PONTIFICIA UNIVERSIDAD CATÓLICA DE CHILE
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# ON THE PROPAGATION OF GRAVITATIONAL WAVES IN AN EXPANDING UNIVERSE

MAURICIO GAMONAL SAN MARTÍN

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Advisor:
PROF. JORGE ALFARO

Examining Committee:
BENJAMIN KOCH PhD
ROLANDO DÜNNER PhD

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*"The most beautiful experience we can have is the mysterious. It is the fundamental emotion that stands at the cradle of true art and true science. Whoever does not know it and can no longer wonder, no longer marvel, is as good as dead, and his eyes are dimmed"*

*– Albert Einstein (1879–1955)*

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[1]A student organization for scientific dissemination. Visit https://fisicaitinerante.cl/.

# Abstract


In this report a complete investigation line developed in [1–4] was reviewed, which shows an influence of cosmological parameters ($\Lambda$ or non-relativistic matter) in the propagation of gravitational waves. This influence is caused by a coordinate transformation between a frame located in the source of a gravitational wave and a cosmological observer.

In order to delve in this line, the main work of this report was to develop a generalization of this effect considering an arbitrary perfect fluid as a background in the Universe. The metrics, coordinate transformations and their linearized forms were found in order to study the effect of other components not studied before, as radiation.

Finally, a numerical analysis of the solutions considering realistic models was done, using the timing residual effect in the pulsars observations as an indicator of the cosmological influences in the propagation of gravitational waves measured from Earth. The results of this analysis imply a potential measurement ($5.3\sigma$ for $\Lambda$CDM model) of a peak in the timing residual due to cosmological influence in the propagation of gravitational waves.



**Keywords:** General Relativity, Gravitational Waves, Expanding Universe, $\Lambda$CDM Model, Pulsar Timing Array.

# Notation and Conventions

In the development of this report, natural units will be used (i.e. $c = G = \hbar = 1$) except where is indicated; with $c$ as the speed of light in vacuum, $G$ as Newton Universal Gravitation constant and $\hbar$ as reduced Planck constant.

In general, latin indices (e.g. $i$, $j$, $k, \ldots$) will correspond to the three-dimensional spatial coordinates and they will take the values $1$, $2$ or $3$ ($x$, $y$ or $z$)

On the other hand, greeks indices (e.g. $\mu$, $\nu, \ldots$) will correspond to the four-dimensional spacetime coordinates and they will take the values $0$, $1$, $2$ o $3$ ($t$, $x$, $y$ o $z$). The component $x^0$ will be generally considered as the temporal coordinate of the system.

Additionally, we will use the Einstein summation convention: The appearance of two repeated indices implies the sum in these indices. For example, $p^\mu p_\mu = \sum_\mu p^\mu p_\mu$.

A metric of spacetime will be denoted by $g_{\mu\nu}$ and the spacetime interval will defined as $\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu$. The metric of Minkowski flat spacetime will be $\eta_{\alpha\beta} \equiv \text{diag}(-1, 1, 1, 1)$.

Some abbreviations used:

**GR**: General Relativity

**EFE**: Einstein Field Equations

$\Lambda$**CDM**: Cosmological Constant + Cold Dark Matter

**FLRW**: Friedmann–Lemaître–Robertson–Walker

**SdS**: Schwarzschild–de Sitter

**SSD**: Spherically Symmetric + Dust

**SSDR**: Spherically Symmetric + Dust + Radiation

**SS**$\omega_i$: Spherically Symmetric + perfect fluid with equation of state $p_i = \omega_i \rho_i$

**GW**: Gravitational Wave

**PTA**: Pulsar Timing Array

# Introduction

At present, mankind has two main pillars on which to base its study on the laws that govern the fundamental behavior of nature. On the one hand, the Quantum Theory, which describes with a high degree of precision three of the four fundamental forces known to date (Weak Nuclear Interaction, Electromagnetic Interaction and Strong Nuclear Interaction). On the other hand, the **Theory of General Relativity** (GR) describes the behavior of the bodies that are affected by the fourth fundamental force: The gravitational interaction.

Both models are mathematically self-consistent and have a high degree of acceptance in the scientific community. However, they are incompatible with each other: Gravitation, as we know it, could not be described in terms of a renormalizable Quantum Field Theory [5]. Due to the numerous and precise experimental verifications at quantum scale, it is considered that GR is a consistent model but that describes gravity from a classic perspective.

Nevertheless, GR still has a great predictive power and provides the framework of many astrophysical events. In that sense, one of the most transcendental observations of last century was the measurement of accelerated expansion of the Universe [6]. The standard cosmological model ($\Lambda$CDM) is the simplest one that provides the well measured properties of the cosmos. This model assumes the validity of GR but it requires an extra component: The addition of a constant, $\Lambda$, into GR field equations. Once this is done, $\Lambda$CDM parameters can be fitted within cosmological observations in order to obtain experimental values [7].

The cosmological constant arises as a necessity to obtain accelerated expansion and its nature is currently unknown. That problem is known as the **Dark Energy** mystery and no complete theory can explain it satisfactorily yet [8]. Until now, the only way of measure $\Lambda$ is from cosmological observations, but the extremely large distances involved implies a lower precision than other astronomical observations.

On the other hand, in the last years one of the most astonishing predictions of GR has gained attention: The discovery and measurement of Gravitational Waves (GW) [9]: Ripples in spacetime caused by astronomical perturbations propagates through the Universe and now can be observed. Then, it is natural to ask ourselves if the accelerated expansion of the Universe could affect the propagation of GW. An investigation line[2], developed in [1–3], has shown that the propagation of Gravitational Waves in an expanding Universe not only is influenced by cosmological constant but also it could serve for local measure of $\Lambda$ at galactic scale.

The core of this proposal is based on fixing a coordinate transformation between the GW source and a cosmological observer (e.g. the Earth). It is in this transformation that the influence of $\Lambda$ appears explicitly in the propagation of GW. In [4], this phenomenon is studied with the addition of non-relativistic matter (i.e. a perfect fluid with no pressure) in the Universe. Thus, in this report we will study a generalization of this situation for a perfect fluid with arbitrary equation of state $p_i = \omega_i \rho_i$, and it will be analyzed numerically for the case in which the radiation density ($\omega_i = 1/3$) is not negligible.

For this, in Chapter 1, foundations of GR will be reviewed. We will derive the Friedmann equations of standard cosmology and the relevant spacetime metrics that will be used later. Then, we will discuss the linearized version of GR and the gravitational wave solution.

In Chapter 2, we will review the work of Espriu et al. from the beginning. We will find the coordinate transformation in each case (only $\Lambda$, only Dust, $\Lambda$ + Dust and the generalization for an arbitrary fluid), the respective linearized equations and the relevance of coordinate transformation in gravitational wave solutions.

Later, in Chapter 3, we will show a setup consisting in a GW source far from Earth and a nearby pulsar. Thus, we will find that the previous equations can be used in the measurements of Timing Residual under this configuration, numerical analysis of equations included. Then, we will use a pulsar catalog to find a reasonable and realistic example showing the potential of this results.

Finally, in Chapter 4, the main conclusions of this report will be indicated and we will present ideas for future work based in this investigation line.

[2]It should be noted that several works are being realized about the propagation of GW in non-vacuum background with the aim to studying the nature of Dark Matter and Dark Energy [10–12].

# 1. Einstein's Theory of General Relativity

Denominated by part of the scientific community as the most *beautiful* of all existing physical theories [13], the Theory of General Relativity –developed by Albert Einstein between 1905 and 1915– definitively changed the way we understand the nature of space and time.

This model enjoys great acceptance, not only for its conceptual simplicity, but also throughout history has been verified experimentally with great accuracy: Through the precession of Mercury perihelion [14], the deflection of light in a solar eclipse [15], the development of GPS [16] and lately the detection of gravitational waves [9].

Next, theory foundations and main equations that will guide the work of this report will be shown in the appendixes . A detailed development of General Relativity can be found in classic textbooks as [17] and [18], which serve as references for this chapter.

## 1.1. Foundations and Field Equations

In 1905 Albert Einstein published the Special Theory of Relativity [19], which quickly gained scientific acceptance due to various experimental verifications. However, this model does not consider the presence of gravitational interaction: Only applies to inertial frames.

In order to generalize Special Relativity, Einstein postulated the *Principle of Equivalence* (an accelerating reference frame is identical to an equivalent gravitational field in small enough regions of space) and the *Principle of General Covariance* (equations must be covariant, preserving their form under general coordinate transformations) [20].

To take advantage of the Principle of Covariance we must know how to transform our equations under general transformations of coordinates. Thus, a good idea is to use a mathematical object called *tensor*. If we consider a general transformation $x' \to x$, then a tensor $m$ times covariant and $n$ times contravariant transforms as

$$T^{i'_1 \cdots i'_n}{}_{j'_1 \cdots j'_m} = \frac{\partial x^{i'_1}}{\partial x^{i_1}} \cdots \frac{\partial x^{i'_n}}{\partial x^{i_n}} \frac{\partial x^{j_1}}{\partial x^{j'_1}} \cdots \frac{\partial x^{j_m}}{\partial x^{j'_m}} T^{i_1 \cdots i_n}{}_{j_1 \cdots j_m}\,, \tag{1.1}$$

so a tensorial equation will be generally covariant. Moreover, the Equivalence Principle induces a relation between free falling and geodesic motion, so we can think spacetime as a curved manifold equipped with a metric tensor $g_{\mu\nu}$, that satisfies

$$g_{\mu\nu} = g_{\nu\mu} \qquad g_{\mu'\nu'} = \frac{\partial x^\rho}{\partial x^{\mu'}} \frac{\partial x^\sigma}{\partial x^{\nu'}} g_{\rho\sigma} \qquad g^{\mu\lambda} g_{\lambda\nu} = \delta^\mu_\nu. \tag{1.2}$$

From the study of curved differential manifolds, we can find that the only free-divergence tensor constructed with $g_{\mu\nu}$ and its first and second derivatives is the *Einstein tensor* defined as

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}, \tag{1.3}$$

where $R_{\mu\nu}$ is the Ricci tensor and $R$ is the scalar curvature. On the other hand, conservation of energy and momentum implies that the divergence of stress-energy tensor is zero, ${T^{\mu\nu}}_{;\nu} = 0$. Einstein conjectured that $G_{\mu\nu} \propto T_{\mu\nu}$ and to find the proportionality constant one can use the Newtonian limit, where the field equations must reduce to $\nabla^2 \phi = 4\pi G \rho$ ($\phi$ is the Newtonian gravitational potential and $\rho$ is the density of matter). After this procedure, we obtain the **Einstein Field Equations** (EFE):

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \kappa T_{\mu\nu}, \tag{1.4}$$

where $\kappa = \frac{8\pi G}{c^4}$. We can note that, if it is added a term proportional to $g_{\mu\nu}$ into (1.3), it also satisfies (1.4). The proportionality constant of this term is known as the *cosmological constant* $\Lambda$. Thus, it is possible to find a generalized EFE of the form

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}. \tag{1.5}$$

This expression comprise a set of ten Partial Differential Equations and describe the gravitational interaction. Therefore, Einstein Field Equations tell us how spacetime is curved by the presence of matter/energy and how matter/energy moves through spacetime.

## 1.2. Schwarzschild and SdS spacetimes

The first exact solution of (1.4) was found by Karl Schwarzschild in 1916 [21]. Let us consider the following assumptions for a spacetime metric:

(i) Static: $g_{\mu\nu}$ is not time dependent.

(ii) Spherically Symmetric: Angular terms of the form $r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2)$.

Thus, the most general metric that satisfies the previous conditions is given by

$$g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -A(r)\mathrm{d}t^2 + B(r)\mathrm{d}r^2 + r^2[\mathrm{d}\theta^2 + \sin^2(\theta)\mathrm{d}\phi^2], \tag{1.6}$$

where $A(r)$ and $B(r)$ are unknown functions that depend only of $r$. In Appendix A it is shown that for an asymptotically flat spacetime, the metric (1.6) becomes

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{r}\right)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1 - \frac{2M}{r}} + r^2\mathrm{d}\Omega^2, \tag{1.7}$$

where $\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2(\theta)\mathrm{d}\phi^2$. This is the **Schwarzschild metric** and describes the deformation of spacetime due to the presence of a spherical mass (e.g. a star or a Black Hole) in an empty Universe.

It is also possible to obtain a similar metric for a *de Sitter* space, which means that the Universe is expanding and gravitation is described by generalized Einstein Field Equations (1.5) with $\Lambda > 0$. In Appendix B it is shown that, following the same steps as before, the metric (1.6) becomes

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{r} - \frac{\Lambda}{3}r^2\right)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1 - \frac{2M}{r} - \frac{\Lambda}{3}r^2} + r^2\mathrm{d}\Omega^2. \tag{1.8}$$

This is the **Schwarzschild–de Sitter (SdS) metric** and describes the deformation of spacetime due to the presence of the same spherical mass but now in an expanding Universe with a positive cosmological constant, a de Sitter space.

### 1.3. FLRW metric and Friedmann equations

Years later another EFE solution was found. The Friedmann-Lemaître-Robertson-Walker (FLRW) metric was developed between 1922 and 1937 [22–25]. It represents an isotropic, homogeneous and expanding Universe through the generic metric

$$g_{\mu\nu}\mathrm{d}x^{\mu}\mathrm{d}x^{\nu} = -\mathrm{d}T^2 + a^2(T)\left(\frac{\mathrm{d}R^2}{1-kR^2} + R^2\mathrm{d}\Omega^2\right), \tag{1.9}$$

where $k$ may be taken as $\{-1, 0, 1\}$ for negative, zero or positive curvature respectively, $a(T)$ is known as the *scale factor* and $\{T, R, \theta, \phi\}$ are called comoving coordinates.

Latest measurements of cosmological parameters show that $k$ is approximately zero [7], so the Universe appears to be spatially flat. From this important consideration, the following metric will be used and it will be denoted as **FLRW metric**

$$g_{\mu\nu}\mathrm{d}x^{\mu}\mathrm{d}x^{\nu} = -\mathrm{d}T^2 + a^2(T)\mathrm{d}l^2, \tag{1.10}$$

where $\mathrm{d}l^2$ is the three-dimensional spacial line element. In order to use Einstein Field Equations, we need an expression for Stress-Energy tensor. If we consider a perfect fluid (i.e. a fluid that has not viscosity and it does not conduce heat) the Stress-Energy tensor takes the following form

$$T_{\mu\nu} = (\rho + p)U_{\mu}U_{\nu} + pg_{\mu\nu}, \tag{1.11}$$

where $\rho$ is the rest energy density, $p$ is the isotropic pressure and $U_{\mu}$ is a four-velocity of the fluid. Moreover, it is common to use a *equation of state* that relate pressure and density by

$$p_i = \omega_i\rho_i, \tag{1.12}$$

where $\omega_i$ is some constant.

In Appendix C it is shown that, for a Universe filled by a perfect fluid with density $\rho_i$ and pressure $p_i$ in a FLRW spacetime described by (1.10), the Einstein Field Equations give

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{\kappa}{3}(\rho_i + \rho_\Lambda) \tag{1.13a}$$

$$\left(\frac{\ddot{a}}{a}\right) = \kappa\left(\frac{\rho_\Lambda}{3} - \frac{\rho_i}{6} - \frac{p_i}{2}\right), \tag{1.13b}$$

where $\rho_\Lambda \equiv \frac{\Lambda}{\kappa}$ and a dot over quantities means derivative respect of time. These are known as the 1st and 2nd **Friedmann Equations** respectively. From taking derivative of (1.13a) with respect to $T$ (comoving time, do not confuse with the trace of Stress-Energy tensor) and replacing into (1.13b), it follows that

$$\begin{aligned}
\frac{\kappa}{3}\dot{\rho}_i &= 2\left(\frac{\dot{a}}{a}\right)\left[\frac{\ddot{a}a - \dot{a}^2}{a^2}\right] = 2\left(\frac{\dot{a}}{a}\right)\left[\kappa\left(\frac{\rho_\Lambda}{3} - \frac{\rho_i}{6} - \frac{p_i}{2}\right) - \frac{\kappa}{3}\left(\rho_i + \rho_\Lambda\right)\right] \\
\dot{\rho}_i &= -3\left(\frac{\dot{a}}{a}\right)\left(p_i + \rho_i\right).
\end{aligned}$$

Using equation of state (1.12), last expression becomes

$$\frac{\mathrm{d}\rho_i}{\mathrm{d}T} = -\rho_i\left[3(\omega_i + 1)\left(\frac{\dot{a}}{a}\right)\right] \to \frac{\mathrm{d}\rho_i}{\rho_i} = -3(\omega_i + 1)\frac{\mathrm{d}a}{a}.$$

After integration we obtain

$$\frac{\rho_i}{\rho_0} = \left(\frac{a(T)}{a_0}\right)^{-3(\omega_i+1)}, \tag{1.14}$$

where $\rho_0 = \rho(T_0)$ and $a_0 = a(T_0)$ are integration constants. Replacing the last expression into (1.13a) provides us a solution of scale factor for the case in which one fluid ($\Lambda \to 0$) with equation of state (1.12) is present,

$$a(T) = a_0\left(\frac{T}{T_0}\right)^{\frac{2}{3(\omega_i+1)}}. \tag{1.15}$$

Combining (1.14) with (1.15) we obtain the general form of density in terms of $T$

$$\rho_i = \frac{4}{3(\omega_i + 1)^2 \kappa T^2}, \tag{1.16}$$

where we used

$$\rho_0 = \frac{4}{3(\omega_i + 1)^2 \kappa T_0^2}. \tag{1.17}$$

Another important solution is found by demanding no material content in the Universe (i.e. $\rho_i = 0$) and $\Lambda \neq 0$. Solving (1.13a), the scale factor becomes

$$a(T) = \exp\left(\sqrt{\frac{\Lambda}{3}}\Delta T\right), \tag{1.18}$$

with $\Delta T = T - T_0$ and $a_0 = 1$. The FLRW metric with this scale factor describes de Sitter spacetime and, therefore, the same physical situation as SdS metric for a large $r$.

It is considered that (1.13a) and (1.13b) are the basis of standard cosmology. Currently, the most accepted model is $\Lambda$CDM: $\Lambda \neq 0$ and Cold Dark Matter. The effective energy density in $\Lambda$CDM model is commonly given by

$$\rho_{\text{eff}} = \rho_\Lambda + \rho_d + \rho_r = \rho_\Lambda + \rho_{d0}\left[\frac{a_0}{a(T)}\right]^3 + \rho_{r0}\left[\frac{a_0}{a(T)}\right]^4, \tag{1.19}$$

where $\rho_{d0}$ is the density of non-relativistic matter (i.e. Cold Dark Matter and baryonic matter, $\omega_d = 0$) at $T_0$ and $\rho_{r0}$ is radiation density ($\omega_r = 1/3$) at $T_0$.

### 1.4. Linearized Gravity and Gravitational Waves

EFE have two interesting features: They show that spacetime curvature is dynamic, but also its behavior is highly non-linear. In order to simplify calculations, we can consider the case where a flat spacetime is perturbed. Thus, the metric can be written as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \tag{1.20}$$

with $|h_{\mu\nu}| \ll 1$. In Appendix D it is shown that linearized EFE can be written as

$$\Box \bar{h}_{\mu\nu} = -2\Lambda\eta_{\mu\nu} - 2\kappa T_{\mu\nu}, \tag{1.21}$$

where we introduced the trace-reversed metric perturbation given by

$$\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h \qquad h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\bar{h} \qquad \bar{h} = -h, \tag{1.22}$$

and where it was chosen the *Lorenz Gauge*, which demands

$$\partial_\beta \bar{h}^{\beta\alpha} = 0. \tag{1.23}$$

The set of equations (1.21) describes the effects due to a weak gravitational field and it is called **linearized gravity**. As discussed in [1, 4], the perturbation $h_{\mu\nu}$ can be decomposed into a gravitational wave contribution $h^{(\text{GW})}_{\mu\nu}$ and a background part $h^{(\text{bg})}_{\mu\nu}$ which will have contributions due to $\Lambda$ and others fluid components of Universe involved (dust or radiation). As they do not interact between each other, we can write the metric as

$$g_{\mu\nu} = \eta_{\mu\nu} + h^{(\text{GW})}_{\mu\nu} + h^{(\Lambda)}_{\mu\nu} + h^{(\text{fluid})}_{\mu\nu}, \tag{1.24}$$

where each contribution satisfies from (1.21)

$$\Box \bar{h}^{(\text{GW})}_{\mu\nu} = 0 \tag{1.25}$$

$$\Box \bar{h}^{(\Lambda)}_{\mu\nu} = -2\Lambda\eta_{\mu\nu} \qquad \Box \bar{h}^{(\text{fluid})}_{\mu\nu} = -2\kappa T_{\mu\nu}. \tag{1.26}$$

In the next sections, we will consider only expansions up to order $\sqrt{\Lambda + \sum_i \rho_i}$, where $\rho_i$ will be the density of a particular component. Thus, we will concentrate only in (1.25), which is an homogeneous wave equation, whose general solution is an harmonic wave, reason why it is called the **Gravitational Wave** (GW) solution. In Cartesian coordinates the metric perturbation takes the following form

$$\bar{h}_{\mu\nu}^{(\mathrm{GW})} = A_{\mu\nu} \exp(ik_\alpha x^\alpha), \tag{1.27}$$

where $A_{\mu\nu}$ are constant components of some tensor and $k_\alpha \equiv (\Omega, \mathbf{k})$, with $\Omega$ as the angular frequency of GW and $\mathbf{k}$ as its wave vector. From (1.25) and (1.27) we can note that

$$\Box \bar{h}_{\alpha\beta} = \eta^{\mu\nu}\partial_\mu \partial_\nu \bar{h}_{\alpha\beta} = -\eta^{\mu\nu} k_\mu k_\nu \bar{h}_{\alpha\beta} = 0 \leftrightarrow \eta^{\mu\nu} k_\mu k_\nu = 0 \rightarrow \Omega^2 = |\mathbf{k}|^2. \tag{1.28}$$

Last expression shows that the dispersion relation of GW is equal to 1, which means that GW propagates at the speed of light. Furthermore, from Lorenz gauge conditions, it follows

$$A^{\alpha\beta} k_\beta = 0, \tag{1.29}$$

which is a restriction on the amplitude $A^{\alpha\beta}$. Nevertheless, the amplitude is not fully restricted because there is still a gauge freedom (see Appendix D). It is common to use the Transverse-Traceless (TT) gauge, where $A^\alpha_\alpha = 0$ and $A_{\alpha\beta} U^\beta = 0$, with $U^\beta$ as some fixed four-velocity. In this gauge $\bar{h}_{\mu\nu} = h_{\mu\nu}$ and it can be shown [26] that

$$A_{\alpha\beta} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A_+ & A_\times & 0 \\ 0 & A_\times & -A_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{1.30}$$

where $A_+$ and $A_\times$ indicates the polarizations of GW. If $r, t$ are the spherical coordinates in which gravitational waves are described from the source, then at a distance large enough –but small compared to cosmological distances– the metric perturbation $h_{\mu\nu}$ will be [2, 4]

$$h_{\mu\nu}^{(\mathrm{GW})} = \frac{1}{r}\Big( E_{\mu\nu} \cos[\Omega(t-r)] + D_{\mu\nu} \sin[\Omega(t-r)] \Big), \tag{1.31}$$

where $E_{\mu\nu}$ and $D_{\mu\nu}$ are of the same form that (1.30) because they belong to TT gauge.

# 2. Gravitational Waves in an expanding Universe

From the previous chapter, it is clear that a linearized version of Einstein Field Equation allows a radiative solution in form of Gravitational Waves. The simplest and standard way to study GW is using the coordinates that emerge from the GW source (i.e. $t, r$) in vacuum.

In this chapter, it will be shown that a specific coordinate transformation will produce changes in the propagation of gravitational waves in a non-empty background (e.g. in the case where $\Lambda$, dust or any perfect fluid is present), which will be useful in cosmology.

## 2.1. Appropriated coordinate systems

Let us consider the following situation: Two super-massive Black Holes are orbiting around a common center of mass and slowly approaching to each other. When they merge, an amount of gravitational radiation is released in all directions.

An appropriate coordinate system to measure the perturbation of spacetime metric from the GW source is the set $\{t, r, \theta, \phi\}$, that represent a spherically symmetric spacetime seen from the source. These coordinates are the same used in (1.31) and, for example, in a vacuum background the geometry of spacetime will be given by Schwarzschild metric (1.7).

On the other hand, it is important to remind that an accelerated expansion of the Universe is measured from Earth, which means that an observer in Earth is a cosmological observer. Thus, it is natural to use FLRW comoving coordinates $\{T, R, \theta, \phi\}$ and the metric

$$\mathrm{d}s^2 = -\mathrm{d}T^2 + a^2(T)\left(\mathrm{d}R^2 + R^2\mathrm{d}\Omega^2\right), \tag{2.1}$$

to describe the perturbation in spacetime due to a GW as seen from Earth. If we are able to find the two metrics that describe the same physical scenario, then the transformation

$$\{t, r, \theta, \phi\} \to \{T, R, \theta, \phi\}$$

will provide us the form of $h_{\mu\nu}^{(\mathrm{GW})}$ as seen from Earth using (1.31).

## 2.2. New metrics and the respectively coordinate transformations

From previous discussion, it is clear that in a vacuum background the geometry of spacetime as seen from the GW source will be Schwarzschild metric. Next, we will consider other background options and derive the respective metrics and coordinate transformations.

**Only $\Lambda$ is present: SdS Case**

This is the case where a gravitational wave propagates through a de Sitter spacetime, which is described by the Schwarzschild-de Sitter metric (1.8). As far from the GW source, the term corresponding to the mass of Black Hole is not relevant, so we can use the restricted SdS metric

$$\mathrm{d}s^2 = -\left(1 - \frac{\Lambda}{3}r^2\right)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1 - \frac{\Lambda}{3}r^2} + r^2\mathrm{d}\Omega^2. \tag{2.2}$$

Otherwise, the FLRW metric for the same physical situation is given by (2.1), with scale factor given by

$$a(T) = \exp\left(\sqrt{\frac{\Lambda}{3}}T\right). \tag{2.3}$$

From (1.2), it follows that metric transform as second rank tensor

$$g_{\mu'\nu'} = \frac{\partial X^\mu}{\partial x^{\mu'}}\frac{\partial X^\nu}{\partial x^{\nu'}}g_{\mu\nu}. \tag{2.4}$$

Thus, for the temporal and radial coordinates

$$g_{TT} = \left(\frac{\partial t}{\partial T}\right)^2 g_{tt} + \left(\frac{\partial r}{\partial T}\right)^2 g_{rr} \tag{2.5}$$

$$g_{RR} = \left(\frac{\partial t}{\partial R}\right)^2 g_{tt} + \left(\frac{\partial r}{\partial R}\right)^2 g_{rr}, \tag{2.6}$$

and for angular coordinates we simply get

$$r^2 \mathrm{d}\Omega^2 \to a^2(T) R^2 \mathrm{d}\Omega^2. \tag{2.7}$$

From (2.5) and (2.6) it follows that

$$-1 = -\left(\frac{\partial t}{\partial T}\right)^2 \left(1 - \frac{\Lambda}{3} r^2\right) + \left(R\sqrt{\frac{\Lambda}{3}} \exp\left(\sqrt{\frac{\Lambda}{3}} T\right)\right)^2 \left(1 - \frac{\Lambda}{3} r^2\right)^{-1} \tag{2.8}$$

$$\exp\left(2\sqrt{\frac{\Lambda}{3}} T\right) = -\left(\frac{\partial t}{\partial R}\right)^2 \left(1 - \frac{\Lambda}{3} r^2\right) + \exp\left(\sqrt{\frac{\Lambda}{3}} T\right)^2 \left(1 - \frac{\Lambda}{3} r^2\right)^{-1}. \tag{2.9}$$

Thereby,

$$\frac{\partial t}{\partial T} = \frac{1}{1 - \frac{\Lambda}{3} R^2 \exp\left(2\sqrt{\frac{\Lambda}{3}} T\right)} \tag{2.10}$$

$$\frac{\partial t}{\partial R} = \frac{\frac{R}{3}\sqrt{\frac{3}{\Lambda}} \exp\left(2\sqrt{\frac{\Lambda}{3}} T\right)}{1 - \frac{\Lambda}{3} R^2 \exp\left(2\sqrt{\frac{\Lambda}{3}} T\right)}. \tag{2.11}$$

Integrating both equations follows that

$$t(T, R) = H(T, R) + T + f(R) \tag{2.12}$$

$$t(T, R) = H(T, R) + g(T), \tag{2.13}$$

where $f(R)$ and $g(T)$ are arbitrary functions of $R$ and $T$ respectively, and

$$H(T, R) = -\frac{1}{2}\sqrt{\frac{3}{\Lambda}} \ln\left[1 - \frac{\Lambda}{3} R^2 \exp\left(2\sqrt{\frac{\Lambda}{3}} T\right)\right] + \text{constant}. \tag{2.14}$$

From (2.12) y (2.13) we note that $f(R) = 0$ and so $g(T) = T$. Then, the solution will be, up to addition by a constant,

$$t(T, R) = H(T, R) + T. \tag{2.15}$$

Therefore, the coordinate transformation between SdS and FLRW spacetimes are

$$r(T, R) = a(T)R \tag{2.16a}$$

$$t(T, R) = T - \sqrt{\frac{\Lambda}{3}} \ln \sqrt{1 - \frac{\Lambda}{3} a(T)^2 R^2}. \tag{2.16b}$$

These transformations agree with those presented in [1, 4].

**Only Dust is present: SSD Case**

Now we will consider a different background, which it is called the matter-dominated era. In this scenario, only non-relativistic matter is present in the Universe, which is denoted as *dust*. Dust is pressureless, so the equation of state reads $p = 0$ and then $\omega_d = 0$. From (1.15) and (1.16) we obtain for this case

$$a(T) = a_0 \left(\frac{T}{T_0}\right)^{\frac{2}{3}} \qquad \rho_d = \frac{4}{3\kappa T^2}, \tag{2.17}$$

where $\rho_d$ is the dust density at $T$. We are looking for a coordinate system where dust is described in a spherically symmetric static coordinates with origin coinciding with GW source. In first place, we note that angular coordinates transform as before

$$r^2 \, \mathrm{d}\Omega^2 \to a(T)^2 R^2 \, \mathrm{d}\Omega^2.$$

From (2.4) and the requirement that metric must be diagonal (i.e. $g_{rt} = g_{tr}$) it follows that

$$0 = \frac{\partial T}{\partial t}\frac{\partial T}{\partial r} g_{TT} + \frac{\partial R}{\partial t}\frac{\partial R}{\partial r} g_{RR}, \tag{2.18}$$

and by explicitly calculating the partial derivatives we obtain

$$\frac{\partial R}{\partial t} = -\frac{2}{3}\frac{r\frac{\partial T}{\partial t}}{a_0 T_0\left(\frac{T}{T_0}\right)^{5/3}} \tag{2.19}$$

$$\frac{\partial R}{\partial r} = \frac{1}{a(T)} - \frac{2}{3}\frac{r\frac{\partial T}{\partial r}}{a_0 T_0\left(\frac{T}{T_0}\right)^{5/3}}. \tag{2.20}$$

From (2.18) we find

$$\frac{\partial T}{\partial r} = \frac{a(T)^2}{\frac{\partial T}{\partial t}}\frac{\partial R}{\partial t}\frac{\partial R}{\partial r} = \frac{2}{9}\frac{2r^2\frac{\partial T}{\partial r} - 3rT}{T^2}. \tag{2.21}$$

Thus,

$$\frac{\partial T}{\partial r} = \frac{-6Tr}{9T^2 - 4r^2} \tag{2.22}$$

To find the new components of the metric, let us use another consequence of (1.2)

$$g_{tt} = \left(\frac{\partial T}{\partial t}\right)^2 g_{TT} + \left(\frac{\partial R}{\partial t}\right)^2 g_{RR} \tag{2.23}$$

$$g_{rr} = \left(\frac{\partial T}{\partial r}\right)^2 g_{TT} + \left(\frac{\partial R}{\partial r}\right)^2 g_{RR}, \tag{2.24}$$

and (2.19), (2.20), (2.22), following that

$$g_{tt} = -\left(\frac{\partial T}{\partial t}\right)^2 \left[\frac{T^2 - 4r^2}{9T^2}\right] \tag{2.25}$$

$$g_{rr} = \frac{9T^2}{T^2 - 4r^2}. \tag{2.26}$$

From (2.17), we note that

$$\frac{\partial T}{\partial t} = \frac{\partial T}{\partial \rho_d}\frac{\partial \rho_d}{\partial t} = -\frac{\rho_d^{-3/2}}{\sqrt{3\kappa}}\frac{\partial \rho_d}{\partial t}, \tag{2.27}$$

and as

$$T = \sqrt{\frac{4}{3\kappa\rho_d}}, \tag{2.28}$$

the equations (2.25) and (2.26) can be expressed in terms of $\rho_d$

$$g_{tt} = -\left(\frac{\partial\rho_d}{\partial t}\right)^2 \left[\frac{1 - \frac{\kappa\rho_d r^2}{3}}{3\kappa\rho_d^3}\right] \qquad g_{rr} = \left(1 - \frac{\kappa\rho_d r^2}{3}\right)^{-1}, \tag{2.29}$$

so the new spherically symmetric metric is of the form

$$\mathrm{d}s^2 = -\frac{(\partial_t\rho_d)^2}{3\kappa\rho_d^3}\left[1 - \frac{\kappa\rho_d r^2}{3}\right]\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1 - \frac{\kappa\rho_d r^2}{3}} + r^2\,\mathrm{d}\Omega^2. \tag{2.30}$$

However, using (2.17) y (2.22) we get

$$\frac{\partial\rho_d}{\partial r} = \frac{\kappa\rho_d^2 r}{1 - \frac{\kappa\rho_d}{3}r^2}. \tag{2.31}$$

Defining $\tilde{\rho}_d \equiv \kappa\rho_d$, last expression becomes

$$\frac{\partial\tilde{\rho}_d}{\partial r} = \frac{\tilde{\rho}_d^2 r}{1 - \frac{\tilde{\rho}_d}{3}r^2}. \tag{2.32}$$

From (2.32), it can be formed conveniently the next expression

$$\frac{\partial}{\partial r}\left[\frac{6 + r^2\tilde{\rho}_d}{\tilde{\rho}_d^{1/3}}\right] = 0 \tag{2.33}$$

Thereby,

$$\frac{6 + r^2\tilde{\rho}_d}{\tilde{\rho}_d^{1/3}} = C(t), \tag{2.34}$$

where $C(t)$ is constant respect to $r$ but not with respect to $t$.

Due to dimensional analysis, we note that in natural units $[\tilde{\rho}_d] = L^{-2}$ and, hence, $[C(t)] = L^{2/3}$. As there is no other parameter involved apart of $t$ and, besides, in natural units $[t] = L$, then we can set $C(t) = At^{2/3}$, where $A$ is an arbitrary constant.

Moreover, it is expected that at a later stage dust will be diluted homogeneously. This implies that for $t \to \infty$, the metric (2.30) is almost Minkowskian. Therefore,

$$\lim_{t\to\infty(\rho_d\to 0)} \frac{(\partial_t \rho_d)^2}{3\kappa\rho_d^3} = 1. \tag{2.35}$$

Then, (2.34) becomes

$$\frac{6 + r^2\kappa\rho_d}{(\kappa\rho_d)^{1/3}} = At^{2/3}. \tag{2.36}$$

Taking time derivative of last equation and solving for $\partial_t(\kappa\rho_d)$

$$\partial_t(\kappa\rho_d) = -\frac{(\kappa\rho_d)^{4/3}}{1 - \frac{\kappa\rho_d}{3}r^2}\frac{A}{3t^{1/3}}. \tag{2.37}$$

Squaring, dividing by $3\kappa\rho_d^3$ and replacing the previous results it follows that

$$\frac{(\partial_t\kappa\rho_d)^2}{3\kappa\rho_d^3} = \frac{\kappa^2 A^3}{27(1 - \frac{\kappa\rho_d}{3}r^2)(6 + r^2\kappa\rho_d)}. \tag{2.38}$$

Applying the limit $\rho_d \to 0$ and canceling all the $\kappa^2$ we can find the value of $A$ in the Minkowskian limit,

$$\lim_{t\to\infty(\rho_d\to 0)} \frac{(\partial_t \rho_d)^2}{3\kappa\rho_d^3} = \frac{A^3}{27\cdot 6} = 1 \;\to\; A = 3\sqrt[3]{6}. \tag{2.39}$$

Replacing last expression into (2.38) gives

$$\frac{(\partial_t \rho_d)^2}{3\kappa\rho_d^3} = \frac{1}{(1 - \frac{\kappa\rho_d}{3}r^2)(1 + \frac{\kappa\rho_d}{6}r^2)}. \tag{2.40}$$

Therefore, the spherically symmetric spacetime which represent the same physical situation as (2.1) with the scale factor (2.17) is given by

$$\mathrm{d}s^2 = -\frac{\mathrm{d}t^2}{(1-\frac{\kappa\rho_d}{3}r^2)(1+\frac{\kappa\rho_d}{6}r^2)} + \frac{\mathrm{d}r^2}{1-\frac{\kappa\rho_d}{3}r^2} + r^2\mathrm{d}\Omega^2 \tag{2.41}$$

and it will be denoted as the **SSD metric**. From (2.17) and defining $\rho_{d0} \equiv \frac{4}{3\kappa T_0^2}$ the scale factor is

$$a(T) = a_0\left(\frac{\rho_{d0}}{\rho_d}\right)^{1/3}. \tag{2.42}$$

Thus, using (2.36) we can finally find the coordinate transformation $(t, r) \to (T, R)$ in terms of $\rho_d$ and $\rho_{d0}$:

$$t = \frac{\left[6 + (\kappa\rho_{d0})^{2/3}(\kappa\rho_d)^{1/3}R^2\right]^{3/2}}{9\sqrt{2\kappa\rho_d}} \tag{2.43a}$$

$$r = a_0\sqrt[3]{\frac{\rho_{d0}}{\rho_d}}R. \tag{2.43b}$$

Equations (2.41) and (2.43) absolutely agree with the results of [4].

### A kind of generalization: SS$\omega_i$ Case

One of the main objectives of this report is to find a generalization of the previous cases in order to study the propagation of gravitational waves in a unknown background that could be useful for other models of gravitation that requires another kind of components.

Starting from (1.15) and (1.16), we can find that for an arbitrary perfect fluid with equation of state $p_i = \omega_i\rho_i$ ($\omega_i \neq -1$), the scale factor and the energy density are given by

$$a(T) = a_0\left(\frac{T}{T_0}\right)^{\frac{2}{3(\omega_i+1)}} \qquad \rho_i = \frac{4}{3(\omega_i+1)^2\kappa T^2}. \tag{2.44}$$

Thus, the physical scenario is a Universe filled by this fluid, so the geometry of spacetime is given by FLRW metric (2.1) with scale factor (2.44). We are looking for a spherically symmetric metric, analogous to Schwarzschild in the case of vacuum, but now with a background filled only by the fluid.

As before, we note that angular coordinates transformation are trivial

$$r^2\,\mathrm{d}\Omega^2 \to a(T)^2 R^2\,\mathrm{d}\Omega^2,$$

so, at the moment, we have

$$\begin{cases} T(t,r) & = \text{unknown function} \\ R(t,r) & = \frac{r}{a(T)}. \end{cases} \tag{2.45}$$

Following the same steps as before, using

$$g_{\mu'\nu'} = \frac{\partial X^\mu}{\partial x^{\mu'}}\frac{\partial X^\nu}{\partial x^{\nu'}} g_{\mu\nu} \tag{2.46}$$

and the requirement that new metric must be diagonal, we obtain the relation

$$0 = \frac{\partial T}{\partial t}\frac{\partial T}{\partial r} g_{TT} + \frac{\partial R}{\partial t}\frac{\partial R}{\partial r} g_{RR}. \tag{2.47}$$

Partial derivatives read

$$\frac{\partial R}{\partial r} = -\frac{1}{3}\frac{2r\frac{\partial T}{\partial r} - 3T(\omega_i+1)}{a(T)(\omega_i+1)T} \tag{2.48}$$

$$\frac{\partial R}{\partial t} = -\frac{2}{3}\frac{r\frac{\partial T}{\partial t}}{a(T)(\omega_i+1)T}. \tag{2.49}$$

From (2.47) we find that

$$\frac{\partial T}{\partial r} = \frac{a(T)^2}{\frac{\partial T}{\partial t}}\frac{\partial R}{\partial t}\frac{\partial R}{\partial r}. \tag{2.50}$$

Solving for $\frac{\partial T}{\partial r}$ from last equation gives

$$\frac{\partial T}{\partial r} = \frac{6rT(\omega_i + 1)}{4r^2 - 9(\omega_i + 1)^2 T^2}. \tag{2.51}$$

To find the components of new metric let us use (2.46)

$$\begin{aligned} g_{tt} &= \left(\frac{\partial T}{\partial t}\right)^2 g_{TT} + \left(\frac{\partial R}{\partial t}\right)^2 g_{RR} \\ g_{rr} &= \left(\frac{\partial T}{\partial r}\right)^2 g_{TT} + \left(\frac{\partial R}{\partial r}\right)^2 g_{RR}, \end{aligned}$$

added to (2.49), (2.48) and (2.51),

$$g_{tt} = -\left(\frac{\partial T}{\partial t}\right)^2 \left[\frac{9(\omega_i + 1)^2 T^2 - 4r^2}{9(\omega_i + 1)^2 T^2}\right] \tag{2.52}$$

$$g_{rr} = \frac{9(\omega_i + 1)^2 T^2}{9(\omega_i + 1)^2 T^2 - 4r^2}. \tag{2.53}$$

Definition of density in (2.44) implies that

$$\frac{\partial \rho_i}{\partial t} = \frac{\partial \rho_i}{\partial T}\frac{\partial T}{\partial t} \qquad T = \frac{2}{3(\omega_i + 1)}\sqrt{\frac{3}{\kappa \rho_i}} \tag{2.54}$$

Hence, the equations (2.52) and (2.53) now are expressed in terms of $\rho_i$,

$$g_{tt} = -\left(\frac{\partial \rho_i}{\partial t}\right)^2 \left[\frac{1 - \dfrac{\kappa \rho_i r^2}{3}}{3\kappa \rho_i^3 (\omega_i + 1)^2}\right] \qquad g_{rr} = \left(1 - \frac{\kappa \rho_i r^2}{3}\right)^{-1}. \tag{2.55}$$

New metric reads

$$\mathrm{d}s^2 = -\frac{(\partial_t \rho_i)^2}{3\kappa \rho_i^3 (\omega_i + 1)^2}\left[1 - \frac{\kappa \rho_i r^2}{3}\right] \mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1 - \frac{\kappa \rho_i r^2}{3}} + r^2\, \mathrm{d}\Omega^2. \tag{2.56}$$

Moreover, using (2.44) and (2.51) we get

$$\frac{\partial \rho_i}{\partial r} = \frac{(\omega_i + 1)\kappa\rho_i^2 r}{1 - \frac{\kappa\rho_i}{3}r^2}. \tag{2.57}$$

If we properly define $\tilde{\rho}_i \equiv \kappa\rho_i$, last expression becomes

$$\frac{\partial \tilde{\rho}_i}{\partial r} = \frac{(\omega_i + 1)\tilde{\rho}_i^2 r}{1 - \frac{\tilde{\rho}_i}{3}r^2}. \tag{2.58}$$

As in the case of dust, it can be noticed that from (2.58) it follows

$$\frac{\partial}{\partial r}\left[\frac{c + r^2\tilde{\rho}_i}{\tilde{\rho}_i^n}\right] = 0, \tag{2.59}$$

where $c$ and $n$ are unknown constants that we suppose exist due to previous cases. Let us find them

$$\begin{aligned}
\frac{\partial}{\partial r}\left[\frac{c + r^2\tilde{\rho}_i}{\tilde{\rho}_i^n}\right] &= \tilde{\rho}_i{}^{-(1+n)}\left[2r\tilde{\rho}_i{}^2 - \left(cn + (n-1)r^2\tilde{\rho}_i\right)\frac{\partial \tilde{\rho}_i}{\partial r}\right] \\
&= \tilde{\rho}_i{}^{-(1+n)}\left[2r\tilde{\rho}_i{}^2 - 3\frac{\left(cn + (n-1)r^2\tilde{\rho}_i\right)\tilde{\rho}_i r(\omega_i + 1)}{3 - r^2\tilde{\rho}_i}\right] \\
&= \frac{\tilde{\rho}_i{}^{-(1+n)}}{3 - r^2\tilde{\rho}_i}\left[2r\tilde{\rho}_i{}^2(3 - r^2\tilde{\rho}_i) - 3(cn + (n-1)r^2\tilde{\rho}_i)r\rho_i(\omega_i + 1)\right] \\
&= \frac{\tilde{\rho}_i{}^{-(1+n)}}{3 - r^2\tilde{\rho}_i}\left[r\tilde{\rho}_i{}^2\{6 - 3cn(\omega_i + 1)\} - r^3\tilde{\rho}_i{}^3\{2 + 3(n-1)(\omega_i + 1)\}\right] = 0.
\end{aligned}$$

As $r$ is a independent variable, we can find a system of equations

$$6 - 3cn(\omega_i + 1) = 0 \tag{2.60}$$

$$2 + 3(n-1)(\omega_i + 1) = 0, \tag{2.61}$$

whose solution is given by

$$c = \frac{6}{3\omega_i + 1} \qquad n = \frac{3\omega_i + 1}{3(\omega_i + 1)}. \tag{2.62}$$

Therefore,

$$\frac{c + r^2\tilde{\rho}_i}{\tilde{\rho}_i^n} = F(t), \tag{2.63}$$

where $F(t)$ is a function only with respect to $r$. By the same dimensional analysis realized in previous section, we note that in natural units $[\tilde{\rho}_i] = L^{-2}$ and thereby $[F(t)] = L^{2n}$. As there is no other parameter involved apart from $t$ and also as $[t] = L$ in natural units, then we set $F(t) = At^{2n}$, with $A$ as a dimensionless arbitrary constant.

For any fluid we can expect that at later stage it will be diluted homogeneously, which implies that for $t \to \infty$ the metric (2.56) is almost flat. Then,

$$\lim_{t\to\infty(\rho_i\to 0)} \frac{(\partial_t\rho_i)^2}{3\kappa\rho_i^3(\omega_i+1)^2} = 1. \tag{2.64}$$

On the other hand, (2.63) can be written as

$$\frac{c + r^2\kappa\rho_i}{(\kappa\rho_i)^n} = At^{2n}. \tag{2.65}$$

Taking derivative with respect to $t$ and solving for $\partial_t\rho_i$,

$$\frac{\partial\rho_i}{\partial t} = -\frac{2nAt^{2n-1}(\kappa\rho_i)^n\rho_i}{\kappa\rho_i n r^2 - r^2\kappa\rho_i + cn}. \tag{2.66}$$

Squaring, dividing by $3\kappa\rho_i^3$ and replacing the previous results it follows that

$$\frac{(\partial_t\rho_i)^2}{3\kappa\rho_i^3(\omega_i+1)^2} = \frac{4n^2A^{1/n}(\kappa r^2\rho_i + c)^{\frac{2n-1}{n}}}{3(\omega_i+1)^2[(n-1)\kappa r^2\rho_i + cn]}. \tag{2.67}$$

Applying the limit $\rho_i \to 0$ we can set the value of $A$

$$\lim_{t\to\infty(\rho_i\to 0)} \frac{(\partial_t\rho_i)^2}{3\kappa\rho_i^3(\omega_i+1)^2} = \frac{4n^2A^{1/n}c^{\frac{2n-1}{n}}}{3(\omega_i+1)^2(cn)^2} = 1 \;\to\; A = c\left(\frac{3}{4}\right)^n(\omega_i+1)^{2n}. \tag{2.68}$$

Replacing the last equation into (2.67) gives us

$$\begin{aligned}
\frac{(\partial_t \rho_i)^2}{3\kappa\rho_i^3(\omega_i+1)^2} &= \frac{4(nc)^2\left[\frac{(4/3)^n(r^2\kappa\rho_i+c)(\omega_i+1)^{-2n}(\kappa\rho_i)^{-n}}{c}\right]^{\frac{2n-1}{n}}(\kappa\rho_i)^{2n-1}(9/16)^n(\omega_i+1)^{4n-2}}{3(\kappa\rho_i r^2(n-1)+cn)^2} \\
&= \frac{(cn)^2\left(\frac{r^2\kappa\rho_i}{c}+1\right)^{\frac{2n-1}{n}}}{(\kappa\rho_i r^2(n-1)+cn)^2} \\
&= \frac{\left(1+\frac{\kappa\rho_i r^2}{c}\right)^{\frac{2n-1}{n}}}{\left(1-\frac{\kappa\rho_i r^2(1-n)}{cn}\right)^2} \quad \text{but} \quad \frac{2n-1}{n} = \frac{3\omega_i-1}{3\omega_i-1} \quad \frac{1-n}{cn} = \frac{1}{3} \\
&= \frac{1}{\left(1+\frac{\kappa\rho_i r^2(3\omega_i+1)}{6}\right)^{\frac{1-3\omega_i}{1+3\omega_i}}\left(1-\frac{\kappa\rho_i r^2}{3}\right)^2}.
\end{aligned}$$

Finally, the spherically symmetric spacetime has the following structure:

$$\mathrm{d}s^2 = -\frac{\mathrm{d}t^2}{\left(1-\frac{\kappa\rho_i r^2}{3}\right)\left(1+\frac{\kappa\rho_i r^2(3\omega_i+1)}{6}\right)^{\frac{1-3\omega_i}{1+3\omega_i}}} + \frac{\mathrm{d}r^2}{1-\frac{\kappa\rho_i r^2}{3}} + r^2\mathrm{d}\Omega^2. \tag{2.69}$$

Using (2.44) and (1.17), scale factor becomes $a(T) = \left(\frac{\rho_{d0}}{\rho_i}\right)^{1/3(\omega_i+1)}$. Finally, from (2.54) we can find the coordinate transformation in terms of $\rho_i$ y $\rho_0$:

$$t = \frac{\left[c+R^2(\kappa\rho_0)^{\frac{2}{3(\omega_i+1)}}(\kappa\rho_i)^{\frac{3\omega_i+1}{3(\omega_i+1)}}\right]^{\frac{1}{2n}}}{\left(A^{\frac{1}{2n}}\right)\sqrt{\kappa\rho_i}} \tag{2.70a}$$

$$r = \left(\frac{\rho_0}{\rho_i}\right)^{\frac{1}{3(\omega_i+1)}} R. \tag{2.70b}$$

Therefore, the metric (2.69) is solution of EFE (1.4) with an arbitrary fluid as background and (2.70) provide us the coordinate transformation with FLRW metric.

## 2.3. Linearized equations

From the discussion in section 1.4, and due to Gravitational Waves are a solution of linearized Einstein Field Equations (1.21), it will be necessary to linearize the metrics and coordinate transformations obtained in the previous section.

### SdS spacetime

Directly from the restricted SdS metric (2.2) we can note that for $\Lambda r^2 \ll 1$ the linearized SdS spacetime will be

$$\mathrm{d}s^2 = -\left(1 - \frac{\Lambda}{3}r^2\right)\mathrm{d}t^2 + \left(1 + \frac{\Lambda}{3}r^2\right)\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2. \tag{2.71}$$

Moreover, from (1.18) we can expand the scale factor at second order

$$a(T) = a_0\left[1 + \Delta T\sqrt{\frac{\Lambda}{3}} + \Delta T^2\frac{\Lambda}{6}\right]R + \mathcal{O}\left(\Lambda^2\right). \tag{2.72}$$

Thereby, we can find the linearized coordinate transformations from (2.16a) and (2.16b)

$$t(T,R) = T + a_0^2\left(\frac{R^2}{2}\sqrt{\frac{\Lambda}{3}} + R^2\Delta T\frac{\Lambda}{3}\right) + \mathcal{O}\left(\Lambda^2\right) \tag{2.73a}$$

$$r(T,R) = a_0\left[1 + \Delta T\sqrt{\frac{\Lambda}{3}} + \Delta T^2\frac{\Lambda}{6}\right]R + \mathcal{O}\left(\Lambda^2\right). \tag{2.73b}$$

Even more, from (2.72), it is possible to find a linearized FLRW metric

$$\mathrm{d}s^2 = -\mathrm{d}T^2 + a_0^2\left(1 + 2\sqrt{\frac{\Lambda}{3}}\Delta T + \frac{2\Lambda}{3}\Delta T^2\right)\left[\mathrm{d}R^2 + R^2\mathrm{d}\Omega^2\right] + \mathcal{O}\left(\Lambda^2\right). \tag{2.74}$$

Hence, the equations (2.71), (2.73a) and (2.73b) will be useful in the discussion of GW propagation.

### SS$\omega_i$ spacetime

The geometry of a spacetime filled by a perfect fluid with equation of state (1.12) is given by (2.69). For $\kappa\rho_i r^2 \ll 1$ the Taylor series expansion becomes

$$\mathrm{d}s^2 = -\left[1 + \frac{\kappa\rho_i r^2}{6}(3\omega_i + 1)\right]\mathrm{d}t^2 + \left[1 + \frac{\kappa\rho_i r^2}{3}\right]\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2. \tag{2.75}$$

Furthermore, the correspondingly coordinate transformations between this metric and FLRW are of the form

$$t = \frac{\left[c + R^2(\kappa\rho_0)^{\frac{2}{3(\omega_i+1)}}(\kappa\rho_i)^{\frac{3\omega_i+1}{3(\omega_i+1)}}\right]^{\frac{1}{2n}}}{\left(A^{\frac{1}{2n}}\right)\sqrt{\kappa\rho_i}} \tag{2.76a}$$

$$r = \left(\frac{\rho_0}{\rho_i}\right)^{\frac{1}{3(\omega_i+1)}} R, \tag{2.76b}$$

where

$$c = \frac{6}{3\omega_i + 1} \qquad n = \frac{3\omega_i + 1}{3(\omega_i + 1)} \to \frac{1}{2n} = \frac{3(\omega_i + 1)}{2(3\omega_i + 1)} \tag{2.77}$$

$$A = c\left(\frac{3}{4}\right)^n (\omega_i + 1)^{2n} \to A^{\frac{1}{2n}} = A = \sqrt{\frac{3}{4}}\left(\frac{6}{3\omega_i + 1}\right)^{\frac{3(\omega_i+1)}{2(3\omega_i+1)}}(\omega_i + 1). \tag{2.78}$$

We can easily linearize $r$ coordinate in terms of $\Delta T = T - T_0$ using (2.44)

$$\begin{aligned}
r &= R\left(\frac{\rho_0}{\rho_i}\right)^{\frac{1}{3(\omega_i+1)}} = R\left(\frac{T}{T_0}\right)^{\frac{2}{3(\omega_i+1)}} = R\left(1 + \frac{\Delta T}{T_0}\right)^{\frac{2}{3(\omega_i+1)}} \\
&= R\left(1 + \frac{2}{3(\omega_i+1)T_0}\Delta T - \frac{3\omega_i + 1}{9(\omega_i+1)^2 T_0^2}\Delta T^2\right) + \mathcal{O}\left(\Delta T^3\right) \qquad T_0 = \frac{1}{(\omega_i + 1)}\sqrt{\frac{4}{3\kappa\rho_0}} \\
&= R\left(1 + \Delta T\sqrt{\frac{\kappa\rho_0}{3}} - \frac{\Delta T^2}{12}(3\omega_i + 1)\kappa\rho_0\right) + \mathcal{O}\left(\Delta T^3\right).
\end{aligned} \tag{2.79}$$

On the other hand, for $t$ in (2.70a) firstly we have to expand $\kappa\rho_i r^2 \ll 1$ and then linearize in terms of $\Delta T$, obtaining

$$
\begin{aligned}
t &\approx T_0 + \frac{R^2}{3(\omega_i+1)}\frac{1}{T_0} + \left[1 - \frac{R^2}{3(\omega_i+1)}\frac{3\omega_i - 1}{3(\omega_i+1)T_0^2}\right]\Delta T + \dots \qquad T_0 = \frac{1}{(\omega_i+1)}\sqrt{\frac{4}{3\kappa\rho_0}} \\
&\approx T + \frac{R^2}{2}\sqrt{\frac{\kappa\rho_0}{3}} + \frac{R^2}{12}(1-3\omega_i)\kappa\rho_0\Delta T + \dots
\end{aligned} \tag{2.80}
$$

Thus, the linearized coordinate transformation between FLRW and SS$\omega_i$ spacetimes for an arbitrary perfect fluid with equation of state $p_i = \omega_i\rho_i$ are given by

$$
t = T + \frac{R^2}{2}\sqrt{\frac{\kappa\rho_0}{3}} + \frac{R^2}{12}(1-3\omega_i)\kappa\rho_0\Delta T + \mathcal{O}\big((\kappa\rho_0)^2\big) \tag{2.81a}
$$

$$
r = R\left(1 + \Delta T\sqrt{\frac{\kappa\rho_0}{3}} - \frac{\Delta T^2}{12}(1+3\omega_i)\kappa\rho_0\right) + \mathcal{O}\big((\kappa\rho_0)^2\big). \tag{2.81b}
$$

These will be the relevant expressions for GW analysis in chapter 3. It can be noted that at first order, the coordinate transformation has the same functional behavior of SdS transformations (2.16).

### SSD spacetime

If in the SS$\omega_i$ spacetime we set $\omega_i = 0$ the solutions for a SSD spacetime should be reproduced. The equation (2.75), by setting $\omega_i = 0$, becomes

$$
\mathrm{d}s^2 = -\left(1 + \frac{\kappa\rho_d r^2}{6}\right)\mathrm{d}t^2 + \left(1 + \frac{\kappa\rho_d r^2}{3}\right)\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2. \tag{2.82}
$$

Additionally, the coordinate transformations (2.81a) and (2.81b), by setting $\omega_i = 0$, become

$$
t = T + \frac{R^2}{2}\sqrt{\frac{\kappa\rho_{d0}}{3}} + \frac{R^2}{12}\kappa\rho_{d0}\Delta T + \mathcal{O}\big((\kappa\rho_0)^2\big) \tag{2.83a}
$$

$$
r = R\left(1 + \Delta T\sqrt{\frac{\kappa\rho_{d0}}{3}} - \frac{\Delta T^2}{12}\kappa\rho_{d0}\right) + \mathcal{O}\big((\kappa\rho_0)^2\big). \tag{2.83b}
$$

These results are important because they perfectly agree with the metric and linearized coordinate transformation between FLRW and SSD spacetime found in [4], so it confirms that the generalization works, at least for this case. In fact, one can note that the generalization also "works" for the case $\omega_i = -1$, because it gives the same functional behavior that SdS linear equations (2.71), (2.73a) and (2.73b). Nevertheless, starting the demonstration we have assumed that $\omega_i \neq -1$, reason why we do not consider this case as a validation.

### Full $\Lambda$CDM spacetime

Next, we will consider a composite spacetime. It will be filled by Dark Energy (due to a cosmological constant contribution, $\omega_i = -1$), non-relativistic matter (baryonic and Dark Matter, $\omega_i = 0$) and radiation (photons and neutrinos, $\omega_i = 1/3$). This spacetime is the basis of $\Lambda$CDM model and modern cosmology.

First of all, it is necessary to obtain a full scale factor for the FLRW spacetime, which will be the solution of 1st Friedmann Equation

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{\kappa}{3}(\rho_d + \rho_\Lambda + \rho_r) = \frac{\kappa}{3}\left(\rho_{d0}\left(\frac{a_0}{a(T)}\right)^3 + \rho_{r0}\left(\frac{a_0}{a(T)}\right)^4 + \rho_\Lambda\right). \tag{2.84}$$

It is difficult to find an analytical solution. However, we can solve it by a series method, at least up to first order in $\Delta T$. Let us use the convention $a_0 = 1$ and the ansatz

$$a(T) = 1 + \varepsilon\Delta T + \mathcal{O}\Big((\Delta T)^2\Big). \tag{2.85}$$

Inserting last expression into (2.84) gives

$$\frac{\varepsilon^2}{(1+\varepsilon\Delta T)^2} = \frac{\kappa}{3}\left(\frac{\Lambda}{\kappa} + \frac{\rho_{d0}}{(1+\varepsilon\Delta T)^3} + \frac{\rho_{r0}}{(1+\varepsilon\Delta T)^4}\right). \tag{2.86}$$

Verifying the zero order at $\Delta T$ we find the relation

$$\varepsilon^2 = \frac{\kappa}{3}\left(\frac{\Lambda}{\kappa} + \rho_{d0} + \rho_{r0}\right) \to \varepsilon = \sqrt{\frac{\Lambda + \kappa\rho_{d0} + \kappa\rho_{r0}}{3}}. \tag{2.87}$$

Thus, the scale factor a first order in $\sqrt{\Lambda}$ is of the form

$$a(T) = 1 + \sqrt{\frac{\Lambda + \kappa\rho_{d0} + \kappa\rho_{r0}}{3}}\Delta T + \mathcal{O}(\Lambda). \tag{2.88}$$

Actually, it is easy to show that for a effective energy density of the form

$$\rho_{\text{eff}}(T) = \sum_{i=1}^{n} \rho_i(T), \tag{2.89}$$

where $\rho_i(T)$ are arbitrary functions of $T$ and full scale factor at first order in $\Delta T$ is given by

$$a(T) = 1 + \Delta T \sqrt{\frac{\kappa}{3}\sum_{i=1}^{n} \rho_{i0}} \qquad \rho_{i0} = \rho_i(T_0). \tag{2.90}$$

Then, we can take advantage of additivity under the square root in the scale factor. Let us take root term as the leading term. Then, from (2.88), (2.81a) and (2.81b), it follows that

$$t = T + \frac{R^2}{2}\sqrt{\frac{\Lambda + \kappa\rho_{r0} + \kappa\rho_{d0}}{3}} + \mathcal{O}(\Lambda) \tag{2.91a}$$

$$r = R\left(1 + \Delta T\sqrt{\frac{\Lambda + \kappa\rho_{r0} + \kappa\rho_{d0}}{3}}\right) + \mathcal{O}(\Lambda). \tag{2.91b}$$

These transformations show that limits for individual fluid are recovered for each respective $\omega_i$: All they satisfy (2.81a) and (2.81b) by itself. Furthermore, from linear additivity of metrics we can construct the full linearized spacetime metric of ΛCDM model

$$\mathrm{d}s^2 = -\left[1 + \frac{\kappa\rho_d r^2}{6} + \frac{\kappa\rho_r r^2}{3} - \frac{\Lambda}{3}r^2\right]\mathrm{d}t^2 + \left[1 + \frac{\kappa\rho_d r^2}{3} + \frac{\kappa\rho_r r^2}{3} + \frac{\Lambda}{3}r^2\right]\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2. \tag{2.92}$$

### 2.4. Relevance for Gravitational Waves

The only remaining piece is the connection between previously derived coordinate transformations and propagation of GW. Using (2.91a) and (2.91b), the harmonic solution of propagating GW (1.31) is expressed in the new coordinates as follows

$$h'^{(\mathrm{GW})}_{\mu\nu} = \frac{1}{R}\left(1 - R\sqrt{\frac{\Lambda + \kappa\rho_{r0} + \kappa\rho_{d0}}{3}}\right)\left(E'_{\mu\nu}\cos\left[w_{\mathrm{eff}}T - k_{\mathrm{eff}}R\right] + D'_{\mu\nu}\sin\left[w_{\mathrm{eff}}T - k_{\mathrm{eff}}R\right]\right), \tag{2.93}$$

where $w_{\mathrm{eff}}$ and $k_{\mathrm{eff}}$ are defined as

$$w_{\mathrm{eff}} \equiv \Omega\left(1 - R\sqrt{\frac{\Lambda + \kappa\rho_{r0} + \kappa\rho_{d0}}{3}}\right) \qquad k_{\mathrm{eff}} \equiv \Omega\left(1 - \frac{R}{2}\sqrt{\frac{\Lambda + \kappa\rho_{r0} + \kappa\rho_{d0}}{3}}\right). \tag{2.94}$$

This result is important: Not only Doppler effect is reproduced in $w_{\mathrm{eff}}$. Also wave number is affected by Universe expansion, which is a phenomenon not considered previously [4].

# 3. Possible measurements using PTA

The main results of previous chapter, i.e. equations (2.93) and (2.94), show that perturbations of spacetime due to gravitational waves, described from Earth using $(T, R)$ coordinates, carry cosmological information if the source is at a large distance enough.

In this chapter it will be shown that this information can be observationally measured. For this, we will use the light coming from a pulsar[1] and the shift in timing arrival of the pulse due to a gravitational wave interaction. We will see this timing residual could be measured by Pulsar Timing Array (PTA) projects in nearly future, so equations developed previously configure potential scientific predictions.

## 3.1. One pulsar configuration

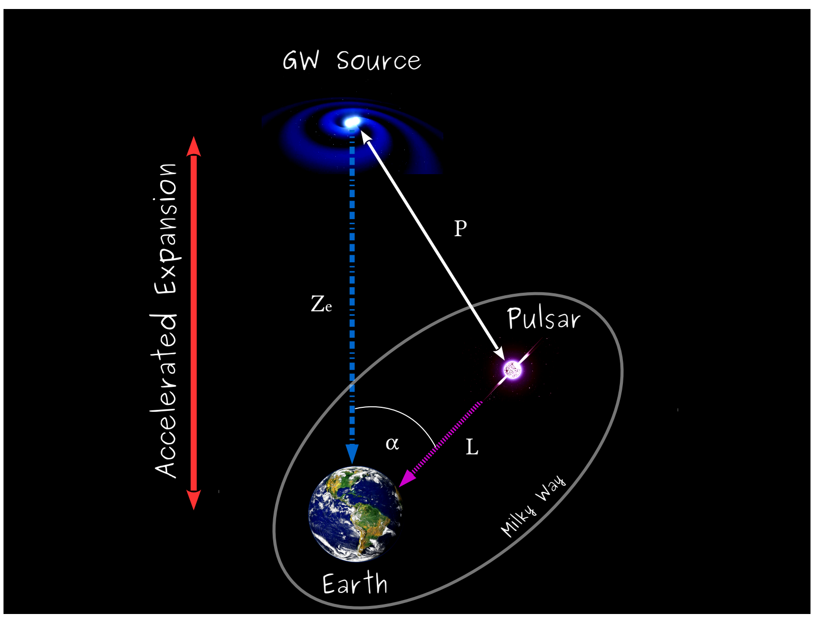


FIGURE 3.1. Setup of the relative situation of a GW source ($R = 0$), Earth ($Z = Z_e$) and a nearby Pulsar located at $\mathbf{P} = (P_X, P_Y, P_Z)$ referred to the source. The Z direction is chosen to be defined by the source-Earth axis. Polar and azimuthal angles are $\alpha$ and $\beta$ respectively, from Z axis. (Self-elaborated image).

Let us consider the set up in Figure 3.1 with a GW source, Earth and a nearby Pulsar. The Earth-pulsar distance is $L$ and the Earth-GW Source distance is $Z_e$. Earth and pulsar

[1]From **pul**sating st**ar**: A magnetized rotating neutron star or white dwarf that emits a beam of radiation.

are gravitationally bounded to Milky Way, so they do not feel Universe expansion. However, GW source and the system Earth-pulsar are not bounded, so they do feel accelerated expansion and, therefore, the discussion and results of previous chapter can be applied.

The Pulsar emits light with a particular electromagnetic field. Denote the time-dependent phase of this field at the pulsar as $\phi_0$. Then, the phase of the electromagnetic field measured from Earth is given by

$$\phi(T) = \phi_0 \left[T - \frac{L}{c} - \tau_0(T) - \tau_{\text{GW}}(T)\right] \tag{3.1}$$

where $c$ the speed of light[2], $\tau_0(T)$ is the timing correction associated to the spacial motion of the Earth respect to the Solar system (which will not be considered) and $\tau_{\text{GW}}(T)$ is the timing correction due to Gravitational Wave influence.

### 3.2. Timing residual due to GW for one pulsar

It can be shown [27, 28] that the last correction of (3.1) is given by,

$$\tau_{\text{GW}}(T) = -\frac{1}{2}\hat{n}^i\hat{n}^j H_{ij}(T) \tag{3.2}$$

where $\hat{n} = (-\sin\alpha\cos\beta, -\sin\alpha\cos\beta, \cos\alpha)$ is a unit vector pointing from Earth towards the pulsar and $H_{ij}$ is the integral of the transverse-traceless metric perturbation along the null geodesic in the path pulsar-Earth, which could be parametrized by

$$\mathbf{R}(x) = \mathbf{P} + L(1+x)\hat{n} \qquad \text{with } x \in [-1, 0].$$

Under this parametrization, $H_{ij}(T)$ takes the following form

$$H_{ij}(T) = \frac{L}{c}\int_{-1}^{0} h'^{(\text{GW})}_{\mu\nu}\left(T + \frac{L}{c}x, |\mathbf{R}(x)|\right) \mathrm{d}x. \tag{3.3}$$

[2]From now we will recover original units.

Note that in our framework, GW source is far from Earth, then we can assume $\frac{L}{Ze} \ll 1$ and then

$$
\begin{aligned}
\mathbf{R}(x) &= \mathbf{P} + L(1+x)\hat{n} = (-xL\sin\alpha\cos\beta, -xL\sin\alpha\sin\beta, Z_e + xL\cos\alpha) \\
|\mathbf{R}(x)| &= (Z_e^2 + 2xLZ_e\cos\alpha + x^2L^2)^{1/2} = Z_e\left(1 + 2\frac{xL}{Z_e}\cos\alpha + \frac{x^2L^2}{Z_e^2}\right)^{1/2} \\
&\approx Z_e\left(1 + 2\frac{xL}{Z_e}\cos\alpha\right)^{1/2} \approx Z_e\left(1 + \frac{xL}{Z_e}\cos\alpha\right) = Z_e + xL\cos\alpha.
\end{aligned}
$$

Thus, we have to evaluate $h'^{(\mathrm{GW})}_{\mu\nu}(T, R)$ from (2.93), with

$$
T \to T_e + \frac{L}{c}x \qquad R \to Z_e + xL\cos\alpha,
$$

where $T_e = \frac{Z_e}{c}$. Now we can contract the equation (3.2) using (3.3). However, in the TT-Lorenz gauge, for a GW propagating through the Z axis, the only non-zero values of $E'_{\mu\nu}$ and $D'_{\mu\nu}$ are in the X,Y components [1].

If, just for simplicity, we additionally assume that $\left|E'_{\mu\nu}\right| = \left|D'_{\mu\nu}\right| \equiv \varepsilon\ \forall\mu,\nu$, then for $\Lambda$CDM spacetime the timing residual is given by

$$
\begin{aligned}
\tau_{\mathrm{GW}}^{\Lambda\mathrm{CDM}} = &-\frac{L\varepsilon}{2c}(\sin^2\alpha\cos^2\beta + 2\sin^2\alpha\cos\beta\sin\beta - \sin^2\alpha\sin^2\beta) \\
&\times \int_{-1}^{0} \frac{1 - \sqrt{\frac{\Lambda+\kappa\rho_{d0}+\kappa\rho_{r0}}{3}}\left[T_e + \frac{xL}{c}\right]}{Z_e + xL\cos\alpha}[\cos\Theta(x) + \sin\Theta(x)]\mathrm{d}x,
\end{aligned} \tag{3.4}
$$

where

$$
\Theta(x) = \Omega\left(1 - \frac{Z_e + xL\cos\alpha}{c}\sqrt{\frac{\Lambda + \kappa\rho_{d0} + \kappa\rho_{r0}}{3}}\right)\left(T_e + \frac{xL}{c}\right) \tag{3.5}
$$

$$
- \Omega\left(1 - \frac{Z_e + xL\cos\alpha}{2c}\sqrt{\frac{\Lambda + \kappa\rho_{d0} + \kappa\rho_{r0}}{3}}\right)\left(\frac{Z_e + xL\cos\alpha}{c}\right). \tag{3.6}
$$

If we consider the simple case of a GW propagating in the same plane of the pulsar emission, then $\beta \to 0$ and $\tau_{\text{GW}}^{\Lambda\text{CDM}}$ takes the form

$$\tau_{\text{GW}}^{\Lambda\text{CDM}}(\alpha) = -\frac{L\varepsilon \sin^2 \alpha}{2c} \int_{-1}^{0} \frac{1 - \sqrt{\frac{\Lambda + \kappa\rho_{d0} + \kappa\rho_{r0}}{3}} \left[ T_e + \frac{xL}{c} \right]}{Z_e + xL \cos\alpha} [\cos\Theta(x) + \sin\Theta(x)] \mathrm{d}x, \tag{3.7}$$

while the case of propagation of GW through Minkowski spacetime ($\Lambda \to 0$) reduces to

$$\tau_{\text{GW}}^{\text{Mink}}(\alpha) = -\frac{L\varepsilon \sin^2 \alpha}{2c} \int_{-1}^{0} \frac{\cos[\Omega \left( T_e + \frac{xL}{c} - \frac{Z_e + xL\cos\alpha}{c} \right)]}{Z_e + xL \cos\alpha} \mathrm{d}x. \tag{3.8}$$

### 3.3. Numerical analysis of timing residual

To perform a numerical analysis we have to choice some reasonable values of the parameters in (3.7) in order to visualize the $\tau_{\text{GW}}$ behavior.

The setup described in Figure 3.1 can be modeled with the following set of values

| Parameters values | | |
|---|---|---|
| $Z_e$ | $3 \times 10^{24}$ m | $\sim 100000$ kpc |
| $T_e$ | $Z_e/c = 10^{16}$s | $\sim 300$ Myr |
| $L$ | $10^{19}$m | $\sim 1000$ ly |
| $\Omega$ | $10^{-8}$ rad$/s$ | |
| $\varepsilon$ | $1.2 \times 10^{9}$ m | |

TABLE 3.1. List of values considered for the parameters in the numerical analysis of timing residual $\tau_{\text{GW}}$. The same values will be used in all cases.

For the GW source, we choose a large distance $Z_e$, but we can note that it is not a cosmological distance near to Big Bang. On the other hand, the distance between Earth and pulsar is within the margin of a galactic scale. It can be seen that $L \ll Z_e$, as required.

The angular frequency is of the expected order for future PTA projects, in particular, for International Pulsar Timing Array. The same argument is used to fix $\varepsilon$, due to that it satisfies $|h| \sim \frac{\varepsilon}{R} \sim 10^{-15}$, where $|h|$ is the expected accuracy of PTA projects [29].

In the first place, we compare the results of Timing Residual in the Minkowski spacetime, with different distances for the GW source.

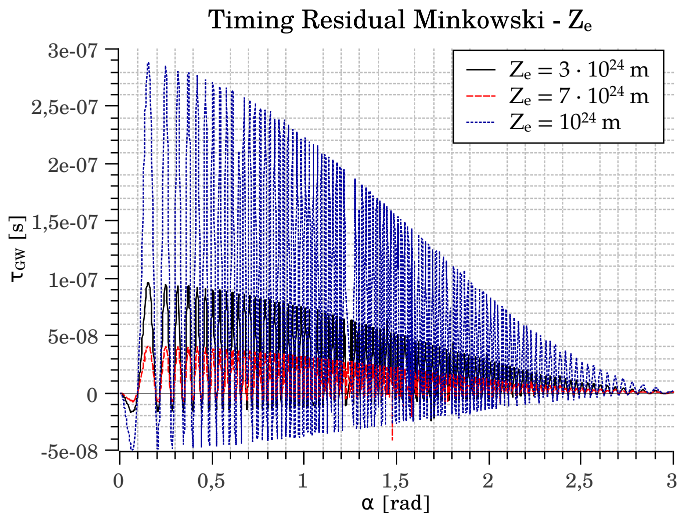


FIGURE 3.2. Raw timing residual for flat spacetime, with different distances of the GW source.

We can note that as the GW source moves away, the magnitude of the time residual decreases and vice versa. Varying now the angular frequency of the GW source we obtain.

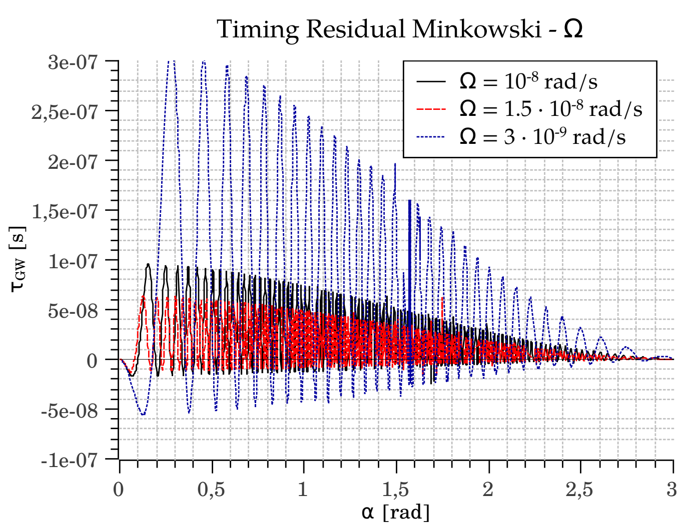


FIGURE 3.3. Raw timing residual for flat spacetime, with different angular frequencies of the GW source.

From the last two figures, we can notice a close relation between the properties of the GW source and the behavior of the Timing Residual of the pulsar emission. Moreover, if we now consider the SdS case ($\Lambda \neq 0$), the relation is even more evident.

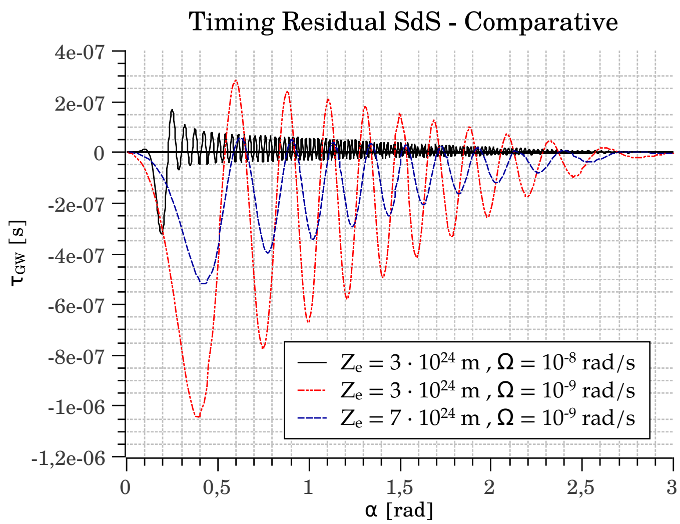


FIGURE 3.4. Raw timing residual for SdS spacetime, with different angular frequencies and distances of the GW source.

We can notice a notorious peak in the residual time. This peak will be observed in each case, with great differences between the kind of components present in the respective Universe.

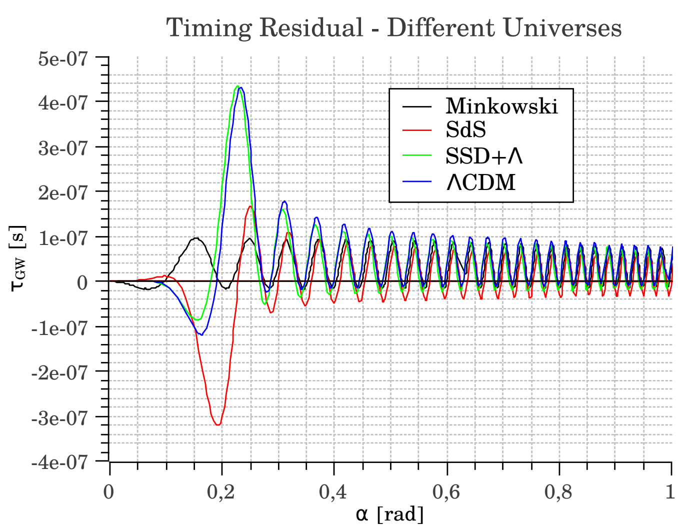


FIGURE 3.5. Raw timing residual for different spacetimes. The source–Earth distance $Z_e$ and the angular frequency $\Omega$ are fixed parameters, given by the table 3.1.

In the last figure we used the appropriated limits of $\Lambda$CDM model in the equation (3.7). For example, the SdS case means $\Lambda \neq 0$ and every other component equals to zero, the SSD case means $\rho_{d0} \neq 0$ and every other component is zero and so on.

The figure 3.5 shows us that the composition of the Universe in the present affects the behavior of timing residual. We can notice that the angular position of the peak, in which the timing residual is maximized, changes at each case. For instance, in the SdS case the peak is located at $\sim 0.19$ rad, while in the $\Lambda$CDM case the peak is located at $\sim 0.23$ rad.

Even more, the amplitude of the time residuals increases in a non depreciable percentage when we add matter into the effective density of the $\Lambda$CDM model. For example, the absolute value of the peak in the case SdS reaches $3.2 \cdot 10^{-7}$ s, while in the $\Lambda$CDM case reaches $4.3 \cdot 10^{-7}$ s, which means a 34% of difference between both cases.

From the discussion in [3], the angular position of the enhancement in the peak of timing residual that appears in the Figure 3.5 could be related with a stationary line of the wave in which the phase of GW remains almost constant along the trajectory from pulsar to Earth. Nevertheless, the nature of the peak and the influence of the cosmological parameters in the behavior of the timing residual is not fully understood yet. We are not able to give a complete qualitative explanation of the phenomenon due to the several factors involved in the measurement of timing residual coming from pulsars. Actually, is one of the missing pieces in the puzzle and it should be studied in the future research.

### 3.4. Using pulsars catalog

In order to make an analysis of the possible signal shown in the previous figures, we will use some pulsars from ATNF catalog [30]. As we know, pulsars are stable clocks whose periods are known with great accuracy. Assuming a modest precision of $\sigma_t = 9.6 \times 10^{-7} s \approx 10^{-6} s$. This value is not arbitrary, it is obtained by averaging the precision achieved of best pulsar in International PTA project [2].

We can define a *statistical significance* of the timing residual, which tell us how good a possible measurement could be if our hypothesis is correct, of the form

$$\sigma = \sqrt{\frac{1}{N_p N_t} \sum_{i=1}^{N_p} \sum_{j=1}^{N_t} \left( \frac{\tau_{\text{GW}}(T_e, L_i, \alpha_i, \beta_i, Z_e, \Omega, \varepsilon, \Lambda)}{\sigma_t} \right)^2}, \tag{3.9}$$

where index $i$ running from 1 to $N_p$ (number of pulsars averaged) and $j$ running from 1 to $N_t$ (number of observations). Assuming we perform measurements every 11 days through 3 years, then $N_t = 101$. The pulsars considered are shown in Table 3.2.

| Pulsars from ATNF Catalogue | $\phi$ | $L_i$ |
|---|---|---|
| J0024-7204E | $-44.89°$ | 4.69 kpc |
| J0024-7204D | $-44.88°$ | 4.69 kpc |
| J0024-7204M | $-44.89°$ | 4.69 kpc |
| J0024-7204G | $-44.89°$ | 4.69 kpc |
| J0024-7204I | $-44.88°$ | 4.69 kpc |

TABLE 3.2. List of pulsars averaged for an hypothetical source at angular separation $\alpha$. It is shown the data given in [30], where $\phi$ is the galactic latitude – transformed to $\beta_i$– and $L_i$ the distance between Earth and pulsar. We can note that this set simplify the computation of $\sigma$ because all pulsars are near each other.

We will keep $\alpha$ as a free parameter and suppose that an hypothetical GW source is located at $\alpha$ radians between Earth and pulsars. Thus, the statistical significance for this set of parameters is given by

$$\sigma(\alpha) = \sqrt{\frac{1}{5 \cdot 101} \sum_{i=1}^{5} \sum_{j=1}^{101} \left( \frac{\tau_{\text{GW}}(\beta_i)}{\sigma_t} \right)^2} \tag{3.10}$$

For the case of Minkowski flat spacetime we obtain

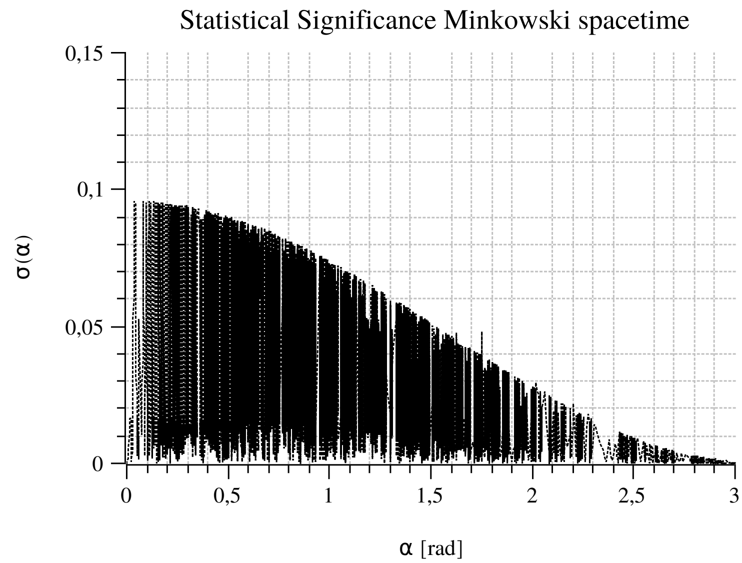


FIGURE 3.6. $\sigma(\alpha)$ in Minkowski spacetime, no relevant signal observed.

If we consider now the other components we get the following plot

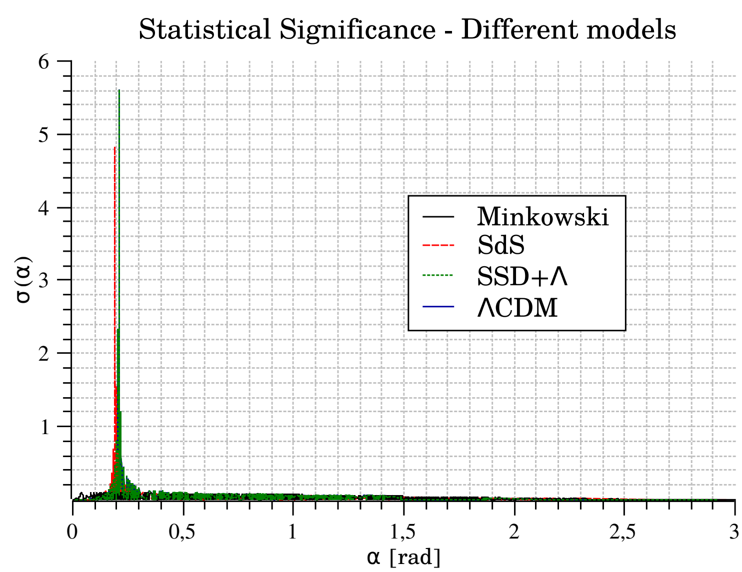


FIGURE 3.7. $\sigma(\alpha)$ in different spacetimes. A huge peak is observed near 0.2 rad. Green and blue curves are almost the same, due to SSD+$\Lambda$ and $\Lambda$CDM models are very close each other.

It is clearly noted that a huge peak of $4.8\sigma$ is observed approximately in $\alpha = 0.19$ rad in the SdS case and a peak of $5.6\sigma$ is observed in $\alpha = 0.21$ rad in the $\Lambda$CDM case, which tell us that the composition of the Universe could be involved in the angular position of the peak of the timing residual. We can note that as we incorporate elements into spacetime, the angle of peak slightly increases. Also, it is very important to note that for a full $\Lambda$CDM spacetime an hypothetical signal with significance of $5.3\sigma$ could be observed. However, this could be a very idealized situation.

Thus, we can develop a more realistic simulation. In the figure 3.7, only a cluster of 5 pulsars were considered and all of them were averaged at the same angle $\alpha$. However, in the actual catalog, many pulsars are present at different angles (in the galactic coordinates). We have considered 13 well distributed groups of 5 pulsars each (65 pulsar in total) and a gravitational wave source located at galactic coordinates $\theta_S = 20°$ and $\phi_S = 15°$. Then, we averaged them using the statistical significance given by

$$\sigma_k = \sqrt{\frac{1}{5 \cdot 101} \sum_{i=1}^{5_k} \sum_{j=1}^{101} \left( \frac{\tau_{\mathrm{GW}}(L_i, \alpha_i, \beta_i)}{10^{-6}} \right)^2} \tag{3.11}$$

and plot it as a function of the average angle of the group, $\bar{\sigma}_k = \sum_{i=1}^{5_k} \frac{\alpha_i}{5}$, as follows

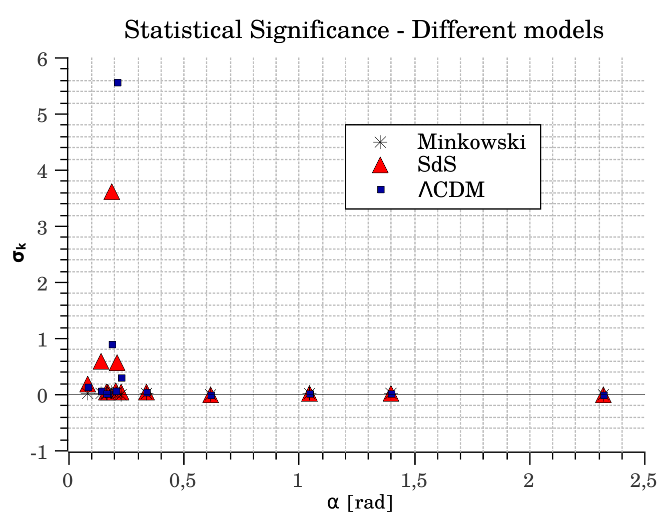


FIGURE 3.8. Statistical significance at different spacetimes using a more realistic set of pulsars. A huge peak is observed again, but both points are referred to a cluster of pulsar located near to the peak of maximum effect.

The last plot shows a very similar behavior of the statistical significance to the figure 3.7. There is a huge peak at certain angles for each kind of Universe (SdS or $\Lambda$CDM) and for the rest of angles the significance goes to almost zero. This means that if a gravitational wave signal is measured and a cluster of pulsars is located just at the angle in which the effect of timing residual is maximized, the significance of the observation could reach $5.5\sigma$ in the case of $\Lambda$CDM model. Obviously, this is a particular situation that not always could happen, so we have to be cautious about the expectation of getting positive and powerful results.

However, one can expect that eventually, if future PTA projects reach an enough accuracy and the catalog of pulsars increases their number, a potential signal of timing residual due to gravitational waves could be measured from Earth with a certain degree of statistical significance.

Moreover, in our proposed physical scenario, no peak is observed in Minkowski spacetime. Therefore, if a signal is measured, it could give us a notion of the cosmological components of the Universe at the present using only local measurements methods, which is a new approach of cosmological observations that should be considered in the future research.

# 4. Conclusions and outlook

A complete investigation line developed in [1–4] was reviewed. It was shown that there is an influence of accelerated expansion of Universe in the propagation of Gravitational Waves due to a coordinate transformation between the GW source coordinates and the cosmological observer ones (i.e. GW described by an observer located in Earth).

The functional behavior of this influence was found and it was generalized for an arbitrary perfect fluid as background of spacetime. A numerical analysis of the exact solutions obtained was done, finding that a signal of this influence in a full $\Lambda$CDM spacetime could be observed with a statistical significance of $5.5\sigma$ in a particular case.

These results are very impressive due to the conceptual simplicity of the hypothesis: Just a coordinate transformation will give us a observational signal that could be measured in the next years. Not only that, also it give us a powerful tool for doing local measurements of cosmological parameters as $\Lambda$ or the components of Universe (e.g. radiation or non-relativistic matter). These results could mean an important and independent validation of cosmological parameters that now are in dispute (as $H_0$). Besides, it could imply a resurgence in interest about PTA observations, which results should arrive in the next years. As a natural extension of this investigation, it is proposed to do an exhaustive and rigorous numerical analysis as it was done in [2] but considering others components of the Universe.

Furthermore, this work can be used to study if the same effects due to coordinates transformation are present in other models of gravitation. For example, $\tilde{\delta}$–Gravity, a model presented in [31, 32], does not need a cosmological constant in order to obtain a accelerated expansion of Universe [33, 34], but it needs a radiation component in the Universe background that was not studied before. In this model it was found that the cosmological constant is actually not always constant [35], therefore, from the work developed in this report it could be studied the effects of GW for other forms of $\Lambda$. To do that, we have to find the metrics that describes the geometry of spacetime from the GW source and the respective coordinate transformation into a FLRW metric using, for instance, $\tilde{\delta}$–Gravity Field Equations.

# References


[1] J. Bernabeu, D. Espriu, and D. Puigdomenech, "Gravitational waves in the presence of a cosmological constant", Physical Review **D84**, [Erratum: Phys. Rev. D86, 069904 (2012)], 063523 (2011), arXiv:1106.4511 [hep-th].

[2] D. Espriu, and D. Puigdomenech, "Local measurement of $\Lambda$ using Pulsar Timing Array", Astrophysical Journal **764**, 163 (2013), arXiv:1209.3724 [gr-qc].

[3] D. Espriu, "Pulsar Timing Arrays and the cosmological constant", AIP Conf. Proc. **1606**, 86–98 (2014), arXiv:1401.7925 [astro-ph].

[4] J. Alfaro, D. Espriu, and L. Gabbanelli, "On the propagation of gravitational waves in a $\Lambda$CDM universe", (2017), arXiv:1711.08315 [hep-th].

[5] A. Shomer, "A pedagogical explanation for the non-renormalizability of gravity", (2007), arXiv:0709.3555 [hep-th].

[6] A. G. Riess, A. V. Filippenko, P. Challis, A. Clocchiatti, A. Diercks, P. M. Garnavich, R. L. Gilliland, C. J. Hogan, S. Jha, R. P. Kirshner, B. Leibundgut, M. M. Phillips, D. Reiss, B. P. Schmidt, R. A. Schommer, R. C. Smith, J. Spyromilio, C. Stubbs, N. B. Suntzeff, and J. Tonry, "Observational evidence from supernovae for an accelerating universe and a cosmological constant", Astronomical Journal **116**, 1009–1038 (1998), arXiv:astro-ph/9805201 [astro-ph].

[7] P. Ade, et al., "Planck 2015 results. XIII. Cosmological parameters", Astron. Astrophys. **594**, A13 (2016), arXiv:1502.01589 [astro-ph].

[8] M. Li, Xiao-Dong, S. Wang, and Y. Wang, "Dark Energy: A brief review", Front. Phys. (Beijing) **8**, 828–846 (2013), arXiv:1209.0922 [astro-ph].

[9] B. Abbott, et al., "Observations of Gravitational Waves from a Binary Black Hole Merger", Physical Review Letters **116**, 061102 (2016), arXiv:1602.03837 [gr-qc].

[10] R. Flauger, and S. Weinberg, "Gravitational Waves in Cold Dark Matter", Physical Review **D97**, 123406 (2018), arXiv:1801.00386 [astro-ph].

[11] V. A. Belinski, and G. V. Verseshchagin, "On the cosmological gravitational waves and cosmological distances", Physics Letters **B778**, 332–338 (2018), arXiv:1710.11588 [gr-qc].

[12] S. Viaggiu, "A possible new cosmological redshift effect due to $\Lambda$ on traveling gravitational waves in Friedmann universes", Int. J. Mod. Phys **D26**, 1750168 (2017), arXiv:1708.00176 [hep-th].

[13] L. Landau, and E. Lifshitz, *The Classical Theory of Fields*, 2nd ed. (Reverté, 1992), p. 318.

[14] A. Einstein, "Erklärung der perihelbewegung des Merkur aus der allgemeinen relativitätstheorie", Sitzungsberichte der Königlich Preußischen Akademie der Wissenschaften, 831–839 (1915).

[15] F. Dyson, A. Eddington, and C. Davidson, "A Determination of the deflection of light by the Sun's Gravitational Field, from observations made at the Total Eclipse of May 29, 1919", Philosophical Transactions of the Royal Society of London Series A **220**, 291–333 (1920).

[16] N. Ashby, "Relativity in the Global Positioning System", Living Reviews Foundations of Gravitation **6**, 1 (2003).

[17] S. Weinberg, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity* (Wiley, 1972).

[18] S. Carroll, *Spacetime and geometry: An Introduction to General Relativity*, 1ra Ed. (Addison Wesley, 2004).

[19] A. Einstein, "Zur elektrodynamik bewegter körper", Annalen der Physik **322**, 891–921 (1905).

[20] A. Einstein, "Die grundlage der allgemeinen relativitätstheorie", Annalen der Physik **354**, 769–822 (1916).

[21] K. Schwarzschild, "Über das Gravitationsfeld eines Massenpunktes nach der Einsteinschen Theorie", Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys.), 189–196 (1916).

[22] A. Friedmann, "Über die krümmung des raumes", Zeitschrift fur Physik **10**, 377–386 (1922).

[23] G. Lemaître, "Un univers homogène de masse constante et de rayon croissant rendant compte de la vitesse radiale des nébuleuses extra-galactiques", Annales de la Société Scientifique de Bruxelles **47**, 47–59 (1927).

[24] H. Robertson, "Kinematics and World-Structure", Astrophysical Journal **82**, 284 (1935).

[25] A. Walker, "On Milne's Theory of World-Structure", Proceedings of the London Mathematical Society **42**, 90–127 (1937).

[26] B. Schutz, *A First Course in General Relativity*, 2nd ed. (Cambridge United Press, 2009), pp. 204–205.

[27] X. Deng, and L. S. Finn, "Pulsar Timing Array observations of gravitational wave source timing parallax", Mon. Not. Roy. Astron. Soc. **414**, 50–58 (2011), arXiv:1008.0320 [astro-ph].

[28] L. S. Finn, "Response of interferometric gravitational wave detectors", Physical Review **D79**, 022002 (2009), arXiv:0810.4529 [gr-qc].

[29] M. Colpi, and A. Sesana, "Gravitational Wave Sources in the Era of Multi-Band Gravitational Wave Astronomy", in *An Overview of Gravitational Waves: Theory, Sources and Detection*, edited by G. Auger, and E. Plagnol, (2017), pp. 43–140, arXiv:1610.05309 [astro-ph].

[30] G. B. Hobbs, A. Teoh, and M. Hobbs, *ATNF pulsar catalogue*, (2005) www.atnf.csiro.au/research/pulsar/psrcat.

[31] J. Alfaro, P. González, and R. Ávila, "A Finite Quantum Gravity Field Theory Model", Class. Quant. Grav. **28**, 215020 (2011), arXiv:1009.2800 [gr-qc].

[32] J. Alfaro, "Delta-gravity and Dark Energy", Physics Letters **B709**, 101–105 (2012), arXiv:1006.5765 [gr-qc].

[33] J. Alfaro, and P. González, "Cosmology in Delta-Gravity", Class. Quant. Grav. **30**, 085002 (2013), arXiv:1210.6107 [gr-qc].

[34] J. Alfaro, and P. González, "$\tilde{\delta}$ Gravity, $\tilde{\delta}$ matter and the accelerated expansion of the Universe", (2017), arXiv:1704.02888 [gr-qc].

[35] M. San Martín, "Cosmología en delta gravity", Práctica de Licenciatura (Pontificia Universidad Católica de Chile, 2016).

# APPENDIX A. DERIVATION OF SCHWARZSCHILD METRIC

The most general metric that is static and spherically symmetric (conditions from section 1.2) is given by

$$g_{\mu\nu}\mathrm{d}x^{\mu}\mathrm{d}x^{\nu} = -A(r)\mathrm{d}t^2 + B(r)\mathrm{d}r^2 + r^2[\mathrm{d}\theta^2 + \sin^2(\theta)\mathrm{d}\phi^2], \tag{A.1}$$

where $A(r)$ and $B(r)$ are unknown functions that depend only of $r$. If we multiply the Einstein Field Equations (1.4) by $g^{\mu\nu}$ and sum over follows that

$$g^{\mu\nu}R_{\mu\nu} - \frac{1}{2}Rg^{\mu\nu}g_{\mu\nu} = \kappa g^{\mu\nu}T_{\mu\nu} \to R - 2R = \kappa T \to R = -\kappa T, \tag{A.2}$$

where we used that $R \equiv g^{\mu\nu}R_{\mu\nu}$, from (1.2) $\delta^{\mu}_{\mu} = 4$ and $T \equiv T^{\mu}_{\mu}$ is the trace of Stress-Energy tensor. Thus, for a vacuum solution $T = 0$, then $R = 0$ and EFE now reads

$$R_{\mu\nu} = 0. \tag{A.3}$$

When computing Ricci tensor components, using (A.1), we obtain

$$R_{rr} = -\frac{A''(r)}{2A(r)} + \frac{1}{4}\left(\frac{A'(r)}{A(r)}\right)\left[\frac{B'(r)}{B(r)} + \frac{A'(r)}{A(r)}\right] + \frac{1}{r}\left(\frac{B'(r)}{B(r)}\right) \tag{A.4}$$

$$R_{tt} = \frac{A''(r)}{2B(r)} - \frac{1}{4}\left(\frac{A'(r)}{B(r)}\right)\left[\frac{B'(r)}{B(r)} + \frac{A'(r)}{A(r)}\right] + \frac{1}{r}\left(\frac{A'(r)}{B(r)}\right) \tag{A.5}$$

$$R_{\theta\theta} = 1 + \frac{r}{2B(r)}\left[\frac{B'(r)}{B(r)} - \frac{A'(r)}{A(r)}\right] - \frac{1}{B(r)} \tag{A.6}$$

$$R_{\phi\phi} = \sin^2(\theta)R_{\theta\theta} \tag{A.7}$$

From (A.3); $R_{rr} = 0$, $R_{tt} = 0$ and $R_{\theta\theta} = 0$. It can be noted that

$$\frac{R_{rr}}{B(r)} + \frac{R_{tt}}{A(r)} = \frac{1}{rB(r)}\left(\frac{B'(r)}{B(r)} + \frac{A'(r)}{A(r)}\right) \stackrel{!}{=} 0, \tag{A.8}$$

then $A(r)B'(r) + B(r)A'(r) = 0$, so

$$\frac{\mathrm{d}}{\mathrm{d}r}\Big(A(r)B(r)\Big) = 0 \rightarrow A(r)B(r) = C, \tag{A.9}$$

with $C$ a constant. If we now impose that the spacetime must be asymptotically flat (i.e $\lim_{r\to\infty} g_{\mu\nu} = \eta_{\mu\nu}$) it follows that $C = 1$. Thus, $A(r) = [B(r)]^{-1}$. Replacing the last expression into (A.6) gives

$$R_{\theta\theta} = 1 - \frac{r}{B(r)}\frac{A'(r)}{A(r)} - \frac{1}{B(r)} = 1 - rA'(r) - A(r) = 1 - \frac{\mathrm{d}}{\mathrm{d}r}\Big(rA(r)\Big) = 0. \tag{A.10}$$

Integrating with respect to $r$, we obtain that $A(r) = 1+\frac{D}{r}$, with $D$ a constant. At Newtonian limit $g_{tt} \approx -1 - 2\phi = -1 + \frac{2M}{r}$. Comparing, can be inferred that $D = -2M$, where $M$ is the mass of gravitational source. Finally, the **Schwarzschild metric** is

$$g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -\left(1 - \frac{2M}{r}\right)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1 - \dfrac{2M}{r}} + r^2\mathrm{d}\Omega^2, \tag{A.11}$$

where $\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2(\theta)\mathrm{d}\phi^2$.

# Appendix B. Derivation of SdS metric

In a de Sitter spacetime we use the generalized Einstein Field Equations. Thus, following the same steps as before, let multiply (1.5) by $g^{\mu\nu}$ and sum over components

$$R - 2R + 4\Lambda = \kappa T \ \rightarrow \ R = 4\Lambda - \kappa T. \tag{B.1}$$

Then, a vacuum solution requires that

$$R_{\mu\nu} = \Lambda g_{\mu\nu}. \tag{B.2}$$

Conveniently we note that

$$\frac{R_{rr}}{B(r)} + \frac{R_{tt}}{A(r)} = \frac{\Lambda g_{rr}}{B(r)} + \frac{\Lambda g_{tt}}{A(r)} = \Lambda \left( \frac{B(r)}{B(r)} - \frac{A(r)}{A(r)} \right) \stackrel{!}{=} 0, \tag{B.3}$$

and as components of Ricci tensor do not change, the same procedure as before can be done. In particular, from last equation is clearly that again $A(r) = [B(r)]^{-1}$. Then, from (A.10) and (B.2) the $\theta\theta$ component of Ricci tensor must satisfy

$$R_{\theta\theta} = 1 - \frac{\mathrm{d}}{\mathrm{d}r}\Big( rA(r) \Big) = \Lambda g_{\theta\theta} = \Lambda r^2. \tag{B.4}$$

Integrating the last expression and applying the appropriate limits, we obtain

$$A(r) = 1 - \frac{2M}{r} - \frac{\Lambda}{3} r^2. \tag{B.5}$$

Finally, the **Schwarzschild-de Sitter metric** (SdS) is given by

$$g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -\left(1 - \frac{2M}{r} - \frac{\Lambda}{3}r^2\right)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1 - \frac{2M}{r} - \frac{\Lambda}{3}r^2} + r^2 \mathrm{d}\Omega^2. \tag{B.6}$$

# APPENDIX C. DERIVATION OF FRIEDMANN EQUATIONS

The components of Stress-Energy tensor for a perfect fluid in thermodynamic equilibrium are given by

$$T_{\mu\nu} = (\rho + p)U_\mu U_\nu + p g_{\mu\nu}, \tag{C.1}$$

where $\rho$ is the rest energy density and $p$ is the isotropic pressure. In the FLRW comoving coordinates it holds that $U^\mu = U_\mu = (-1, 0, 0, 0)$ due to normalization condition $g_{\mu\nu}U^\mu U^\nu = -1$. Thus, in FLRW coordinates it follows that

$$[T_{\mu\nu}] = \begin{bmatrix} \rho & 0 & 0 & 0 \\ 0 & p\,a^2 & 0 & 0 \\ 0 & 0 & p\,a^2 & 0 \\ 0 & 0 & 0 & p\,a^2 \end{bmatrix}. \tag{C.2}$$

Additionally, for several non-interacting fluid components we can define a *equation of state* that relates pressure and energy density

$$p_i = \omega_i \rho_i, \tag{C.3}$$

where $i$ will label the fluid component and $\omega_i$ is a constant which characterizes the type of fluid. For example, $\omega = 0$ corresponds to Dust and $\omega = 1/3$ to Radiation.

For the metric given by (1.10), the components of Ricci Tensor and scalar curvature in Cartesian coordinates ($\mathrm{d}l^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$) are given by

$$R_{00} = -3\frac{\ddot{a}}{a} \quad R_{ii} = a\ddot{a} + 2\dot{a}^2 \quad R_{ij} = 0 \quad R = 6\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2\right] \tag{C.4}$$

where $a = a(T)$ is the factor scale and a dot means derivative respect $T$. From Stress-Energy tensor of perfect fluid (C.2) and the components of Ricci tensor (C.4) it follows that the $00$ component of generalized Einstein Field Equations (1.5) takes the form

$$
\begin{aligned}
R_{00} - \frac{1}{2} g_{00} R + \Lambda g_{00} &= \kappa T_{00} \\
-3\frac{\ddot{a}}{a} + 3\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2\right] - \Lambda &= \kappa \rho_i \\
\left(\frac{\dot{a}}{a}\right)^2 &= \frac{\kappa \rho_i + \Lambda}{3}.
\end{aligned}
$$

Defining $\rho_\Lambda \equiv \frac{\Lambda}{\kappa}$, we obtain the **1st Friedmann Equation**

$$
\left(\frac{\dot{a}}{a}\right)^2 = \frac{\kappa}{3}\left(\rho_i + \rho_\Lambda\right). \tag{C.5}
$$

For $ii$ component of generalized EFE (1.5), it follows that

$$
\begin{aligned}
R_{ii} - \frac{1}{2} g_{ii} R + \Lambda g_{ii} &= \kappa T_{ii} \\
a\ddot{a} + 2\dot{a}^2 - 3a^2\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2\right] + \Lambda a^2 &= \kappa\, p_i\, a^2 \\
-2\ddot{a}a - \dot{a}^2 + \Lambda a^2 &= \kappa\, p_i\, a^2.
\end{aligned}
$$

Dividing by $a^2$ and using $\rho_\Lambda$ we get

$$
2\left(\frac{\ddot{a}}{a}\right) + \left(\frac{\dot{a}}{a}\right)^2 = \kappa(\rho_\Lambda - p_i).
$$

When replacing (C.5) into last equation the **2nd Friedmann Equation** is obtained

$$
\left(\frac{\ddot{a}}{a}\right) = \kappa\left(\frac{\rho_\Lambda}{3} - \frac{\rho_i}{6} - \frac{p_i}{2}\right). \tag{C.6}
$$

# Appendix D. Derivation of linearized EFE

In linearized Gravity we consider a little perturbation of flat spacetime given by

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \qquad |h_{\mu\nu}| \ll 1 \tag{D.1}$$

From now we will consider only contributions at first order in $h_{\mu\nu}$, which will imply that the indices will be raised and lowered by means of the $\eta^{\mu\nu}$ and $\eta_{\mu\nu}$ respectively. For example,

$$h^{\beta}_{\alpha} = \eta^{\beta\mu} h_{\mu\alpha} \qquad h^{\alpha\beta} = \eta^{\alpha\mu} h^{\beta}_{\mu} = \eta^{\alpha\mu}\eta^{\beta\nu} h_{\mu\nu} \tag{D.2}$$

Let us start linearizing Christoffel symbols

$$\begin{aligned}
\Gamma^{\mu}{}_{\alpha\beta} &\equiv \frac{1}{2} g^{\mu\nu}\Big(\partial_\alpha g_{\beta\nu} + \partial_\beta g_{\nu\alpha} - \partial_\nu g_{\alpha\beta}\Big) \\
&= \frac{1}{2}\eta^{\mu\nu}\Big(\partial_\alpha h_{\beta\nu} + \partial_\beta h_{\nu\alpha} - \partial_\nu h_{\alpha\beta}\Big) + \mathcal{O}\Big(h^2\Big) \\
&= \frac{1}{2}\Big(\partial_\alpha h^{\mu}_{\beta} + \partial_\beta h^{\mu}_{\alpha} - \eta^{\mu\nu}\partial_\nu h_{\alpha\beta}\Big) + \mathcal{O}\Big(h^2\Big)
\end{aligned} \tag{D.3}$$

Therefore, linearizing Riemann curvature tensor gives

$$\begin{aligned}
R_{\mu\nu\rho\sigma} &\equiv g_{\rho\lambda}\Big(\partial_\mu \Gamma^{\lambda}{}_{\nu\sigma} - \partial_\nu \Gamma^{\lambda}{}_{\mu\sigma} + \Gamma^{\lambda}{}_{\mu\eta}\Gamma^{\eta}{}_{\nu\sigma} - \Gamma^{\lambda}{}_{\nu\eta}\Gamma^{\eta}{}_{\mu\sigma}\Big) \\
&= \eta_{\rho\lambda}\Big(\partial_\mu \Gamma^{\lambda}{}_{\nu\sigma} - \partial_\nu \Gamma^{\lambda}{}_{\mu\sigma}\Big) + \mathcal{O}\Big(h^2\Big) \\
&= \frac{1}{2}\Big(\partial_\rho\partial_\nu h_{\mu\sigma} + \partial_\sigma\partial_\mu h_{\nu\rho} - \partial_\sigma\partial_\nu h_{\mu\rho} - \partial_\rho\partial_\mu h_{\nu\sigma}\Big) + \mathcal{O}\Big(h^2\Big)
\end{aligned} \tag{D.4}$$

Next, linearizing Ricci tensor gives

$$
\begin{aligned}
R_{\mu\nu} &\equiv g^{\rho\sigma} R_{\rho\mu\sigma\nu} = \eta^{\rho\sigma} R_{\rho\mu\sigma\nu} + \mathcal{O}\Big(h^2\Big) \\
&= \frac{1}{2}\Big(\partial_\alpha \partial_\mu h^\alpha_\nu + \partial_\alpha \partial_\nu h^\alpha_\mu - \partial_\mu \partial_\nu h - \Box h_{\mu\nu}\Big) + \mathcal{O}\Big(h^2\Big),
\end{aligned} \tag{D.5}
$$

where $h \equiv \eta^{\alpha\beta} h_{\alpha\beta}$ is the trace of metric perturbation and $\Box$ is the d'Alambertian operator defined in flat Minkowski spacetime as $\Box \equiv \eta^{\mu\nu}\partial_\mu \partial_\nu = -\partial_t^2 + \partial_x^2 + \partial_y^2 + \partial_z^2$. The scalar curvature also must be linearized

$$
R \equiv g^{\mu\nu} R_{\mu\nu} = \eta^{\mu\nu} R_{\mu\nu} + \mathcal{O}\Big(h^2\Big) \tag{D.6}
$$

$$
= \partial_\mu \partial_\nu h^{\mu\nu} - \Box h + \mathcal{O}\Big(h^2\Big). \tag{D.7}
$$

Thus, using the expressions in (D.5) and (D.7) into (1.3) give us a linearized version of Einstein tensor

$$
\begin{aligned}
G_{\mu\nu} &\equiv R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R \eta_{\mu\nu} + \mathcal{O}\Big(h^2\Big) \\
&= \frac{1}{2}\Big(\partial_\mu \partial_\alpha h^\alpha_\nu + \partial_\nu \partial_\alpha h^\alpha_\mu - \partial_\mu \partial_\nu h - \Box h_{\mu\nu} + \eta_{\mu\nu}(\Box h - \partial_\alpha \partial_\beta h^{\alpha\beta})\Big) + \mathcal{O}\Big(h^2\Big).
\end{aligned} \tag{D.8}
$$

If we introduce the trace-reversed metric perturbation

$$
\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h \qquad h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\bar{h} \qquad \bar{h} = -h, \tag{D.9}
$$

the linearized Einstein tensor becomes

$$
G_{\mu\nu} = \frac{1}{2}\Big(\partial_\mu \partial_\alpha \bar{h}^\alpha_\nu + \partial_\nu \partial_\alpha \bar{h}^\alpha_\mu - \Box \bar{h}_{\mu\nu} - \eta_{\mu\nu}\partial_\alpha \partial_\beta \bar{h}^{\alpha\beta}\Big) + \mathcal{O}\Big(h^2\Big). \tag{D.10}
$$

Thus, linearized EFE becomes

$$\partial_\mu \partial_\alpha \bar{h}^\alpha_\nu + \partial_\nu \partial_\alpha \bar{h}^\alpha_\mu - \Box \bar{h}_{\mu\nu} - \eta_{\mu\nu} \partial_\alpha \partial_\beta \bar{h}^{\alpha\beta} = 2\kappa T_{\mu\nu} + \mathcal{O}\Big(h^2\Big). \tag{D.11}$$

The last set of equations is linear in $h_{\mu\nu}$ but it is still difficult to solve. In order to simplify even more (D.11) we can use a very important property: The *gauge invariance*. Let us consider the most general coordinate transformation that leaves the field weak, it will be a *infinitesimal gauge transformation* of the form

$$x^\alpha \to x'^\alpha = x^\alpha + \xi^\alpha(x^\beta), \tag{D.12}$$

where the functions $\xi^\alpha$ are small in the sense that $|\partial_\beta \xi^\alpha| \ll 1$. From (1.2), the transformed metric components $g'_{\alpha\beta}$ can be written as

$$g'_{\alpha\beta} = \eta_{\alpha\beta} + h'_{\alpha\beta} + \mathcal{O}\Big((\partial\xi)^2\Big), \tag{D.13}$$

with $h'_{\alpha\beta} \equiv h_{\alpha\beta} - \partial_\alpha \xi_\beta - \partial_\beta \xi_\alpha$ and then $\left|h'_{\alpha\beta}\right| \ll 1$. Using (D.9) and applying the gauge transformation it follows that

$$\bar{h}'_{\alpha\beta} = \bar{h}_{\alpha\beta} - \partial_\alpha \xi_\beta - \partial_\beta \xi_\alpha + \eta_{\alpha\beta} \partial_\mu \xi^\mu. \tag{D.14}$$

We can note if $\partial_\beta \bar{h}^{\alpha\beta} = 0$ the linearized EFE simplifies considerably. To impose these conditions (they are four equations) we have to show that it is possible to find new coordinates where conditions hold. Thus, for the new coordinates it follows that

$$\partial_\beta \bar{h}'^{\beta\alpha} = \partial_\beta \bar{h}^{\alpha\beta} + \frac{1}{2} \Box \xi^\alpha. \tag{D.15}$$

It is evident that $\partial_\beta \bar{h}'^{\beta\alpha} = 0$ implies a Poisson equation for $\xi^\alpha$,

$$\Box \xi^\alpha = -2 \partial_\beta \bar{h}^{\alpha\beta}, \tag{D.16}$$

whose solution always can be found. Therefore, linearized EFE is gauge invariant and we can impose the previous conditions in order to simplify linearized EFE.

The set of four conditions

$$\partial_\beta \bar{h}^{\beta\alpha} = 0, \tag{D.17}$$

are called the *Lorenz Gauge*. If we take a set of coordinates that satisfies the Lorenz gauge conditions, from (D.10), the linearized Einstein Field Equations (1.5) will given by

$$\Box \bar{h}_{\mu\nu} = -2\Lambda\eta_{\mu\nu} - 2\kappa T_{\mu\nu} + \mathcal{O}\Big(h^2\Big). \tag{D.18}$$

We also can note that any set of coordinates $\zeta^\alpha$ that satisfy

$$\Box \zeta^\alpha = 0, \tag{D.19}$$

also satisfy the Poisson equation

$$\Box(\xi^\alpha + \zeta^\alpha) = -2\partial_\beta \beta \bar{h}^{\alpha\beta} \tag{D.20}$$

and instantaneously will satisfy gauge conditions. Therefore, the Lorenz gauge do not completely fix the gauge. In fact, we will say that it is a class of gauges.