Gravitational Waves Background in an Expanding Universe

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Abstract

In this report we review the predictions about the stochastic Gravitational Wave Background which corresponds to a perturbation in the space time due to the effect of Gravitational Waves coming from all directions. In this framework, Hellings and Downs show that exist a correspondence between the angular separation of a pulsar pair and the amplitude of the metric perturbation measured in Earth using Pulsar Timing Data.

In the framework of Gravitational Waves in an accelerated expanding universe ([1], [2] and [3]), we intend to find this correspondence between the angular separation and the amplitude of the metric perturbation. During the development of this work, in addition, we derived a expression for the plane wave solution of GW which generalizes the propagation direction of the GW.

Finally we find that the angular factor that accounts for the mentioned correspondence is exactly the same as the one found by Hellings and Downs, hence, this angular dependence it is not affected by the accelerated expansion of the universe.

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Chapter 1

Introduction

Einstein's General Theory of Relativity is one of the better known theories in the physics environment. This theory meant a big breakthrough in our understanding and interpretation of physics, specially of the Universe. Einstein's work gives a new interpretation of gravity describing it as a geometric property of the spacetime [4]. Here, the presence of mass curves the space and the behavior or movement of bodies is given by the curvature of space, thus, in this sense, the mass tells the spacetime how to curve and the curvature tells the mass how to move in spacetime.

This theory has been greatly accepted by the scientific community due to its precise predictions for example of the perihelion of Mercury [5] and the predictions of the deflection of sunlight tested during a solar eclipse [6]. However, this theory is incompatible with the Quantum Theory because it could not be described as a renormalizable Quantum Field Theory [7] thus it is said that GR describes gravity consistently but in a classic perspective.

Other issues on the theory appears when studying the universe at cosmological scales where to explain the accelerated expansion of the universe [8] [9] it is necessary to introduce a constant Λ into the field equations, then GR plus this constant is known as the standard cosmological model (ΛCDM) and it is the simplest one that explains this behavior of the Universe. Since the cosmological constant Λ is added "by hand" to reproduce the accelerated expansion, it is normal to ask one self, what does this constant represents? This unknown nature of the cosmological constant is known as the *Dark Energy* problem.

One interesting aspect about GR is that it predicts the existence of Gravitational Waves (GW)[10], perturbations in the space-time produced by disturbances due to massive accelerating objects. In this framework some authors theorized about a stochastic gravitational wave background [11][12] that could be measured using Pulsar data. In particular, Hellings and Downs [13] derived an expression for an angular factor, depending on the angles between two pulsars, that scales the strength of the background signal when correlating the two pulsar data.

Recent investigations intend to study if the accelerated expansion of the Universe affects the propagation of Gravitational Waves. In [1] [2] it is shown that the accelerated expansion does affect the propagation of Gravitational Waves. In particular, [3] generalizes the problem to a Universe with all its components.

In this particular work, we are interested in studying this predicted stochastic background of GW for an expanding universe; that is considering the previous work mentioned in the previous paragraph. We hope to derive an angular factor analogue to the one derived by Hellinngs and Downs and compare them to search for any differences.

In order to do this, we start in Chapter 2 by revising Einstein's derivation for Gravitational Waves, using the linearized Einstein Field Equations. Also, we will mention about the detection of GW and emphasize on the Pulsar timing method. Finally we will explain about the Gravitational Wave Background starting from Hellings and Downs prescription and later on a more general one.

In Chapter 3 we will review the work in [3] to check the coordinate transformation and the expression for propagating GW in an accelerating universe.

Later, in Chapter 4 we study the GW background in an accelerating universe aiming to find the angular expression and compare with the standard case.

Finally in Chapter 5 we draw conclusions and mention ideas for future work.

Chapter 2

Gravitational Waves

We can describe Gravitational Waves (GW) as ripples in the spacetime caused by astronomical perturbations that propagates through the universe at the speed of light. Predicted by Albert Einstein during his development of the General Theory of Relativity for the first time in 1916 [10]. In order to understand the GW we first need to review the basics of General Relativity.

2.1 General Relativity

Starting from the Special Theory of Relativity, Einstein continued to develop the theory to be able to incorporate gravity into it. It was not until 1915 that Einstein found workable field equations [14]

$$G_{\mu\nu} = \kappa T_{\mu\nu} \tag{2.1}$$

where $\kappa = \frac{8\pi G}{c^4}$ and

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \tag{2.2}$$

is the Einstein tensor defined from the Ricci tensor $R_{\mu\nu}$ and the Ricci scalar R. Finally, in the next year Einstein had consolidated his General Theory of Relativity by writing "*The Foundation of the General Theory of Relativity*" [4]. Several years later, Einstein considered the need for a "Cosmological Constant" to solve the cosmological problem of GR [15], proving that if a term proportional to $g_{\mu\nu}$ is added to the left side of Eq (2.1) the equation is also satisfied. Thus, we can write the general Einstein Field Equation (EFE) as

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu} \tag{2.3}$$

This equation describes how the spacetime is curved by the presence of matter or energy and how matter or energy moves through spacetime.

2.2 Linearized Einstein Field Equations

We already know that Einstein's theory of gravitation includes the possibility of propagation of gravitational waves but since the EFE behaves in a non linear way, the calculations becomes highly complicated. Then, it is useful to linearize the theory in order to simplify the calculations.

Let us consider a flat spacetime which is perturbed, then the metric is expressed as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad |h_{\mu\nu}| << 1 \tag{2.4}$$

where $\eta_{\mu\nu}$ corresponds to the Minkowski metric. It is now useful to introduce the *trace-reversed* metric perturbation, where:

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h, \quad \bar{h} = -h \tag{2.5}$$

using the Lorenz Gauge $\partial_\nu \bar{h}^{\nu\mu} = 0$ and only keeping the therms linear in $h_{\mu\nu}$ we'll have that the linearized EFE will be

$$\Box \bar{h}_{\mu\nu} = -2\Lambda\eta_{\mu\nu} - 2\kappa T_{\mu\nu} \tag{2.6}$$

where $\Box$ is the d'Alambertian operator defined as $\Box = -\partial_t + \nabla^2$ in a flat spacetime. The equations (2.6) are also known as the *Weak-Field Einstein Equations* since they describe the effects due to a weak gravitational field [16]. In equation (2.4), we can decompose the perturbation term into a gravitational wave (GW) contribution $h^{(GW)}$, a Λ contribution and another one due to the other components of the universe. Since they do not interact with each other, we can write the full perturbed metric as

$$g_{\mu\nu} = \eta_{\mu\nu} + h^{(GW)} + h^{(\Lambda)} + h^{(fluid)} \tag{2.7}$$

and each contribution of the full perturbation will satisfy, from (2.6):

$$\Box \bar{h}_{\mu\nu}^{(GW)} = 0 \tag{2.8}$$

$$\Box \bar{h}_{\mu\nu}^{(\Lambda)} = -2\Lambda\eta_{\mu\nu} \tag{2.9}$$

$$\Box \bar{h}_{\mu\nu}^{(fluid)} = -2\kappa T_{\mu\nu} \tag{2.10}$$

Since we are interested in the GW contribution, we'll be working with Eq (2.8). This is an homogeneous wave equation , whose general solution is an harmonic wave. Equation (2.8) has plane wave solutions of the form

$$\bar{h}_{\mu\nu} = A_{\mu\nu} e^{ik_\alpha x^\alpha} \tag{2.11}$$

where $A_{\mu\nu}$ is a constant symmetric contravariant tensor, and $k_\alpha = (\omega, \mathbf{k})$. Introducing this solution into the wave equations (2.8), we obtain:

$$\Box \bar{h}_{\mu\nu}^{(GW)} = \eta^{\alpha\beta}\partial_\alpha\partial_\beta \bar{h}_{\mu\nu} = -\eta^{\alpha\beta}k_\alpha k_\beta \bar{h}_{\mu\nu} = 0 \iff k_\alpha k^\alpha = 0 \implies |\vec{k}|^2 = \omega^2 \tag{2.12}$$

This implies that k^α is a null four-vector as light rays, thus, the gravitational plane waves propagate with the speed of light c in vacuum.

Now, recalling from the condition of the Lorenz Gauge, we'll have:

$$\partial_\nu \bar{h}^{\nu\mu} = i(k_\nu A^{\mu\nu})e^{ik_\alpha x^\alpha} = 0 \implies k_\nu A^{\mu\nu} = 0 \tag{2.13}$$

This means that the harmonic gauge condition (Lorenz Gauge condition) requires that $A^{\mu\nu}$ is orthogonal (transverse) to the direction of wave vector k_ν. Recalling from the definition, $A^{\mu\nu}$ is an arbitrary rank-2 tensor in a four dimensional space, this means it has 16 independent components. Since this tensor is symmetric ($A^{\mu\nu} = A^{\nu\mu}$) the number independent components gets reduced to 10. From the result of the gauge condition (2.13) the number of independent components gets reduced to only six. Now, making use of the *Gauge Invariance*, let us apply the following gauge transformation

$$h^{\mu\nu} \to \tilde{h}^{\mu\nu} = h^{\mu\nu} - \partial^\mu\xi^\nu - \partial^\nu\xi^\mu \tag{2.14}$$

then, if we want the condition of the Lorenz gauge to be preserved, since

$$\partial_\mu \tilde{h}^{\mu\nu} = \partial_\mu h^{\mu\nu} - \Box\xi^\nu - \partial^\nu(\partial_\mu\xi^\mu) = 0 \tag{2.15}$$

then, the four-vector ξ^{ν} must satisfy $\Box\xi^{\nu} = 0$ and $\partial_{\mu}\xi^{\mu} = 0$,then

$$\partial_{\mu}\tilde{h}^{\mu\nu} = \partial_{\mu}h^{\mu\nu} = 0 \tag{2.16}$$

A four-vector that satisfies this can be written in the form:

$$\xi^{\mu} = iC^{\mu}e^{ik_{\alpha}x^{\alpha}} \tag{2.17}$$

where C^{μ} is a constant vector perpendicular to the wave vector k^{μ}. It is easy to check that this vector holds for the conditions required, then Eq. (2.16) is satisfied.

With this conditions, and gauge elections, we've reduced the number of independent components of $A^{\mu\nu}$ to just three. Finally, we can adopt the so called *transverse traceless (TT) gauge* where we require that:

$$A^{\mu}_{\mu} = 0 \tag{2.18}$$

which will give the final condition on one of the three remaining independent components.In the TT gauge we also have $\bar{h} = h = 0$ then, we can write the amplitude

$$A_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A_{+} & A_{\times} & 0 \\ 0 & A_{\times} & -A_{+} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.19}$$

Where $A_{+,\times}$corresponds to the two different polarization states of GW traveling in z-direction.

Then, the plane wave solution has the form:

$$h_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_{+} & h_{\times} & 0 \\ 0 & h_{\times} & -h_{+} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} e^{i\omega(t-z)} \tag{2.20}$$

The two polarization states can be described by the two polarization tensors:

$$\varepsilon^{+}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \varepsilon^{\times}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.21}$$

the plane wave solution can be finally expressed as:

$$h_{\mu\nu} = (h_+ \varepsilon^+_{\mu\nu} + h_\times \varepsilon^\times_{\mu\nu}) e^{i\omega(t-z)} \tag{2.22}$$

We can write this solution in spherical coordinates from the source of GW. Then at a distance large enough but small compared to cosmological distances, we can write (2.22) as

$$h_{\mu\nu} = \frac{e_{\mu\nu}}{r} \cos[\omega(t-r)] \tag{2.23}$$

where $e_{\mu\nu} = h_+ \varepsilon^+_{\mu\nu} + h_\times \varepsilon^\times_{\mu\nu}$ is called the *polarization tensor* of the GW.

2.3 Detection of Gravitational Waves

Now that we have the theory, we need to think about how to detect this gravitational radiation. The limitation on doing this is that the amplitude of the metric perturbations $h_{\mu\nu}$ that are expected from distant sources of GW are very small, thus, hard to detect.

Almost every astrophysical phenomena produces gravitational waves [16] and we are interested in the most violent ones because they emit a big amount of this radiation, which make the detection a little bit easier.

Maybe, the most known method for GW detection are the laser interferometers due to the recent first direct detection of this kind of signal announced on 2016 [17]. This method uses highly precise lasers to detect interference patterns on the final light beam which are associated to GW influence that makes the light on one arm of the interferometer to travel a longer distance.

On the other hand, we can detect GW indirectly by observing astrophysical phenomena caused by the influence of GW. Some methods are *Spacecraft Tracking* - by comparing the fluctuations in the travel time of the radio signal from Earth to an interplanetary spacecraft -, *CMB Temperature Fluctuations* - study of the CMB temperature distribution to find indicators of signatures of GW from the Big Bang. It is very difficult to measure.

Finally, other indirect evidence for GW comes from Pulsars, rotating neutron stars that emits a light beam which can be observed only when it points towards Earth. This stars have very short and precise rotational periods and some of them (millisecond Pulsars) are so precise that can be used to detect the variations on the pulse train (the signal) due to

GW influence. In particular, by looking correlated data of this timing variations with Pulsar Timing Arrays (PTA), one can detect the influence of a common signal of GW discussed in detail in Section 2.4.

2.4 Gravitational Waves Background

In the late 70's and early 80's, some authors began to consider the possibility that the Universe is filled with a stochastic background of gravitational radiation [12] [11] and it was first pointed out by Detweiler [18] that a Pulsar-Earth system could be used to create a GW antenna.

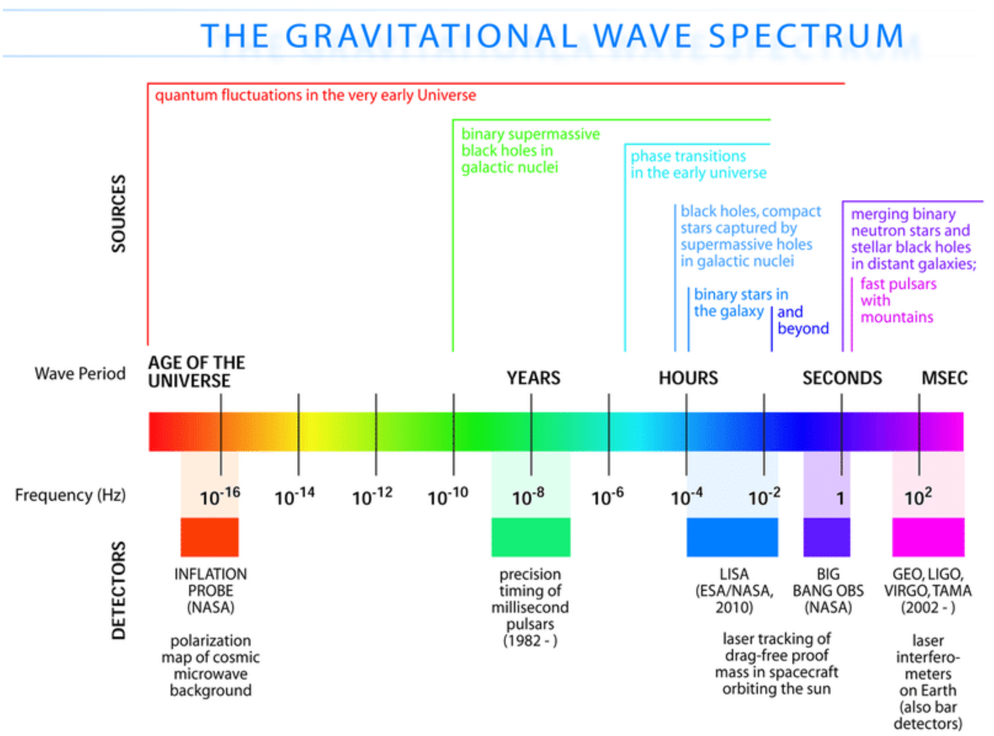

Fig. 2.1 Gravitational-wave spectrum, together with potential sources and relevant detectors[1].

Before continuing, we must characterize this stochastic background of gravitational waves. We can define this stochastic background as any random GW signal produced by the combination of a large number of weak, independent and unresolved sources. Different astrophysical processes produces GW signals in different frequencies (see Fig 2.1) that allow

[1] Image credit: Institute of Gravitational Research/ University of Glasgow.

us to study the origins and history of our Universe, depending on which frequency we are looking. For example, GW can give us information far beyond the time of last scattering, which is the limit for electromagnetic radiation.

This stochastic background is hard to detect because it is a source of noise in a detector, therefore, the problem is to distinguish between the GW noise and instrumental or any other type of noise present in our detection.

In this work we will focus on the stochastic signals produced by Super Massive Black Holes Binaries (SMBHBs), present in different galactic nuclei, which can be detected using the Pulsar Timing Arrays (PTAs).

It is in this framework that Hellings and Downs propose to correlate Pulsar Timing data to find an upper limit to this stochastic background of GW [13] and in the process, they derive an angle factor that scales the strength of the signal depending on the angle between two pulsars. To determine this, we need to express the fractional change in frequency (redshift) in the signal of a pulsar, observed from Earth, caused by a GW. In particular, for a plane wave traveling in the z direction with amplitude $h(t-z)$, [13] express this as

$$z(t) = \frac{\Delta\nu}{\nu} = \frac{1}{2}\cos 2\phi\,[1-\cos\theta] \times [h(t) - h(t-l-l\cos\theta)] \tag{2.24}$$

where l is the distance to the pulsar from Earth at an angle θ and ϕ is the angle to the projection on the $x-y$ plane (Fig 2.2). From this, it is stated that the effect of a GW is to induce fluctuations proportional to $h(t)$ and the general idea is to correlate data in order to find a common signal that can be related to a GW source.

2.4.1 Cross Correlating Pulsar Data

Equation (2.24) can be rewritten as

$$\frac{\Delta_i\nu}{\nu_i} = \alpha_i h(t) + n_i(t) \tag{2.25}$$

where $h(t)$ is the GW signal, common to all pulsars. The factor α_i accounts for the angular factor in (2.24) for the i-th pulsar and $n_i(t)$ represents all the intrinsic fluctuations for each pulsar.

Finally, when correlating the data from a pulsar pair, one obtains:

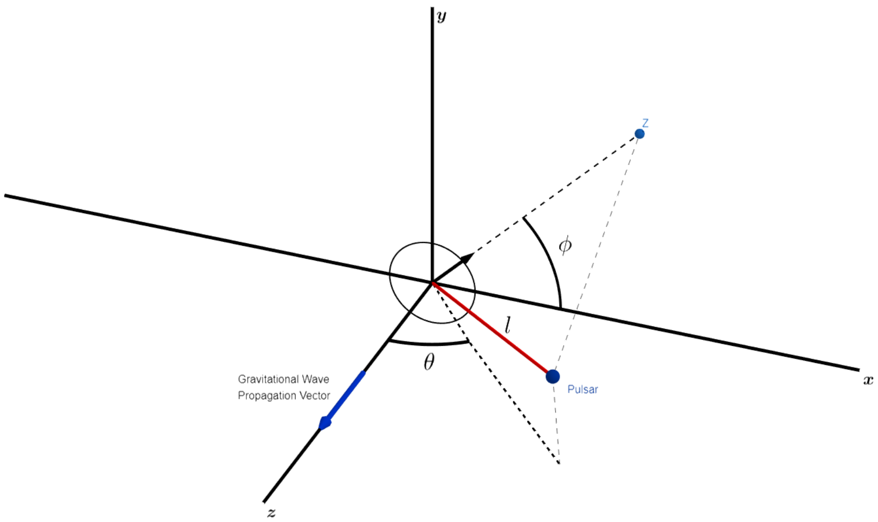

Fig. 2.2 System configuration centered at the Earth, for one pulsar and a propagating GW.

$$C_{ij}(\tau) = \alpha_i \alpha_j \left\langle h^2 \right\rangle + \alpha_i \left\langle h n_j \right\rangle + \alpha_j \left\langle n_i h \right\rangle + \left\langle n_i n_j \right\rangle \tag{2.26}$$

where it's used the mean α_{ij} of the angular factors $\alpha_i \alpha_j$ defined as

$$\alpha_{ij} = \frac{1}{4\pi} \int \alpha_i \alpha_j d\Omega = \frac{1 - \cos\gamma_{ij}}{2} \left[\ln \left(\frac{1 - \cos\gamma_{ij}}{2} \right) - \frac{1}{6} \right] + \frac{1}{3} \tag{2.27}$$

Where γ_{ij} is the angular separation of a pulsar pair observed from Earth. Equation (2.27) is the actual factor that is used to compute the Hellings and Downs Curve which we can see in Fig 2.3

2.4.2 Alternative derivation

In other works [19] [20] a more complete and general derivation of this result is done. It is convenient to recall from these works the expression for the redshift or variational change in frequency of the pulsar signal, which is written as:

$$z(t, \hat{n}) = \frac{1}{2} \frac{\hat{p}^i \hat{p}^j}{1 + \hat{n} \cdot \hat{p}} \Delta h_{ij} \tag{2.28}$$

where Δh_{ij} is the difference in the metric perturbation traveling in the direction $\hat{n}$ at the pulsar and at the Solar System. When choosing a particular coordinate system, placing the Solar System at the origin and the pulsar at some distance away:

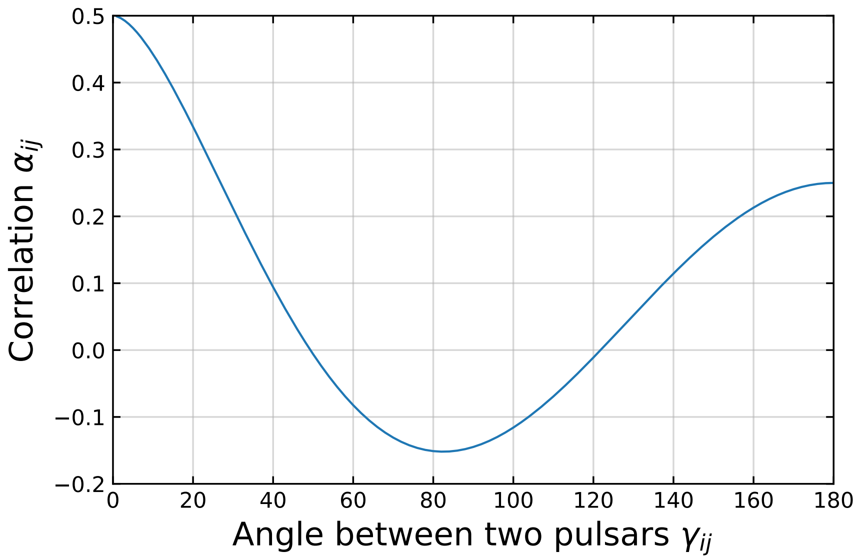

Fig. 2.3 This Corresponds to the Hellings and Downs curve derived by the angular factor on the correlation of the timing data of two pulsars α_{ij}

$$t_p = t_e - L \equiv t - L, \tag{2.29}$$

$$\vec{x}_e = 0, \tag{2.30}$$

$$\vec{x}_p = L\hat{p} \tag{2.31}$$

we find that

$$\Delta h_{ij} = \int_{-\infty}^{\infty} d\omega \quad e^{i\omega t} \left(e^{-i\omega L(1+\hat{n}\hat{p})} - 1 \right) \sum_A h_A(\omega, \hat{n}) e_{ij}^A(\hat{n}) \tag{2.32}$$

and finally taking the Fourier transform of this quantity we can write the redshift in terms of the frequency as

$$\tilde{z}(\omega, \hat{n}) = \left(e^{-i2\pi\omega L(1+\hat{n}\cdot\hat{p})} - 1 \right) \sum_A h_A(\omega, \hat{n}) F^A(\hat{n}) \tag{2.33}$$

It is important to note that this definition for the redshift is written as function of the GW frequency. Additionally, the A corresponds to the different polarization states of the GW $(+, \times)$ and $F^A(\hat{n})$ it is defined as

$$F^A(\hat{n}) \equiv e_{ij}^A(\hat{n}) \frac{1}{2} \frac{\hat{p}^i \hat{p}^j}{1 + \hat{n} \cdot \hat{p}} \tag{2.34}$$

where $h_A(\omega,\hat{n})$ corresponds to the amplitude for each polarization state which depends on the frequency and propagation direction of the GW. Finally, the total redshift is obtained by summing all the contributions comming from every direction, this is

$$\tilde{z}(\omega) = \int_{S^2} d\hat{n}\tilde{z}(\omega,\hat{n}) \tag{2.35}$$

Cross Correlation

To calculate the cross correlation for the Pulsar timing data, let us consider that the signal from the two pulsars is

$$S_i(t) = z_i(t) + n_i(t) \tag{2.36}$$

where $z_i(t)$ corresponds to the fractional change in frequency, produced by the presence of GW (2.35), as function of time and $n_i(t)$ is all the noise intrinsic to each pulsar. With this kind of signal, when calculating the correlation, as in the original work of Hellings and Downs, we expect that the only term remaining in the correlation will be $\left\langle z_i z_j \right\rangle$ because the intrinsic noise for each pulsar is not correlated thus, $\left\langle n_i n_j \right\rangle$ will vanish as well as the $\left\langle z_i n_j \right\rangle$ term.

Since we are interested in using a high number of pulsar pairs, we will average the cross-correlation statistics. To calculate this mean, will imply that we need to evaluate $\langle \bar{z}_1(\omega) z_2(\omega) \rangle$. Taking the definition from (2.35) the expectation value of the redshift can be written as

$$\langle \bar{z}_1 z_2 \rangle = \int_{S^2} d\hat{n} \left(e^{i2\pi\omega L_1(1+\hat{n}\cdot\hat{p}_1)} - 1 \right) \left(e^{-i2\pi\omega L_2(1+\hat{n}\cdot\hat{p}_2)} - 1 \right) \sum_A \left\langle \bar{h}_1^A(\omega,\hat{n}) h_2^A(\omega,\hat{n}) \right\rangle F_1^A(\hat{n}) F_2^A(\hat{n}) \tag{2.37}$$

Given that we are assuming that this stochastic background is isotropic, unpolarized and stationary the term $\left\langle \bar{h}_1^A(\omega,\hat{n}) h_2^A(\omega,\hat{n}) \right\rangle$ will only depend on the frequency ω of the GW, then we can leave it out of the integral. For simplicity we will write it as $\left\langle h^2(\omega) \right\rangle$, then

$$\langle \bar{z}_1 z_2 \rangle = \frac{\left\langle h^2(\omega) \right\rangle}{\beta} \Gamma(\omega) \tag{2.38}$$

where $\Gamma(\omega)$ is defined as

$$\Gamma(\omega) = \beta \sum_{A} \int_{S^2} d\hat{n} \left(e^{i2\pi\omega L_1(1+\hat{n}\cdot\hat{p}_1)} - 1 \right) \left(e^{-i2\pi\omega L_2(1+\hat{n}\cdot\hat{p}_2)} - 1 \right) F_1^A(\hat{n}) F_2^A(\hat{n}) \tag{2.39}$$

this is an angular factor also known as the pulsar analogue of the *Overlap Reduction Function* which is related (or even a generalized form) to the angular factor (2.27) derived by Helling and Downs in 1983.

For convenience, to calculate this, we will fix the coordinates for each pulsar pair in which $\hat{p}_1$ is parallel to the z-axis and $\hat{p}_2$ is in the $x-z$ plane (Fig. 2.4),

$$\hat{p}_1 = (0,0,1), \qquad \hat{p}_2 = (\sin\xi, 0, \cos\xi) \tag{2.40}$$

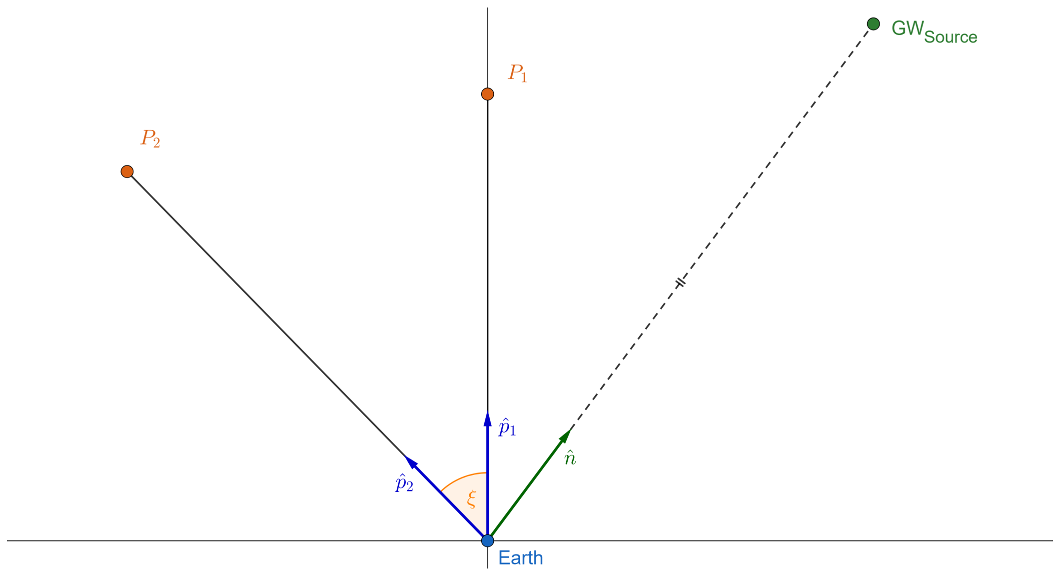

Fig. 2.4 System configuration for a pulsar pair where one of them is on the z direction and the other is on the $x-z$ plane with a separation angle ξ between them.

where ξ is the angular separation between the two pulsars. With this coordinate choice for the system, the $\times$ polarization $F^{\times}(\hat{n})$ (2.34) vanishes, then the sum in (2.39) only contains the term corresponding to the $+$ polarization state. Moreover, it is shown in [20] that the pulsar timing experiments are in a regime where the exponential factors in (2.39) can be neglected. Thus the Overlap Reduction Function can be approximated as

$$\Gamma_0 = \beta \int_{S^2} d\hat{n} F_1^{+}(\hat{n}) F_2^{+}(\hat{n}) \tag{2.41}$$

Where $F_i^{+}(\hat{n})$ is defined in (2.34). The plus polarization tensor, is defined in [20] as

$$e^+_{ij}(\hat{n}) = \hat{\phi}_i\hat{\phi}_j - \hat{\theta}_i\hat{\theta}_j \tag{2.42}$$

with

$$\hat{\theta} = (\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta) \tag{2.43}$$

$$\hat{\phi} = (\sin\phi, -\cos\phi, 0). \tag{2.44}$$

thus, the integrand in (2.41) will be

$$F_1^+(\hat{n})F_2^+(\hat{n}) = \\ -\frac{\sin^2\theta\left[\sin^2\xi\sin^2\phi - \sin^2\xi\cos^2\theta\cos^2\phi - \cos^2\xi\sin^2\theta + 2\sin\xi\cos\xi\sin\theta\cos\theta\cos\phi\right]}{(1+\cos\theta)(1+\cos\xi\cos\theta + \sin\xi\sin\theta\cos\phi)} \tag{2.45}$$

in this coordinate system. Finally, by computing the integral, you finally get

$$\Gamma_0 = 4\pi\beta\left(\frac{1-\cos\xi}{2}\left[\ln\left(\frac{1-\cos\xi}{2}\right) - \frac{1}{6}\right] + \frac{1}{3}\right) \tag{2.46}$$

where it is easy to notice that if we take the normalization factor $\beta = \frac{1}{4\pi}$ we will get the same expression that (2.27) derived by Hellings and Downs.

Chapter 3

Gravitational Waves in a ΛCDM Universe

In the previous section, we derived the plane wave solution for Gravitational Waves. This was for gravitational waves in vacuum, then, the simplest way to study the propagation of GW is by using the coordinates that emerge from the GW source (t, r) as in Eq. (2.23).

3.1 Appropriate Coordinate Choice

This will be different if we consider a full ΛCDM universe, which it is the subject of this work. It is important to define the appropriate coordinate system. For the GW source, it is convenient to use the coordinates $\{t, r, \theta, \phi\}$ that will represent a spherically symmetrical spacetime seen from the source. This are the same coordinates used in (2.23).

We need to consider that from Earth, or a cosmological observer, we see an Universe with an accelerated expansion then it is natural to use a FLWR comoving coordinates $\{T, R, \theta, \phi\}$ to measure the metric perturbation due to a GW seen from Earth.

As we see, to actually predict how $h_{\mu\nu}^{(GW)}$ will be measured from earth, it is necessary to find the transformation from one coordinate system to the other

$$\{t, r, \theta, \phi\} \to \{T, R, \theta, \phi\} \tag{3.1}$$

3.1.1 Coordinate Transformation in ΛCDM

In this kind of universe, the coordinates from the GW source to the cosmological observer (Earth) will transform as [3]:

$$r = 1 + \Delta T \sqrt{\frac{\Lambda + \kappa\rho_{d0} + \kappa\rho_{r0}}{3}} + \mathscr{O}(\Lambda) \tag{3.2}$$

$$t = T + \frac{R^2}{2}\sqrt{\frac{\Lambda + \kappa\rho_{d0} + \kappa\rho_{r0}}{3}} + \mathscr{O}(\Lambda) \tag{3.3}$$

Notice that the term $\sqrt{\frac{\Lambda+\kappa\rho_{d0}+\kappa\rho_{r0}}{3}} \equiv H_0$. In the following we express it as this.

3.2 Gravitational Waves in ΛCDM

Under this coordinate transformation (3.3) (3.2), the solution for the propagating GW will be expressed (2.23) in the new coordinates as [2]:

$$h'^{(GW)}_{\mu\nu} = \frac{e'_{\mu\nu}}{R}(1 - H_0 T)\cos\left[\omega(T - R) + \omega H_0\left(\frac{R^2}{2} - TR\right)\right] \tag{3.4}$$

where we can define the quantities

$$\omega_{eff} \equiv \omega(1 - RH_0) \quad k_{eff} \equiv \omega\left(1 - \frac{R}{2}H_0\right) \tag{3.5}$$

in this way eq (3.4) is written as

$$h'^{(GW)}_{\mu\nu} = \frac{e'_{\mu\nu}}{R}(1 - H_0 T)\cos\left[\omega_{eff}T - k_{eff}R\right] \tag{3.6}$$

Chapter 4

Gravitational Wave Background in ΛCDM

Now that we know how the coordinates transform from the GW source to the cosmological observer (3.3) (3.2), since we want to obtain a expression for the GW background we need to begin to think in how generalize the solution of propagating GW to an arbitrary direction.

4.1 Gravitational Waves in an Arbitrary Direction

Let us start with the solution of an homogeneous wave equation (2.11)

$$h_{\mu\nu} = e_{\mu\nu} e^{ik_\alpha x^\alpha} \tag{4.1}$$

where $k_\alpha = (\omega, -\vec{k})$ and $x^\alpha = (t, \vec{r})$ and here, $\vec{k} = \omega\hat{n}$ and $\vec{r} = r\hat{r}$. With this notation, the last equation becomes

$$h_{\mu\nu} = \frac{1}{r} e_{\mu\nu} e^{i\omega(t - r\hat{n}\cdot\hat{r})} \tag{4.2}$$

Now, applying the coordinate transformations (3.3) and (3.2), we will obtain

$$h'_{\mu\nu} = \frac{e'_{\mu\nu}}{R}(1 - H_0 T) e^{i(\omega_{eff} T - k_e f f R)} \tag{4.3}$$

where in this case we'll have

$$\omega_{eff} = \omega\left(1 - RH_0\hat{n}\cdot\hat{r}\right) \tag{4.4}$$

$$k_{eff} = \omega\left(\hat{n}\cdot\hat{r} - \frac{R}{2}H_0\right) \tag{4.5}$$

this is a kind of generalization of (3.6). Finally, since we want to consider all contributions of GW sources, (4.3) must be integrated over all the frequencies, obtaining

$$h'_{\mu\nu} = \int_{-\infty}^{\infty} d\omega \quad \frac{e'_{\mu\nu}}{R}(1 - H_0 T)e^{i(\omega_{eff}T - k_effR)} \tag{4.6}$$

Now that we know the solution of propagating GW in a full ΛCDM Universe for a general propagation direction, we need to compute the redshift for this particular case, but, before we need to express the polarization tensor in a convenient way $e'_{\mu\nu}$.

4.1.1 Polarization Tensor Transformation

Let us start, by noticing that the polarization tensor in the equation (3.6) and (4.6) it is actually transformed to the coordinates (T,R) but it's full definition it's omitted [2] since it only matters the amplitude of each polarization when we are considering GW propagating in the z direction and its value it is often replaced by a characteristic amplitude of the polarization.

Even if we do not know the exact expression for $e_{\mu\nu}$ we know that it is still in the TT Gauge [1], that is, the only components different from zero are in the X, Y components [3]. With this considerations we can write the transformed polarization tensor as

$$e'_{\mu\nu} = h'_{+}\varepsilon^{+}_{\mu\nu} + h'_{\times}\varepsilon^{\times}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h'_{+} & h'_{\times} & 0 \\ 0 & h'_{\times} & -h'_{+} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{4.7}$$

where $\varepsilon^{+,\times}_{\mu\nu}$ are defined in (2.21). Now, this is for GW traveling in the z direction. Since we want to consider a background of GW from all directions it is necessary to generalize this to every direction. For now, we'll just focus on the polarization tensor since it is the one that will provide the information for the Hellings and Downs Curve.

The way to generalize the polarization tensor is by using the general form of the rotation matrix using the *pitch-roll-yaw* convention (xyz) [21] and considering $\psi = 0$ (see Fig 4.1),

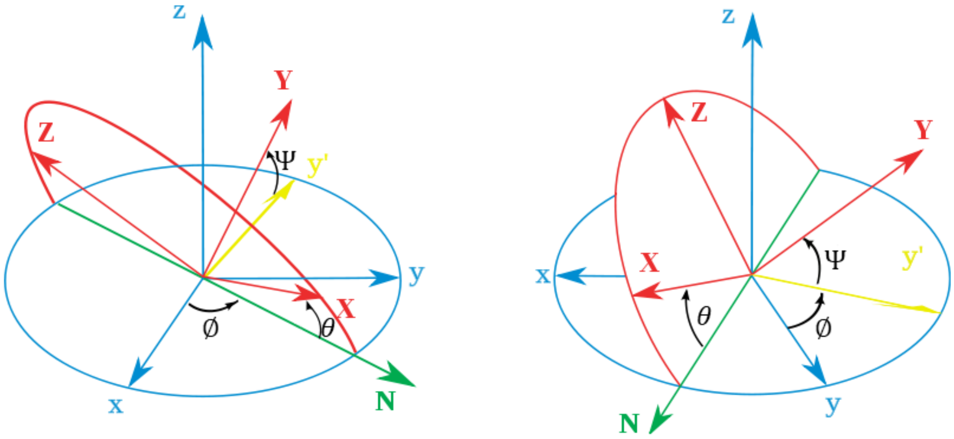

Fig. 4.1 Relation between the original system and the rotated system. This convention is also known as Tait-Bryan angles, where each rotation is about a different axis, starting with a rotation about the z-axis, then, about an intermediary y-axis and finally about the final x-axis.

then the rotation matrix is:

$$R^{\alpha}_{\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\phi) & \sin(\phi) & 0 \\ 0 & -\cos(\theta)\sin(\phi) & \cos(\theta)\cos(\phi) & \sin(\theta) \\ 0 & \sin(\theta)\sin(\phi) & -\sin(\theta)\cos(\phi) & \cos(\theta) \end{pmatrix} \tag{4.8}$$

and the two polarization tensors transform as

$$\varepsilon^{+,\times\prime}_{\alpha\beta} = (R^{\gamma}_{\alpha})^{-1}\varepsilon^{+,\times}_{\gamma\delta}R^{\delta}_{\alpha} \tag{4.9}$$

and after this transformation they are expressed as:

$$\varepsilon^{+'}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \cos^2\theta\cos^2\phi + \cos^2\phi - 1 & \sin\phi\cos\phi\cos^2\theta + \sin\phi\cos\phi & -\sin\theta\cos\theta\cos\phi \\ 0 & \sin\phi\cos\phi\cos^2\theta + \sin\phi\cos\phi & \cos^2\theta - \cos^2\theta\cos^2\phi - \cos^2\phi & -\sin\theta\cos\theta\sin\phi \\ 0 & -\sin\theta\cos\theta\cos\phi & -\sin\theta\cos\theta\sin\phi & \sin^2\theta \end{pmatrix}, \tag{4.10}$$

$$\varepsilon^{\times'}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -2\sin\phi\cos\theta\cos\phi & 2\cos\theta\cos^2\phi - \cos\theta & \sin\theta\sin\phi \\ 0 & 2\cos\theta\cos^2\phi - \cos\theta & 2\sin\phi\cos\theta\cos\phi & -\sin\theta\cos\phi \\ 0 & \sin\theta\sin\phi & -\sin\theta\cos\phi & 0 \end{pmatrix} \tag{4.11}$$

4.2 Redshift for the Transformed Perturbation

Recall from Chapter 2 that the variational change in frequency of the pulsar or the redshift can be written as (2.28)

$$z'(T,\hat{n}) = \frac{1}{2}\frac{\hat{p}^i\hat{p}^j}{1+\hat{n}\cdot\hat{p}}\Delta h'_{ij} \tag{4.12}$$

where in this case, $\Delta h'_{ij}$ corresponds to the diference in the transformed metric perturbation at the pulsar and at the Solar System from eq (4.6). We notice that it is not trivial to write explicitly $\Delta h'_{ij}$ because the metric perturbation that we used is written from the GW source, then, when writing the expression for each position (Pulsar and Solar System) we need to fix the distance to the GW source. Our motivation is to take an arbitrary GW source then we'll need to generalize that situation by integrating now over the whole space to consider GW sources from any direction at any distance from us.

However, at this time we are only interested in the angular terms to compare with the original work of Helling and Downs. Thus, we can group all the complicated terms and considering that we can write the polarization tensor in $\Delta h'_{ij}$ as $e'_{\mu\nu} = h'_+\varepsilon^{+'}_{\mu\nu} + h'_\times\varepsilon^{\times'}_{\mu\nu}$ we can express (4.12) as

$$z'(T,\hat{n}) = \sum_A \frac{1}{2}\frac{\hat{p}^i\hat{p}^j}{1+\hat{n}\cdot\hat{p}}\varepsilon^{A'}_{\mu\nu}\Delta h''_A$$

and we can define $F'^A = \varepsilon^{A'}_{\mu\nu}\frac{1}{2}\frac{\hat{p}^i\hat{p}^j}{1+\hat{n}\cdot\hat{p}}$, then we can finally write:

$$z'(T,\hat{n}) = \sum_A \Delta h''_A F'^A \tag{4.13}$$

4.3 Cross Correlation of Pulsar Data

Our assumptions for the data will be the same that in Chapter 2. Specifically from eq: 2.36, we will have that now, the expectation value of the redshift can be expressed as:

$$\left\langle \bar{z}'_1 z'_2 \right\rangle = \sum_A \int_{S^2} d\hat{n} \quad \left\langle \Delta\bar{h}_1^{A'}(\omega,\hat{n})\Delta h_2^{A'}(\omega,\hat{n}) \right\rangle F_1'^A(\hat{n})F_2'^A(\hat{n}) \tag{4.14}$$

Again, from our assumptions for the GW background, and considering the regime for Pulsar Timing experiments, we can express the previous equation as

$$\left\langle \bar{z}'_1 z'_2 \right\rangle = \frac{\left\langle h'^2(\omega) \right\rangle}{\beta}\Gamma'_0(\omega) \tag{4.15}$$

with

$$\Gamma'_0 = \beta \sum_A \int_{S^2} d\hat{n} \quad F_1'^A(\hat{n})F_2'^A(\hat{n}) \tag{4.16}$$

Again, to calculate this, we will fix the coordinates for each pulsar pair in which $\hat{p}_1$ is parallel to the z-axis and $\hat{p}_2$ is in the $x-z$ plane as we see in Fig 2.4 (we use the same configuration),

$$\hat{p}_1 = (0,0,1), \qquad \hat{p}_2 = (\sin\xi, 0, \cos\xi) \tag{4.17}$$

where, as we explained before, ξ is the angular separation between the two pulsars. With this coordinate choice, when calculating $\hat{p}^i\hat{p}^j\hat{p}^k\hat{p}^l\varepsilon_{ij}^{\times'}\varepsilon_{kl}^{\times'}$ (the numerator for the $\times$ polarization) we notice that this term vanishes. Then, again, only the $+$ polarization contributes to (4.16). Then we will express it as

$$\Gamma'_0(\xi) = \beta \int_{S^2} d\hat{n} \quad F_1'^+(\hat{n})F_2'^+(\hat{n}) \tag{4.18}$$

4.4 Hellings & Downs Curve in an Expanding Universe

Direct evaluation of the last equation, considering $\hat{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$, will result in

$$\Gamma_0'(\xi) =$$
$$\beta \int_{S^2} d\Omega \quad \frac{\sin^2\theta\left[-\sin^2\xi\sin^2\phi + \sin^2\xi\cos^2\theta\cos^2\phi + \cos^2\xi\sin^2\theta - 2\sin\xi\cos\xi\sin\theta\cos\theta\cos\phi\right]}{(1+\cos\theta)(1+\cos\xi\cos\theta + \sin\xi\sin\theta\cos\phi)} \tag{4.19}$$

where $d\Omega = \sin\theta d\theta d\phi$ is the solid angle. Solving this integral over all directions will result in

$$\Gamma_0'(\xi) = \frac{4\pi}{3}\beta\left(\frac{3}{2}(1-\cos\xi)\left[\ln\left(\frac{1-\cos\xi}{2}\right) - \frac{1}{6}\right] + 1\right) \tag{4.20}$$

which is the exact same solution found in equation (2.46).

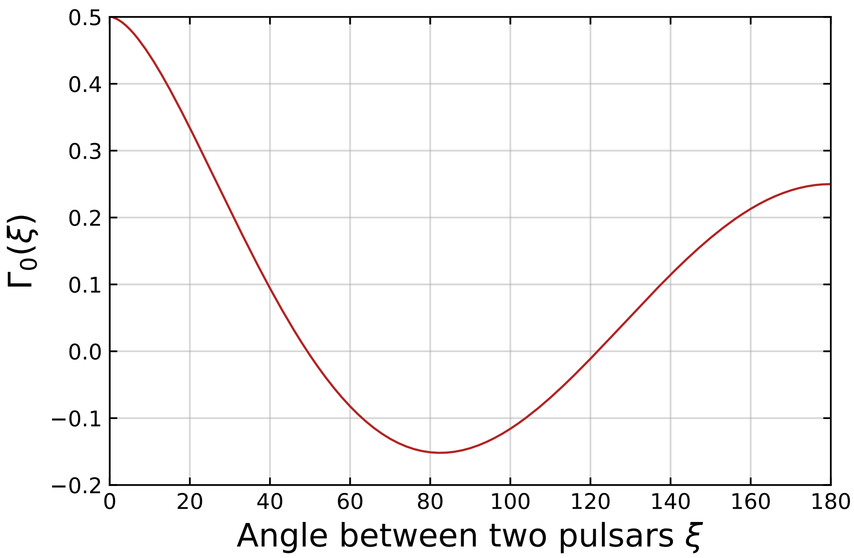

Fig. 4.2 Overlap Reduction Function (or Hellings and Downs Curve) in the transformed coordinate system.

We can see in Figure 4.2 the actual plot for the derived $\Gamma_0'(\xi)$ which is the same as in the usual case. Thus, we can finally say that the angular factor originally derived by Hellings and Downs in 1983 does not get affected by the expansion of the Universe. At first this is expected because the coordinate transformations leaves the angles untransformed, then, given that this angular factor only depends on the angles, it is expected that it will not change at all.

Chapter 5

Conclusions and Future Work

In this work we studied the behavior of the angular function originally derived by Hellings and Downs in 1983 under coordinate transformations to account for the expansion of the universe within the framework of ΛCDM. It is found that this angular factor does not change when is considered that the metric perturbation is affected by the accelerated expansion (3.6).

Nevertheless, it is expected that the actual amplitude or strain for the GW will be affected by the expansion of the universe. In order to verify this, a complete expression for $\Delta h''_A$ is needed.

During the development of this work, we have also found a expression for the metric perturbation in a more general way (4.3) which generalizes the propagation direction of the GW.

We have already mentioned that we continued the investigation line developed by [1],[2] and lately [3]. One of their motivations was to study the propagation of GW in other models of gravitation but they needed this starting point with ΛCDM. Continuing on their motivation, it could be of interest to study this angular dependence on different gravitation models. For example in *Delta Gravity* [22], which is particularly interesting because it does not need a cosmological constant to account for the accelerated expansion of the universe [23], [24],[25]. It would be interesting, to study the effects of GW in this theory and check if there is any change in this angular function.

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