elements become

$$ds^2 = -dUdV, \quad ds^2 = -e^{-a(v-u)}dudv \tag{G.4}$$

The future and past horizons of the Rindler spacetime described in figure B.1 now are given by $dU = 0$ and $dV = 0$, respectively. The lightcone coordinate transformations between the Rindler and the Minkowski spacetimes satisfy that $U = U(u)$, and $V = V(v)$. We can prove this as follows

$$U = T - X \Rightarrow U = -a^{-1}e^{ax}\left(\cosh(at) - \sinh(at)\right)$$

$$\Rightarrow U = -a^{-1}\exp(a(x-t)) \Rightarrow U = -a^{-1}e^{-au}$$

Similarly, it is possible to prove that $V = a^{-1}e^{av}$. This property of the lightcone coordinate transformations will be useful later because it will allow us to separate the right-moving and left-moving quantum modes.

The next step is to consider a massless scalar field in the spacetime as follows

$$S = -\frac{1}{2}\int d^2x\sqrt{-g}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi \tag{G.5}$$

This action is conformally invariant because we are working in 1+1 dimensions. Therefore, the action becomes

$$S = -2\int dUdV\partial_U\phi\partial_V\phi = -2\int dudv\partial_u\phi\partial_v\phi$$

The equations of motion from the action principle are

$$\partial_U\partial_V\phi = 0, \quad \partial_u\partial_v\phi = 0$$

The solutions of these equations are

$$\phi = R(U) + L(V), \quad \phi = \tilde{R}(u) + \tilde{L}(v)$$

Where R, L, $\tilde{R}$ and $\tilde{L}$ are arbitrary smooth functions. The letter R is for right-moving mode, and L is for left-moving mode. These modes never mix between them thanks to, in part, $U = U(u)$, and $V = V(v)$. Besides, the action (G.5) has the standard form. Therefore, in the overlap region $x > |t|$, we can expand the field operator $\hat{\phi}$ in a canonical way

$$\hat{\phi} = \int_0^\infty \frac{d\omega}{\sqrt{2\pi}}\frac{1}{\sqrt{2\omega}}\left[e^{-i\omega U}\hat{a}^-_\omega + e^{i\omega U}\hat{a}^+_\omega\right] + LM \tag{G.6}$$

$$\hat{\phi} = \int_0^\infty \frac{d\Omega}{\sqrt{2\pi}}\frac{1}{\sqrt{2\Omega}}\left[e^{-i\Omega u}\hat{b}^-_\Omega + e^{i\Omega u}\hat{b}^+_\Omega\right] + LM \tag{G.7}$$

Where $e^{-i\omega U}$ is the right-moving mode with a positive frequency with respect to Minkowski time T, while $e^{-i\Omega u}$ is the right-moving mode with a positive frequency with respect to Rindler time t. We denote by LM the left-moving modes $e^{\pm i\omega V}$ and $e^{\pm i\omega v}$, which never mix with the right-moving quantum modes. For this reason, we do not explicitly write down the form of left-moving modes. The canonical commutation relations are

$$[\hat{a}^-_\omega, \hat{a}^+_{\omega'}] = \delta(\omega - \omega'), \quad [\hat{b}^-_\Omega, \hat{b}^+_{\Omega'}] = \delta(\Omega - \Omega') \tag{G.8}$$

The Minkowski vacuum state $|0_M>$ and the Rindler vacuum state $|0_R>$ are given by

$$\hat{a}^-_\omega|0_M>= 0, \quad \hat{b}^-_\Omega|0_R>= 0, \quad |0_M>\neq |0_R> \tag{G.9}$$

The vacuum states are different. This fact is the key to understand the Unruh effect. The Minkowski vacuum $|0_M>$ will be, from the point of view of the accelerated observer, a state that contains particles.

In order to connect the operators $\hat{b}^{\pm}_{\Omega}$ and $\hat{a}^{\pm}_{\omega}$, we use Bogolyubov transformation

$$\hat{b}^{-}_{\Omega} = \int_0^{\infty} d\omega(\alpha_{\Omega\omega}\hat{a}^{-}_{\omega} - \beta_{\Omega\omega}\hat{a}^{+}_{\omega}) \tag{G.10}$$

Introducing this expression and its conjugate in the commutation relation (G.8), one obtains the following condition over the coefficients $\alpha_{\Omega\omega}$ and $\beta_{\Omega\omega}$

$$\int_0^{\infty} d\omega(\alpha_{\Omega\omega}\alpha^{*}_{\Omega'\omega} - \beta_{\Omega\omega}\beta^{*}_{\Omega'\omega}) = \delta(\Omega - \Omega') \tag{G.11}$$

We want to compute the number of Rindler particles $\hat{N}_{\Omega} \equiv \hat{b}^{+}_{\Omega}\hat{b}^{-}_{\Omega}$ in the Minkowski vacuum. Then, we must compute that

$$< \hat{N}_{\Omega} > \equiv < 0_M|\hat{b}^{+}_{\Omega}\hat{b}^{-}_{\Omega}|0_M > \tag{G.12}$$

Using the Bogolyubov transformation (G.10), and $\hat{a}^{-}_{\omega}|0_M >= 0$, we obtain

$$\Rightarrow < \hat{N}_{\Omega} > = \int_0^{\infty} d\omega|\beta_{\Omega\omega}|^2 \tag{G.13}$$

All the calculation reduces to obtain this integral. We start using equation (G.10) in (G.7) to obtain

$$\hat{\phi} = \int_0^{\infty} \frac{d\omega}{\sqrt{2\pi}} \frac{1}{\sqrt{2\omega}} \int_0^{\infty} d\Omega\sqrt{\frac{\omega}{\Omega}} \left[\hat{a}^{-}_{\omega}\left(e^{-i\Omega u}\alpha_{\omega\Omega} - e^{i\Omega u}\beta^{*}_{\omega\Omega}\right) + \hat{a}^{+}_{\omega}\left(e^{i\Omega u}\alpha^{*}_{\omega\Omega} - e^{-i\Omega u}\beta_{\omega\Omega}\right)\right] + LM$$

The field operator $\hat{\phi}$ is in terms of $\hat{a}^{\pm}_{\omega}$ just as (G.6). Therefore, demanding that both expressions are equal, we obtain

$$\Rightarrow e^{-i\omega U} = \int_0^{\infty} d\Omega\sqrt{\frac{\omega}{\Omega}}\left(e^{-i\Omega u}\alpha_{\omega\Omega} - e^{i\Omega u}\beta^{*}_{\omega\Omega}\right)$$

We can invert this expression using the typical tricks of Dirac delta. Then, we obtain the following expressions for the coefficients $\alpha_{\Omega\omega}$ and $\beta_{\Omega\omega}$

$$\alpha_{\Omega\omega} = \int_{-\infty}^{\infty} e^{-i\omega U + i\Omega u} du, \quad \beta_{\Omega\omega} = \int_{-\infty}^{\infty} e^{i\omega U + i\Omega u} du \tag{G.14}$$

From these equations, one can prove the following identity

$$|\alpha_{\Omega\omega}|^2 = e^{2\pi\Omega/a}|\beta_{\Omega\omega}|^2$$

Finally, inserting this result in (G.11), and taking $\Omega = \Omega'$, we reach the desired result

$$\int_0^{\infty} d\omega|\beta_{\Omega\omega}|^2 = \frac{\delta(0)}{\exp(\frac{2\pi\Omega}{a}) - 1} \tag{G.15}$$

Comparing with (G.13), we obtain that

$$< \hat{N}_{\Omega} > = \frac{\delta(0)}{\exp(\frac{2\pi\Omega}{a}) - 1} \tag{G.16}$$

The divergence $\delta(0)$ appears because we are considering all the spacetime. If we made the integral in a volume V, $\delta(0)$ would be replace by V. Then, defining $n \equiv \frac{<\hat{N}_\Omega>}{\delta(0)}$, we obtain the mean density of the Rindler particles

$$n_\Omega = \frac{1}{\exp(\frac{2\pi\Omega}{a}) - 1} \tag{G.17}$$

This result is the Bose-Einstein distribution. Consequently, we can recognize the Unruh temperature T_u as follows

$$T_u = \frac{a}{2\pi} \tag{G.18}$$

Therefore, an accelerated observer will see a thermal bath of particles. The agent that causes the Rindler observer acceleration is the energy source to excite particles in the Minkowski vacuum. We must notice that the computation was carried out for right-moving modes. One arrives at the same distribution (G.17) performing the calculation with left-moving modes.

G.2 Hawking Radiation

The calculation of Hawking radiation is the same as the Unruh effect, once we identify the vacuum states that we will need to perform this computation. To do this, we will identify two coordinate systems related to two different observers. We start with the Schwarzschild solution in 1+1 dimensions, which is

$$ds^2 = -\left(1 - \frac{r_g}{r}\right) dt^2 + \left(1 - \frac{r_g}{r}\right)^{-1} dr^2$$

Where we have defined $r_g \equiv 2M$. Using the tortoise coordinate r^* defined in (2.15), we can obtain the following coordinate transformation

$$\Rightarrow r^* = r + r_g \ln\left(\frac{r}{r_g} - 1\right) - r_g \tag{G.19}$$

This coordinate transformation is valid for $r > r_g$. Then, it describes the exterior of the black hole. When $r \to r_g$, we have that $r^* \to -\infty$. Besides, when $r >>> r_g$, we have that $r^* \sim r$. Thus, the tortoise coordinate is useful to describe a observer at rest at infinity. Using (G.19) the line element becomes

$$ds^2 = -\left(1 - \frac{r_g}{r(r^*)}\right)(dt^2 - dr^{*2})$$

Where r is an implicit function of r^* defined through equation (G.19). As before, we introduce lightcone coordinates: $u \equiv t - r^*$ and $v \equiv t + r^*$. Then, we obtain

$$ds^2 = -\left(1 - \frac{r_g}{r(u,v)}\right) du dv \tag{G.20}$$

Later, we will use the observer related to this line element to define one of the vacuum states that we will need later.

Now, we will describe the other line element related to the other vacuum state. We start expressing the equation (G.19) as follows

$$\Rightarrow 1 - \frac{r_g}{r} = \frac{r_g}{r} \exp\left(\frac{v-u}{2r_g}\right) \exp\left(1 - \frac{r}{r_g}\right)$$

Using this result in equation (G.20), we have

$$\Rightarrow ds^2 = -\frac{r_g}{r(u,v)} \exp\left(1 - \frac{r(u,v)}{r_g}\right) \left(\exp\left(\frac{-u}{2r_g}\right) du\right) \left(\exp\left(\frac{v}{2r_g}\right) dv\right)$$

In this equation, we will use the following coordinate transformations

$$U = -2r_g \exp\left(-\frac{u}{2r_g}\right), \quad V = 2r_g \exp\left(\frac{v}{2r_g}\right) \tag{G.21}$$

Finally, we arrive at the lightcone Kruskal coordinate system, which is

$$ds^2 = -\frac{r_g}{r(U,V)} \exp\left(1 - \frac{r(U,V)}{r_g}\right) dU dV \tag{G.22}$$

This coordinate system will define another vacuum state. As the case of the Unruh effect, we also have that $U = U(u)$, and $V = V(v)$. Therefore, left-moving and right-moving modes will never mix between them. We can also see from (G.21) that $u \in (-\infty, 0)$ and $v \in (0, \infty)$. Then, we are still describing the exterior of the black hole. However, we can extend $u \in (-\infty, \infty)$ and $v \in (-\infty, \infty)$ because the line element (G.22) is still well-defined with this analytical extension.

Equations (G.20) and (G.22) are our two coordinate systems related by the coordinate transformations (G.21). The proper time of an observer at rest at infinity coincides with t. This result comes from the following line element

$$ds^2 = -\left(1 - \frac{r_g}{r(u,v)}\right) du dv \Rightarrow ds^2 \to -du dv = -dt^2 + dr^*, \quad (r \to \infty)$$

In the region where $r \to \infty$, we can expand the field operator $\hat{\phi}$ as follows

$$\hat{\phi} = \int_0^\infty \frac{d\Omega}{\sqrt{2\pi}} \frac{1}{\sqrt{2\Omega}} \left[e^{-i\Omega u} \hat{b}_\Omega^- + e^{i\Omega u} \hat{b}_\Omega^+\right] + LM$$

The vacuum state is defined by $b_\Omega^- |0_B >= 0$. This state is called the Boulware vacuum state. The tortoise coordinate covers only the exterior part of the black hole; in this sense, they are similar to the Rindler coordinates because both coordinate systems describe only a certain part of the whole spacetime. Besides, we notice that the right-moving positive frequency mode $e^{-i\Omega u}$ propagates away from the black hole.

The Kruskal coordinates are regular at the event horizon. Thus, we have that

$$ds^2 = -\frac{r_g}{r(U,V)} \exp\left(1 - \frac{r(U,V)}{r_g}\right) dU dV \Rightarrow ds^2 \to -dU dV = -dT^2 + dR^2, \quad (r \to r_g)$$

Where we have defined in the canonical way T and R through the lightcone: $U \equiv T - X$ and $U \equiv T + X$. The Kruskal coordinates cover the whole Schwarzschild black hole; in this sense, they are similar to the Minkowski coordinates. In the region where $r \to r_g$, we can expand the field operator $\hat{\phi}$ as follows

$$\hat{\phi} = \int_0^\infty \frac{d\omega}{\sqrt{2\pi}} \frac{1}{\sqrt{2\omega}} \left[e^{-i\omega U} \hat{a}_\omega^- + e^{i\omega U} \hat{a}_\omega^+ \right] + LM$$

The term $e^{-i\omega U}$ is the right-moving mode with a positive frequency with respect to the time T near the horizon. The Kruskal vacuum state is defined by $a_\omega^- |0_K >= 0$. Now, we have all the ingredients to perform the computation of Hawking radiation. At this point, the calculation becomes similar to the one carried out for the Unruh effect.

In summary, the Kruskal vacuum state $|0_K >$ plays the role of Minkowski vacuum $|0_M >$, and the Boulware vacuum state $|0_B >$ plays the role of Rindler vacuum $|0_R >$. Besides, the surface gravity κ is equivalent to the acceleration a. Then, we want to compute the number of Boulware particles $\hat{N}_\Omega \equiv \hat{b}_\Omega^+ \hat{b}_\Omega^-$ in the Kruskal vacuum

$$< \hat{N}_\Omega > \equiv < 0_K | \hat{b}_\Omega^+ \hat{b}_\Omega^- | 0_K > \Rightarrow < \hat{N}_\Omega > = \int_0^\infty d\omega |\beta_{\Omega\omega}|^2$$

Doing precisely the same math that (G.13)-(G.17), we obtain the mean density of the particles, which is given by

$$n_\Omega = \frac{1}{\exp(\frac{2\pi\Omega}{\kappa}) - 1} \tag{G.23}$$

This equation is the Bose-Einstein distribution again. Hence, we can recognize the Hawking temperature T_h as follows

$$T_u = \frac{\kappa}{2\pi} \tag{G.24}$$

An eternal black hole is described by the analytical extension made in the Kruskal coordinates. In this case, there are right-moving and left-moving modes because we have two horizons $du = 0$ [future-horizon of the black hole] and $dv = 0$ [past-horizon of the white hole]. The outgoing particles are the right-moving modes, and the incoming particles are the left-moving modes. From this picture, we conclude that an eternal black hole must be in a thermal reservoir at temperature T_h, because the black hole will absorb particles, so it will have to radiate particles to be in thermal equilibrium. On the other hand, a black hole, which is a product of gravitational collapse, does not have a past-horizon. Then, the black hole does not have left-moving modes, and it will only radiate evaporating its mass. Thus, we arrive at the Hawking radiation.

In the 3+1 dimensions, we must consider the spherical part of the metric. However, we can use separable variables in the scalar field as follows

$$\Phi(t, r, \phi, \theta) = \sum_{l,m} \phi_{lm}(t, r) Y_{lm}(\phi, \theta) \tag{G.25}$$

One can prove that the wave equation of the scalar field becomes

$$\left[g^{ab} \partial_a \partial_b \phi + \left(1 - \frac{r_g}{r}\right) \left(\frac{r_g}{r^3} + \frac{l(l+1)}{r^2} \right) \right] \phi_{lm}(r, t) = 0 \tag{G.26}$$

Where the indices $a, b \in (t, r)$. Therefore, the wave equation acquires an effective potential. This extra term in the wave equation introduces the gray factor Γ_{GR} to the mean density of particles as follows

$$n_\Omega = \frac{\Gamma_{GF}}{\exp(\frac{2\pi\Omega}{\kappa}) - 1} \tag{G.27}$$

We will not prove this result here. The details are in (10). We notice that $\Gamma_{GR} < 1$ because the quantum modes must overcome the aforementioned effective potential to escape to infinity. The key idea is that the sector (t, r) is responsible for the Hawking radiation, not the spherical part of the metric. Thus, the analysis in 1+1 dimensions is enough to show the quantum nature of the Hawking radiation.

Finally, we recall the Stefan-Boltzmann law, which is given by

$$L = \Gamma_{GF}\sigma T_h^4 A$$

Where $\sigma = \frac{\pi^2}{60}$ is the Stefan-Boltzmann constant in natural units $[\sigma = \frac{\pi^2 k_b^4}{60\hbar^3 c^2}$ in SI units], and Γ_{GF} is the gray factor correction aforementioned. The mass of the black hole decreases with time as follows

$$\frac{dM}{dt} = -L$$

By combining the last two equations and using the area A and Hawking temperature T_h of the Schwarzschild black hole, we obtain

$$\frac{dM}{dt} = -\frac{\Gamma_{GF}}{15360\pi M^2} \tag{G.28}$$

This result is the Hawking's radiation power formula.

Bibliography

[1] T. Padmanabhan, Gravitation, Foundations and Frontier, Cambridge University Press, Cambridge, UK, 2010. [ISBN: 9780521882231]

[2] M. P. Hobson, G. P. Efstathiou, A. N. Lasenby, General Relativity, an introduction for physicists, Cambridge University Press, Cambridge, UK, 2006. [ISBN: 9780521829519]

[3] L. Susskind, J. Lindesay, An introduction to Black holes, information and the string theory revolution, the holographic principle, World Scientific, 2005. [ISBN: 9789812560834]

[4] R. M. Wald, General Relativity, Chicago University Press, Chicago, USA, 1984. [ISBN: 9780226870335]

[5] P.K. Townsend, Black Holes, DAMTP, University of Cambridge, Silver St., Cambridge, UK, 1997. [arXiv:gr-qc/9707012]

[6] E. Poisson, A Relativist's Toolkit, the mathematics of Black Hole Mechanics, Cambridge University Press, Cambridge, UK, 2004. [ISBN: 9780511606601]

[7] T. Padmanabhan, Quantum Field Theory, The Why, What and How, Springer Press, 2016. [ISBN: 978-3-319-28173-5]

[8] V. Mukhanov, S. Winitzki, Introduction to quantum effects in gravity, 2007. [Online ISBN: 9780511809149]

[9] Øyvind Grøn, Sigbjorn Hervik, Einstein's general theory of relativity, 2010. [DOI :10.1007/978-0-387-69200-5]

[10] Leonard Parker, David Toms, Quantum Field Theory in Curved Spacetime, 2009. [ISBN: 9780511813924]

[11] J. M. Bardeen, B. Carter S. W. Hawking, The Four Laws of Black Hole Mechanics, 1973. [DOI: https://doi.org/10.1007/BF01645742]

[12] Arthur Komar, Positive-Definite Energy Density and Global Consequences for General Relativity, 1963. [DOI: https://doi.org/10.1103/PhysRev.129.1873]

[13] Larry Smarr, Mass Formula for Kerr Black Holes, 1973. [DOI: https://doi.org/10.1103/PhysRevLett.30.71]

[14] J. D. Bekenstein, Black holes and the second law, 1972. [DOI: https://doi.org/10.1007/BF02757029]

[15] J. D. Bekenstein, Black Holes and Entropy, 1973. [DOI: https://doi.org/10.1103/PhysRevD.7.2333]

[16] S. W. Hawking, Particle creation by Black Holes, 1975. [DOI: https://doi.org/10.1007/BF02345020]

[17] J. D. Bekenstein, Generalized second law of thermodynamics in black-hole physics, 1974. [DOI: 10.1103/PhysRevD.9.3292]

[18] Robert M. Wald, Black Hole Entropy is Noether charge, 1993. [DOI: 10.1103/PhysRevD.48.R3427] [arXiv:gr-qc/9307038v1]

[19] James W. York, Black-hole thermodynamics and the Euclidean Einstein action, 1986. [DOI: https://doi.org/10.1103/PhysRevD.33.2092]

[20] Richard C. Tolman, On the Weight of Heat and Thermal Equilibrium in General Relativity, 1930. [DOI: https://doi.org/10.1103/PhysRev.35.904]

[21] D. Lynden-Bell, Roger Wood, the gravo-thermal catastrophe in isothermal spheres and the onset of red-giant structure for stellar systems, 1967. [Published in: Mon.Not.Roy.Astron.Soc. 138 (1968) 495]

[22] David C. Wright, Black holes and the Gibbs-Duhem relation, 1979. [DOI: https://doi.org/10.1103/PhysRevD.21.884]

[23] S. W. Hawking and D.N. Page, Thermodynamics of black holes in anti-de Sitter space, 1983. [DOI: https://doi.org/10.1007/BF01208266]

[24] Unruh, Notes on black-hole evaporation, 1976. [DOI: https://doi.org/10.1103/PhysRevD.14.870]

[25] G. W. Gibbons and S. W. Hawking, Cosmological event horizons, thermodynamics, and particle creation, 1977. [DOI:https://doi.org/10.1103/PhysRevD.15.2738]

[26] Dieter H. Gross, Microcanonical Thermodynamics, Phase Transitions in 'Small' Systems, 2001. [ISBN: 978-981-02-4215-2]

[27] T. Dauxois, S. Ruffo, E. Arimondo, M. Wilkens, Dynamics and Thermodynamics of Systems with Long-Range Interactions, 2002. [ISBN: 978-3-540-45835-7]

[28] Constantino Tsallis, Introduction to nonextensive statistical mechanics approaching a complex world, 2009. [ISBN: 978-0-387-85359-8]

[29] Ted Jacobson, Entropy from Carnot to Bekenstein, 2018. [arXiv:1810.07839]

[30] Ahmed Almheiri, Thomas Hartman, Juan Maldacena, Edgar Shaghoulian, Amirhossein Tajdini, The entropy of Hawking radiation, 2020. [arXiv:2006.06872v1]

[31] David Wallace, The case for black hole thermodynamics part I, 2018. [arXiv:1710.02724v2]

[32] David Wallace, The case for black hole thermodynamics part II, 2018. [arXiv:1710.02725v2]

[33] David Wallace, Why black hole information loss is paradoxical, 2017. [arXiv:1710.03783v2]

[34] Claudio Teitelboim, The cosmological constant as a thermodynamic black hole parameter, 1985. [DOI: https://doi.org/10.1016/0370-2693(85)91186-4]

[35] J. David York, Claudio Teitelboim, Neutralization of the cosmological constant by membrane creation, 1988. [DOI: https://doi.org/10.1016/0550-3213(88)90559-7]

[36] J. Creighton, R.B. Mann, Quasilocal Thermodynamics of Dilaton Gravity coupled to Gauge Fields, 1995. [DOI:https://doi.org/10.1103/PhysRevD.52.4569] [arXiv:gr-qc/9505007]

[37] David Kastor, Sourya Ray and Jennie Traschen, Enthalpy and the Mechanics of AdS Black Holes, 2009. [DOI: 10.1088/0264-9381/26/19/195011] [arXiv:0904.2765]

[38] Brian P. Dolan, The cosmological constant and black hole thermodynamic potentials, 2018. [DOI:10.1088/0264-9381/28/12/125020] [arXiv:1008.5023v4]

[39] Brian P. Dolan, Where is the P dV term in the first law of black hole thermodynamics, 2016. [DOI:10.5772/52455] [arXiv:1209.1272v3]

[40] C. V. Johnson, Holographic Heat Engines, 2014. [DOI:10.1088/0264-9381/31/20/205002] [arXiv:1404.5982v2]

[41] C. V. Johnson, An Exact Efficiency Formula for Holographic Heat Engines, 2016. [DOI: 10.3390/e18040120] [arXiv:1602.02838v2]

[42] David Kubiznak, Robert B. Mann, Mae Teo, Black hole chemistry thermodynamics with Lambda, 2017. [DOI:10.1088/1361-6382/aa5c69] [arXiv:1608.06147v2]

[43] T. Padmanabhan, Thermodynamics and/of horizon: A comparison of Schwarzschild, Rindler and de Sitter spacetimes, 2002. [DOI: 10.1142/S021773230200751X] [arXiv:gr-qc/0202078v1]

[44] T. Padmanabhan, Gravity and the Thermodynamics of Horizons, 2005. [DOI:10.1016/j.physrep.2004.10.003] [arXiv:gr-qc/0311036v2]

[45] T. Padmanabhan, A Dialogue on the Nature of Gravity, 2009. [arXiv:0910.0839v2]

[46] Ted Jacobson, Thermodynamics of Spacetime: The Einstein Equation of State, 1995. [DOI: 10.1103/PhysRevLett.75.1260] [arXiv:gr-qc/9504004v2]

[47] T. Padmanabhan, Classical and Quantum Thermodynamics of horizons in spherically symmetric spacetime, Padmanabhan, 2002. [DOI:10.1088/0264-9381/19/21/306] [arXiv:gr-qc/0204019v2]

[48] Dawood Kothwala, Sudipta Sarkar, Thanu Padmanabhan, Einstein's equations as a thermodynamic identity The cases of stationary axisymmetric horizons and evolving spherically symmetric horizons, 2006. [DOI:10.1016/j.physletb.2007.07.021] [arXiv:gr-qc/0701002v1]

[49] Charles W. Misner, David H. Sharp, Relativistic Equations for Adiabatic, Spherically Symmetric Gravitational Collapse, 1964. [DOI: https://doi.org/10.1103/PhysRev.136.B571]

[50] Sean A. Hayward, Gravitational energy in spherical symmetry, 1996. [DOI:https://doi.org/10.1103/PhysRevD.53.1938] [arXiv:gr-qc/9408002v1]

[51] L. Kirkby's Physics: A Student Companion, Scion Publishing Ltd, 2011. [ISBN-13: 978-1904842682]

[52] Gia Dvali and Cesar Gomez, Black Hole's Quantum N-Portrait, 2011. [DOI: https://doi.org/10.1002/prop.201300001] [arXiv:1112.3359v1]

[53] Gia Dvali and Cesar Gomez, Black Hole's 1/N Hair, 2012. [DOI: 10.1016/j.physletb.2013.01.020] [arXiv:1203.6575v1]

[54] Gia Dvali and Cesar Gomez, Black Holes as Critical Point of Quantum Phase Transition, 2012. [DOI: 10.1140/epjc/s10052-014-2752-3] [arXiv:1207.4059v1]

[55] Jorge Alfaro, Domenec Espriu, Luciano Gabbanelli, Bose-Einstein condensate in a Schwarzschild Black Hole, 2018. [DOI:10.1088/1361-6382/aa9771] [arXiv:1609.01639v3]

[56] Jorge Alfaro, Domenec Espriu, Luciano Gabbanelli, Condensates beyond the horizons, 2019. [DOI: 10.1142/S0217751X20500943] [arXiv:1905.01080v1]

[57] C. J. Pethick and H. Smith, Bose-Einstein condensation in dilute gases, Cambridge University Press, Cambridge, UK, 2008. [ISBN: 978-0521846516]

[58] Abhay Ashtekar, Javier Olmedo, and Parampreet Singh, Quantum extension of the Kruskal spacetime, 2018. [DOI: 10.1103/PhysRevD.98.126003] [arXiv:1806.02406v3]

[59] Abhay Ashtekar, Javier Olmedo, Properties of a recent quantum extension of the Kruskal geometry, 2020. [arXiv:2005.02309]

[60] G. Chaplin, E. Hohfeld, R. B. Laughlin, D. I. Santiago, Quantum Phase Transitions and the Breakdown of Classical General Relativity, 2000. [DOI: 10.1142/S0217751X03016380] [arXiv:gr-qc/0012094]

[61] Pawel O. Mazur, Emil Mottola, Surface Tension and Negative Pressure Interior of a Non-Singular 'Black Hole', 2015. [DOI: 10.1088/0264-9381/32/21/215024] [arXiv:1501.03806v1]

[62] Celine Cattoen, Tristan Faber, Matt Visser, Gravastars must have anisotropic pressures, 2005. [DOI: 10.1088/0264-9381/22/20/002] [arXiv:gr-qc/0505137v2]

[63] Ram Brustein, A. J. M. Medved, Resisting collapse, How matter inside a black hole can withstand gravity, 2019. [DOI:10.1103/PhysRevD.99.064019] [arXiv:1805.11667v2]

[64] Ram Brustein, A. J. M. Medved, Black holes as collapsed polymers 2016. [DOI: 10.1002/prop.201600114] [arXiv:1602.07706v1]

[65] Cenalo Vaz, Quantum gravitational dust collapse does not result in a black hole, 2014. [DOI: 10.1016/j.nuclphysb.2014.12.021] [arXiv:1407.3823]

[66] André LeClair, Extensions of the Schwarzschild solution into regions of non-zero energy density and pressure, 2019. [arXiv:1905.06799]

[67] P. S. Letelier, Clouds of String in General Relativity, 1979. [DOI: 10.1103/PhysRevD.20.1294]

[68] D. Barbosa, V. B. Bezerra, On the rotating Letelier spacetime, 2016. [DOI: https://doi.org/10.1007/s10714-016-2143-1]

[69] Marcos M. Dias e Costa, J. M. Toledo, V. B. Bezerra, On the Letelier spacetime with quintessence, solution, thermodynamics and Hawking radiation, 2018. [DOI: 10.1142/S0218271819500743] [arXiv:1811.12585v1]

[70] V. V. Kiselev, Quintessence and Black Holes, 2002. [DOI:10.1088/0264-9381/20/6/310] [arXiv:gr-qc/0210040v3]

[71] Matt Visser, The Kiselev black holes is neither perfect fluid, nor is it quintessence, 2019. [DOI:10.1088/1361-6382/ab60b8] [arXiv:1908.11058v1]

[72] K. Ghaderi, B. Malakolkalami, Thermodynamics of the Schwarzschild and the Reissner-Nordstrom black holes with quintessence, 2015. [DOI:https://doi.org/10.1016/j.nuclphysb.2015.11.019]

[73] Amritendu Haldar, Ritabrata Biswas, Thermodynamic studies of Different Type of Black Holes General Uncertainty Principle Approach, 2019. [DOI: 10.1142/S0217732318502000] [arXiv:1903.12481v1]

[74] J. M. Toledo, V. B. Bezerra, Some remarks on the thermodynamics of charged AdS black holes with cloud of strings and quintessence, 2019. [DOI: https://doi.org/10.1140/epjc/s10052-019-6616-8]

[75] Gu-Qiang Li, Effects of dark energy on P-V criticality of charged AdS black holes, 2014. [DOI: 10.1016/j.physletb.2014.06.047] [arXiv:1407.0011v1]

[76] J. Sadeghi, B. Pourhassan, M. Rostami, Z. Nekouee, Kiselev-CFT correspondence and black hole thermodynamics, 2019. [arXiv:1812.05479v4]