Copyright.

Dedication.

To my daughter, the light of my eyes, that teach me the science is not all.

Acknowledgements.

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Contents

Summary.

We present a model of the gravitational field based on two symmetric tensors, $g_{\mu\nu}$ and $\tilde{g}_{\mu\nu}$. Besides, we have a new matter fields given by $\tilde{\phi}_I = \tilde{\delta}\phi_I$, where ϕ_I are the original matter fields. We call them $\tilde{\delta}$ matter fields. This theory have a excellent properties in a quantum level. It lives at one loop only, the classical equations of motion to the original fields are conserved and it is finite quantum theory in the vacuum. We call this theory $\tilde{\delta}$ gravity. Then, we find that massive particles do not follow a geodesic while massless particles trajectories are null geodesics of an effective metric. We analyze some cases to study the effect of the new gravitational fields. In first place, we see the Schwarzschild case, where we get a modified deflection of the light produced by the sun. In second place, we see the Cosmological case, where we get an accelerated expansion of the Universe without dark energy. We obtain a different age of the universe and we obtain that a Big-Rip is necessary to explain the expansion. Finally, we see the Non-Relativistic case, where we obtain the Post-Newtonian limit. We do a little analysis in the Newtonian limit to interpret the dark matter like $\tilde{\delta}$ matter.

Introduction.

We know that general relativity (GR) has been tested on scales larger than a millimeter to the solar-system scale [1, 2]. Nevertheless, its quantization has proved to be difficult, though. The theory is non-renormalizable, which prevents its unification with the other forces of nature. Trying to make sense of quantum GR is the main physical motivation of string theories, Loop Quantum Gravity and others [3]-[10], but none has been accepted as the correct and final answer to the problem of quantum gravity. Moreover, recent discoveries in cosmology have revealed that most part of matter is in the form of unknown matter, dark matter [11]-[19], and that the dynamics of the expansion of the Universe is governed by a mysterious component that accelerates the expansion, dark energy [20]-[22]. Although GR is able to accommodate both dark matter and dark energy, the interpretation of the dark sector in terms of fundamental theories of elementary particles is problematic [23]. Although some candidates exists that could play the role of dark matter, none have been detected yet . Also, an alternative explanation based on the modification of the dynamics for small accelerations cannot be ruled out [24, 25].

In GR, dark energy can be explained if a small cosmological constant (Λ) is present. In early times, this constant is irrelevant, but at the later stages of the evolution of the Universe Λ will dominate the expansion, explaining the acceleration. Such small Λ is very difficult to generate in quantum field theory (QFT) models, because Λ is the vacuum energy, which is usually very large [26].

One of the most important mysteries in cosmology and cosmic structure formation is to understand the nature of dark energy in the context of a fundamental physical theory [27, 28]. In recent years there has been various proposals to explain the observed acceleration of the Universe. They include some additional fields in approaches like quintessence, chameleon, vector dark energy or massive gravity; The addition of higher order terms in the Einstein-Hilbert action, like $f(R)$ theories and Gauss-Bonnet terms and finally the introduction of extra dimensions for a modification of gravity on large scales (See [29]).

Less widely explored, but interesting possibilities, are the search for non-trivial ultraviolet fixed points in gravity (asymptotic safety [30]) and the notion of induced gravity [31]-[34]. The first possibility uses exact renormalization-group techniques [35]-[38] together with lattice and numerical techniques such as Lorentzian triangulation analysis [39]. Induced gravity proposes that gravitation is a residual force produced by other interactions.

Recently, in [40, 41] a field theory model explores the emergence of geometry by the spontaneous symmetry breaking of a larger symmetry where the metric is absent. Previous work in this direction can be found in [42]-[48].

In this paper, we present a model of gravitation that is very similar to classical GR, but could make sense at the quantum level. In the construction, we consider two different points. The first is that GR is finite on shell at one loop in the vacuum [3], so renormalization is not necessary at this level. The second is a type of gauge theories, $\tilde{\delta}$ gauge theories (DGT), presented in [49, 50], which main properties are: (a) New kind of fields are created, $\tilde{\phi}_I$, from the originals ϕ_I. (b) The classical equations of motion of ϕ_I are satisfied in the full quantum theory. (c) The model lives at one loop. (d) The action is obtained through the extension of the original gauge symmetry of the model, introducing an extra symmetry that we call $\tilde{\delta}$ symmetry, since it is formally obtained as the variation of the original symmetry. When we apply this prescription to GR we obtain $\tilde{\delta}$ gravity. Quantization of $\tilde{\delta}$ gravity is discussed in [51].

Here, we study the classical effects of $\tilde{\delta}$ gravity. In first place, we will study the Schwarzschild case outside the matter like a simple example. Then, we will use this solution, in a Newtonian approximation, to calculate the deflection of light produced by the sun with $\tilde{\delta}$ gravity to compare this result with GR. This difference must be very small to explain the experimental result and be agreed with GR outside the matter in a solar system scale. The exact solution could be used in black holes. In second place, we will study the cosmological case to explain the accelerate expansion of the universe without dark matter. For this, we will assume that the Universe only has two kind of components, non relativistic matter and radiation (massless particles), which satisfy a fluid-like equation $p = \omega\rho$. In contrast to [52], where an approximation is discussed, in this work we find the exact solution of the equations corresponding to the above suppositions. This solution is used to fit the supernova data and we obtain a physical reason for the accelerated expansion of the Universe within the model: the existence of massless particles. If massless particles were absent, the expansion of the Universe would be the same as in GR without a cosmological constant. The calculus done in [53] and here is the same with the difference that we incorporate $\tilde{\delta}$ matter, plus a delta gauge fixing, in this work. For this, the final result is different, however the reason to explain the expansion of the universe is the same. In the Conclusions we speculate on a possible physical mechanism that could stop the accelerated expansion and prevent the appearance of a Big Rip. Finally, we will study the Non-Relativistic case, where we obtain the Newtonian and Post-Newtonian limit. We verify that, a Newtonian level, $\tilde{\delta}$ gravity is very similar to GR if $\tilde{\delta}$ matter is negligible. This is coincident with the Schwarzschild result in a solar system scale, however in a different scale, like the a galaxy scale $\tilde{\delta}$ matter could be important such that explain dark matter. We obtain a relation between the ordinary density and $\tilde{\delta}$ matter density.

We can say that the main properties of this model at the classical level are: (a) We can be agree with GR, far from the sources. In particular, the causal structure of $\tilde{\delta}$ gravity in vacuum is the same as in general relativity. (b) The Schwarzschild solution suggest that we have a new physics in black holes (c) When we study the evolution of the Universe, it predicts acceleration without a cosmological constant or additional scalar fields. The Universe ends in a Big Rip, similar to the scenario considered in [54]. (d) The scale factor agrees with the standard cosmology at early times and show acceleration only at later times. Therefore we expect that density perturbations should not have large corrections. (e) $\tilde{\delta}$ matter could explain dark matter in the galaxy scale.

It was noted in [50] that the Hamiltonian of delta models is not bounded from below. Phantoms cosmological models [54]-[58] also have this property, although it is not clear whether this problem will subsist or not in a diffeomorphism-invariant model as $\tilde{\delta}$ gravity. Phantom fields are used to explain the expansion of the Universe. So, even if it could be said that our model works on similar grounds, the accelerated expansion of the Universe is really produced by a reduced quantity of a radiation component in the Universe, not by a phantom field.

It should be remarked that $\tilde{\delta}$ gravity is not a metric model of gravity because massive particles do not move on geodesics. Only massless particles move on null geodesics of a linear combination of both tensor fields.

On **Chapter 1**, we will introduce the $\tilde{\delta}$ theories in general and their properties. We will define the $\tilde{\delta}$ variation to clarify the basic natation. We will introduce the new transformation produced by the $\tilde{\delta}$ variation. We focus in the general coordinate transformation because the Einstein-Hilbert action is invariant under this transformation. Finally, we define the modified action that represent $\tilde{\delta}$ theories in general. This action is invariant under our extended transformation. On **Chapter 2**, we will present the $\tilde{\delta}$ gravity action that it is invariant under extended general coordinate transformation. we will find the equations of motion of this action. We will see that the Einstein's equation are valid yet and we will obtain a new equation to $\tilde{g}_{\mu\nu}$. In this equations are defined two energy momentum tensors, $T_{\mu\nu}$ and $\tilde{T}_{\mu\nu}$. Finally, we will find this tensor to the perfect fluid. On **Chapter 3**, we will find the equation of motion to the free particle. We distingue the massive case, where the equation is not a geodesic, and the massless case, where we have a null geodesic with a effective metric. On **Chapter 4**, we will study the Schwarzschild Case. We will solve the equation of motion of $g_{\mu\nu}$ and $\tilde{g}_{\mu\nu}$ with appropriate boundary conditions. Then, we will use this solution to calculate the deflect of light by the sun. On **Chapter 5**, we will study the cosmological case. This chapter is the most important of this work. We will solve the equation to FRW metric and then, we will assume an universe without dark energy, only have non-relativistic matter and radiation to explain the accelerate expansion of the universe with $\tilde{\delta}$ gravity, assuming that we have $\tilde{\delta}$ matter. We will see that the most relevant element is the fraction between radiation and non-relativistic matter density in the present. Finally, on **Chapter 6**, we will introduce the non-relativistic case, where we calculate the Newtonian and Post-Newtonian limit. We will verify the Schwarzschild result and we will do an initial analysis to introduce the possibility to explain dark matter with $\tilde{\delta}$ gravity, using $\tilde{\delta}$ matter.

On **Appendix A**, we will calculate some relations to the equations of motion to $\tilde{\delta}$ gravity and show the variations of $T_{\mu\nu}$ necessary to the perfect fluid. On **Appendix B**, we will calculate the variation of $\tilde{\delta}G_{\mu\nu}$ to calculate the equation of motion of $\tilde{g}_{\mu\nu}$ and demonstrate that this equation is $\tilde{\delta}$ of Einstein's equation. On **Appendix C**, we will write the relation between $g_{\mu\nu}$ and $\tilde{g}_{\mu\nu}$ in term of the Vierbein e^a_μ and the Vierbein tilde $\tilde{e}^a_\mu$. We need this to develop the perfect fluid analysis. Finally, on **Appendix D**, we will show how fix the gauge in $g_{\mu\nu}$ and $\tilde{g}_{\mu\nu}$ for all the case that we solve in this work. For this, we will use the extended harmonic gauge.

It is important to notice that we work with the $\tilde{\delta}$ modification to General Relativity, based on the Einstein-Hilbert theory. From now on, we will refer to this model as $\tilde{\delta}$ Gravity.

For notation, we will use the Riemann Tensor:

$$R^{\alpha}_{\ \beta\mu\nu} = \partial_\mu \Gamma_{\nu\beta}^{\ \ \alpha} - \partial_\nu \Gamma_{\mu\beta}^{\ \ \alpha} + \Gamma_{\mu\gamma}^{\ \ \alpha}\Gamma_{\nu\beta}^{\ \ \gamma} - \Gamma_{\nu\gamma}^{\ \ \alpha}\Gamma_{\mu\beta}^{\ \ \gamma} \tag{1}$$

where the Ricci Tensor given by $R_{\mu\nu} = R^{\alpha}_{\ \ \mu\alpha\nu}$, the Ricci scalar $R = g^{\mu\nu}R_{\mu\nu}$ and:

$$\Gamma_{\mu\nu}^{\ \ \alpha} = \frac{1}{2}g^{\alpha\beta}(\partial_\nu g_{\beta\mu} + \partial_\mu g_{\nu\beta} - \partial_\beta g_{\mu\nu}) \tag{2}$$

is the usual Christoffel symbol. Finally, the covariant derivative is given by:

$$D_\nu A_\mu \equiv A_{\mu;\nu} = A_{\mu,\nu} - \Gamma_{\mu\nu}^{\ \ \alpha} A_\alpha \tag{3}$$

So, it is defined with the usual metric $g_{\mu\nu}$.

Chapter 1

$\tilde{\delta}$ Theories.

In this work we will analyze, in a classical level, a modified gravity theory named $\tilde{\delta}$ Gravity. But before, we need define the $\tilde{\delta}$ Theories in general and their properties. For more detail of this chapter, see [51].

1.1 $\tilde{\delta}$ Variation.

These theories consist in the application of a variation that we will define as $\tilde{\delta}$. As a variation it will have all properties of an usual variation such as:

$$\begin{aligned}
\tilde{\delta}(AB) &= \tilde{\delta}(A)B + A\tilde{\delta}(B) \\
\tilde{\delta}\delta A &= \delta\tilde{\delta}A \\
\tilde{\delta}(\Phi_{,\mu}) &= (\tilde{\delta}\Phi)_{,\mu}
\end{aligned} \tag{1.1}$$

The particular point with this variation is that when applied to a field (function, tensor, etc.) it will give a new elements that we define as $\tilde{\delta}$ fields which is an entire new independent object from the original $\tilde{\Phi} = \tilde{\delta}(\Phi)$. In this moment, we use the convention that a tilde tensor is equal to the $\tilde{\delta}$ transformation of the original tensor associated to it when all its indexes are covariant. So:

$$\tilde{S}_{\mu\nu\alpha\ldots} \equiv \tilde{\delta}\left(S_{\mu\nu\alpha\ldots}\right) \tag{1.2}$$

and we raise and lower indexes using the metric $g_{\mu\nu}$. Therefore:

$$
\begin{aligned}
\tilde{\delta}\left(S^{\mu}{}_{\nu\alpha\ldots}\right) &= \tilde{\delta}(g^{\mu\rho}S_{\rho\nu\alpha\ldots}) \\
&= \tilde{\delta}(g^{\mu\rho})S_{\rho\nu\alpha\ldots} + g^{\mu\rho}\tilde{\delta}\left(S_{\rho\nu\alpha\ldots}\right) \\
&= -\tilde{g}^{\mu\rho}S_{\rho\nu\alpha\ldots} + \tilde{S}^{\mu}{}_{\nu\alpha\ldots} \qquad (1.3)
\end{aligned}
$$

Where we used that $\delta(g^{\mu\nu}) = -\delta(g_{\alpha\beta})g^{\mu\alpha}g^{\nu\beta}$.

1.2 $\tilde{\delta}$ Transformation.

With the previous notation in mind, we can define how transform the tilde element given by (1.2). In general, if we have an element Φ_i that transform:

$$\bar{\delta}\Phi_i = \Lambda_i^j\Phi_j \qquad (1.4)$$

Then $\tilde{\Phi}_i = \tilde{\delta}\Phi_i$ transform:

$$\bar{\delta}\tilde{\Phi}_i = \tilde{\Lambda}_i^j\Phi_j + \Lambda_i^j\tilde{\Phi}_j \qquad (1.5)$$

Where we used that $\tilde{\delta}\bar{\delta}\Phi_i = \bar{\delta}\tilde{\delta}\Phi_i = \bar{\delta}\tilde{\Phi}_i$. Now, we considerate general coordinate transformations or diffeomorphism in its infinitesimal form:

$$
\begin{aligned}
x'^{\mu} &= x^{\mu} - \xi_0^{\mu}(x) \\
\bar{\delta}x^{\mu} &= -\xi_0^{\mu}(x) \qquad (1.6)
\end{aligned}
$$

Where $\bar{\delta}$ is the general coordinate transformation from now. Defining:

$$\xi_1^{\mu}(x) \equiv \tilde{\delta}\xi_0^{\mu}(x) \qquad (1.7)$$

and using (1.5), we can see how transform some element:

I) A scalar Φ:

$$
\begin{aligned}
\bar{\delta}\Phi &= \xi_0^{\mu}\Phi_{,\mu} \qquad (1.8) \\
\bar{\delta}\tilde{\Phi} &= \xi_1^{\mu}\Phi_{,\mu} + \xi_0^{\mu}\tilde{\Phi}_{,\mu} \qquad (1.9)
\end{aligned}
$$

II) A vector V_μ:

$$\begin{aligned}
\bar{\delta} V_\mu &= \xi_0^\beta V_{\mu,\beta} + \xi_{0,\mu}^\alpha V_\alpha && (1.10)\\
\bar{\delta} \tilde{V}_\mu &= \xi_1^\beta V_{\mu,\beta} + \xi_{1,\mu}^\alpha V_\alpha + \xi_0^\beta \tilde{V}_{\mu,\beta} + \xi_{0,\mu}^\alpha \tilde{V}_\alpha && (1.11)
\end{aligned}$$

III) Rank two Covariant Tensor $M_{\mu\nu}$:

$$\begin{aligned}
\bar{\delta} M_{\mu\nu} &= \xi_0^\rho M_{\mu\nu,\rho} + \xi_{0,\nu}^\beta M_{\mu\beta} + \xi_{0,\mu}^\beta M_{\nu\beta} && (1.12)\\
\bar{\delta} \tilde{M}_{\mu\nu} &= \xi_1^\rho M_{\mu\nu,\rho} + \xi_{1,\nu}^\beta M_{\mu\beta} + \xi_{1,\mu}^\beta M_{\nu\beta} + \xi_0^\rho \tilde{M}_{\mu\nu,\rho} + \xi_{0,\nu}^\beta \tilde{M}_{\mu\beta} + \xi_{0,\mu}^\beta \tilde{M}_{\nu\beta} && (1.13)
\end{aligned}$$

This new transformation is the basis of $\tilde{\delta}$ theories. Particulary, in gravitation we have a model with two fields. The first is just the usual gravitational field $g_{\mu\nu}$ and a second one $\tilde{g}_{\mu\nu}$. Then, we will have two gauge transformations associated to general coordinate transformation, given by:

$$\begin{aligned}
\bar{\delta} g_{\mu\nu} &= \xi_{0\mu;\nu} + \xi_{0\nu;\mu} && (1.14)\\
\bar{\delta} \tilde{g}_{\mu\nu}(x) &= \xi_{1\mu;\nu} + \xi_{1\nu;\mu} + \tilde{g}_{\mu\rho}\xi_{0,\nu}^\rho + \tilde{g}_{\nu\rho}\xi_{0,\mu}^\rho + \tilde{g}_{\mu\nu,\rho}\xi_0^\rho && (1.15)
\end{aligned}$$

where we used (1.12) and (1.13). Now, we can introduce the $\tilde{\delta}$ theories.

1.3 Modified Action.

In the last section, the general coordinate transformations were extended. So, we can look for an invariant action now. We start by considering a model which is based on a given action $S_0[\phi_I]$ where ϕ_I are generic fields, then we add to it a piece which is equal to an $\tilde{\delta}$ variation with respect to the fields and we let $\tilde{\delta}\phi_J = \tilde{\phi}_J$ so that we have:

$$S[\phi, \tilde{\phi}] = S_0[\phi] + \kappa_2 \int d^4x \frac{\delta S_0}{\delta \phi_I(x)}[\phi]\tilde{\phi}_I(x) \tag{1.16}$$

with κ_2 an arbitrary constant and the indexes I can represent any kind of indexes. This new action shows the standard structure which is used to define any modified element or function for $\tilde{\delta}$ type theories. In fact, this action is invariant under our extended general coordinate transformations developed in **section 1.2**. For this, you can see [59].

A first important property of this action is that the classical equations of the original fields are preserved. We can see this when (1.16) is varied with respect to $\tilde{\phi}_I$. That is:

$$\frac{\delta S_0}{\delta \phi_I(x)}[\phi] = 0 \tag{1.17}$$

Obviously, we have new equations when varied with respect to ϕ_I. This equations give us $\tilde{\phi}_I$ and they can be reduced to:

$$\int d^4x \frac{\delta^2 S_0}{\delta \phi_I(y)\delta \phi_J(x)}[\phi]\tilde{\phi}_J(x) = 0 \tag{1.18}$$

Another important property of these theories is in quantum level. In [51] is demonstrated that the effective action is

$$\Gamma(\Phi, \tilde{\Phi}) = S_0(\Phi) + \int d^N x \frac{\delta S_0}{\delta \Phi_I(x)} \tilde{\Phi}_I(x) + i\mathrm{Tr}\left(\log\left(\frac{\delta^2 S_0}{\delta \Phi_I(x)\delta \Phi_J(y)}\right)\right) \tag{1.19}$$

This expression is exact because the $\tilde{\delta}$ theories live only to one loop, so higher corrections simply do not exist. Finally, if we compare equation (16.42) of [60] with equation (1.19), we see that the one loop contribution is twice the original theory contribution. In general, $\mathrm{Tr}\left(\log\left(\frac{\delta^2 S_0}{\delta \Phi_I(x)\delta \Phi_J(y)}\right)\right)$ could be divergent and need to be renormalized (See [61]). From equation (1.19), we see that $\tilde{\delta}$ model will be renormalizable if the original theory is renormalizable. But, originally non-renormalizable theories could be finite or renormalizable in the $\tilde{\delta}$ version of it. Particulary, gravity is non-renormalizable, but it is known that is finite to one loop in the vacuum [3]. This means that $\tilde{\delta}$ gravity is finite in the vacuum and it could be renormalizable if we can control the infinities. This is one important motivation to study $\tilde{\delta}$ gravity in classical level like an effective theory and apply it in phenomenology. In the next chapter, we will develop the dynamic of $\tilde{\delta}$ gravity.

Chapter 2

$\tilde{\delta}$ Gravity.

Until now, we have studied $\tilde{\delta}$ theories in general. We found the invariant action to an extended general coordinate transformations, given by (1.16), with the classical equations of motion (1.17) and (1.18). In this chapter, we will present the action of $\tilde{\delta}$ gravity and then we will study the equation of motion. Finally, we will analyze the effect in a perfect fluid to apply in some particular cases in the next chapters.

2.1 Equations of Motion.

Now, we are ready to study the modifications to gravity. For this, let us consider the Einstein-Hilbert Action:

$$S_0 = \int d^4x\sqrt{-g}\left(-\frac{R}{2\kappa} + L_M\right) \tag{2.1}$$

So, using (1.16), this action involves:

$$S = \int d^4x\sqrt{-g}\left(-\frac{R}{2\kappa} + L_M + \frac{\kappa_2}{2\kappa}\left(G^{\alpha\beta} - \kappa T^{\alpha\beta}\right)\tilde{g}_{\alpha\beta} + \kappa_2\tilde{L}_M\right) \tag{2.2}$$

Where $\kappa = \frac{8\pi G}{c^4}$, $\tilde{g}_{\mu\nu} = \tilde{\delta}g_{\mu\nu}$ and:

$$T^{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta}{\delta g_{\mu\nu}}\left[\sqrt{-g}L_M\right] \tag{2.3}$$

$$\tilde{L}_M = \tilde{\phi}_I\frac{\delta L_M}{\delta\phi_I} + (\partial_\mu\tilde{\phi}_I)\frac{\delta L_M}{\delta(\partial_\mu\phi_I)} \tag{2.4}$$

with ϕ_I and $\tilde{\phi}_I = \tilde{\delta}\phi_I$ are the matter fields and $\tilde{\delta}$ matter fields respectively. From this action, we can obtain the equations of motion of $g_{\mu\nu}$ and $\tilde{g}_{\mu\nu}$. It is easy to see that the Einstein's equation are valid yet. Beside, the equation to $\tilde{g}_{\mu\nu}$ is:

$$\begin{aligned} F^{(\mu\nu)(\alpha\beta)\rho\lambda} D_\rho D_\lambda \tilde{g}_{\alpha\beta} &+ \frac{1}{2} R^{\alpha\beta} \tilde{g}_{\alpha\beta} g^{\mu\nu} + \frac{1}{2} R \tilde{g}^{\mu\nu} - R^{\mu\alpha} \tilde{g}^{\nu}_{\alpha} - R^{\nu\alpha} \tilde{g}^{\mu}_{\alpha} + \frac{1}{2} \tilde{g}^{\alpha}_{\alpha} G^{\mu\nu} \\ &= \frac{\kappa}{\sqrt{-g}} \frac{\delta}{\delta g_{\mu\nu}} \left[\sqrt{-g} \left(T^{\alpha\beta} \tilde{g}_{\alpha\beta} - 2\tilde{L}_M \right) \right] \end{aligned} \quad (2.5)$$

with:

$$\begin{aligned} F^{(\mu\nu)(\alpha\beta)\rho\lambda} &= P^{((\rho\mu)(\alpha\beta))} g^{\nu\lambda} + P^{((\rho\nu)(\alpha\beta))} g^{\mu\lambda} - P^{((\mu\nu)(\alpha\beta))} g^{\rho\lambda} - P^{((\rho\lambda)(\alpha\beta))} g^{\mu\nu} \\ P^{((\alpha\beta)(\mu\nu))} &= \frac{1}{4} \left(g^{\alpha\mu} g^{\beta\nu} + g^{\alpha\nu} g^{\beta\mu} - g^{\alpha\beta} g^{\mu\nu} \right) \end{aligned} \quad (2.6)$$

where $(\mu\nu)$ denotes that the μ and ν are in a totally symmetric combination. An important thing to notice is that both equations are of second order in derivatives which is needed to preserve causality. To simplify this equation is useful rewrite:

$$\begin{aligned} \frac{1}{\sqrt{-g}} \frac{\delta}{\delta g_{\mu\nu}} \left[\sqrt{-g} \left(T^{\alpha\beta} \tilde{g}_{\alpha\beta} - 2\tilde{L}_M \right) \right] &= -\frac{2}{\sqrt{-g}} \tilde{g}_{\alpha\beta} \frac{\delta^2}{\delta g_{\mu\nu} \delta g_{\alpha\beta}} \left[\sqrt{-g} L_M \right] \\ &\quad -\frac{2}{\sqrt{-g}} \frac{\delta}{\delta g_{\mu\nu}} \left[\sqrt{-g} \tilde{L}_M \right] \\ &= -\frac{2}{\sqrt{-g}} \tilde{g}_{\alpha\beta} \frac{\delta}{\delta g_{\alpha\beta}} \left[\frac{\delta}{\delta g_{\mu\nu}} \left[\sqrt{-g} L_M \right] \right] \\ &\quad -\frac{2}{\sqrt{-g}} \frac{\delta}{\delta g_{\mu\nu}} \left[\sqrt{-g} \tilde{L}_M \right] \\ &= \frac{1}{\sqrt{-g}} \tilde{g}_{\alpha\beta} \frac{\delta}{\delta g_{\alpha\beta}} \left[\sqrt{-g} T^{\mu\nu} \right] - \frac{2}{\sqrt{-g}} \frac{\delta}{\delta g_{\mu\nu}} \left[\sqrt{-g} \tilde{L}_M \right] \\ &= \tilde{g}_{\alpha\beta} \frac{\delta T^{\mu\nu}}{\delta g_{\alpha\beta}} + \frac{1}{2} \tilde{g}^{\alpha}_{\alpha} T^{\mu\nu} - \frac{2}{\sqrt{-g}} \frac{\delta}{\delta g_{\mu\nu}} \left[\sqrt{-g} \tilde{L}_M \right] \end{aligned} \quad (2.7)$$

Where we used (2.3), (2.4) and:

$$\begin{aligned} -\frac{2}{\sqrt{-g}} \frac{\delta}{\delta g_{\mu\nu}} \left[\sqrt{-g} \tilde{L}_M \right] &= -\frac{2}{\sqrt{-g}} \frac{\delta}{\delta g_{\mu\nu}} \left[\sqrt{-g} \left(\tilde{\phi}_I \frac{\delta L_M}{\delta \phi_I} + (\partial_\alpha \tilde{\phi}_I) \frac{\delta L_M}{\delta(\partial_\alpha \phi_I)} \right) \right] \\ &= \tilde{\phi}_I \frac{\partial}{\partial \phi_I} \left[-\frac{2}{\sqrt{-g}} \frac{\delta}{\delta g_{\mu\nu}} \left[\sqrt{-g} L_M \right] \right] \\ &\quad +(\partial_\alpha \tilde{\phi}_I) \frac{\partial}{\partial(\partial_\alpha \phi_I)} \left[-\frac{2}{\sqrt{-g}} \frac{\delta}{\delta g_{\mu\nu}} \left[\sqrt{-g} L_M \right] \right] \\ &= \tilde{\phi}_I \frac{\partial T^{\mu\nu}}{\partial \phi_I} + (\partial_\alpha \tilde{\phi}_I) \frac{\partial T^{\mu\nu}}{\partial(\partial_\alpha \phi_I)} \end{aligned} \quad (2.8)$$

Therefore:

$$\begin{aligned}
\frac{1}{\sqrt{-g}}\frac{\delta}{\delta g_{\mu\nu}}\left[\sqrt{-g}\left(T^{\alpha\beta}\tilde{g}_{\alpha\beta}-2\tilde{L}_M\right)\right] &= \tilde{g}_{\alpha\beta}\frac{\delta T^{\mu\nu}}{\delta g_{\alpha\beta}}+\frac{1}{2}\tilde{g}^{\alpha}_{\alpha}T^{\mu\nu}+\tilde{\phi}_I\frac{\partial T^{\mu\nu}}{\partial\phi_I}+(\partial_\alpha\tilde{\phi}_I)\frac{\partial T^{\mu\nu}}{\partial(\partial_\alpha\phi_I)} \\
&= \tilde{\delta}T^{\mu\nu}+\frac{1}{2}\tilde{g}^{\alpha}_{\alpha}T^{\mu\nu} \\
&= \tilde{T}^{\mu\nu}-\tilde{g}^{\mu}_{\alpha}T^{\alpha\nu}-\tilde{g}^{\nu}_{\alpha}T^{\alpha\mu}+\frac{1}{2}\tilde{g}^{\alpha}_{\alpha}T^{\mu\nu} \qquad (2.9)
\end{aligned}$$

With $\tilde{T}_{\mu\nu}=\tilde{\delta}T_{\mu\nu}$, so $\tilde{T}^{\mu\nu}=g^{\mu\alpha}g^{\nu\beta}\tilde{\delta}T_{\alpha\beta}$. In conclusion, we can say that the equations of motion are simplified to (See **Appendix A**):

$$\begin{aligned}
G^{\mu\nu} &= \kappa T^{\mu\nu} \qquad (2.10) \\
F^{(\mu\nu)(\alpha\beta)\rho\lambda}D_\rho D_\lambda\tilde{g}_{\alpha\beta}+\frac{1}{2}g^{\mu\nu}R^{\alpha\beta}\tilde{g}_{\alpha\beta}-\frac{1}{2}\tilde{g}^{\mu\nu}R &= \kappa\tilde{T}^{\mu\nu} \qquad (2.11)
\end{aligned}$$

Besides, it is possible to demonstrate that (See **Appendix B**):

$$\tilde{\delta}\left[G_{\mu\nu}\right] = F^{(\alpha\beta)\rho\lambda}_{(\mu\nu)}D_\rho D_\lambda\tilde{g}_{\alpha\beta}+\frac{1}{2}g_{\mu\nu}R^{\alpha\beta}\tilde{g}_{\alpha\beta}-\frac{1}{2}\tilde{g}_{\mu\nu}R \qquad (2.12)$$

This means that $(2.11)_{\mu\nu}=\tilde{\delta}\left[(2.10)_{\mu\nu}\right]$.

On the other side, the action (1.16) is invariant under (1.14) and (1.15). These transformations produce two conservation rules:

$$\begin{aligned}
D_\nu T^{\mu\nu} &= 0 \qquad (2.13) \\
D_\nu\tilde{T}^{\mu\nu} &= \frac{1}{2}T^{\alpha\beta}D^\mu\tilde{g}_{\alpha\beta}-\frac{1}{2}T^{\mu\beta}D_\beta\tilde{g}^{\alpha}_{\alpha}+D_\beta(\tilde{g}^{\beta}_{\alpha}T^{\alpha\mu}) \qquad (2.14)
\end{aligned}$$

It is easy to see that (2.14) is $\tilde{\delta}\left(D_\nu T^{\mu\nu}\right)=0$. In conclusion, the equations of our model are (2.10), (2.11), (2.13) and (2.14).

2.2 Perfect Fluid.

To describe a perfect fluid, an usual action is [62]:

$$S_0=\int d^4x\sqrt{-g}\left(-\frac{R}{2\kappa}+\mathbf{r}(1+\varepsilon(\mathbf{r}))+\lambda_1(u^a u_a+1)+\lambda_2 D_\alpha(\mathbf{r}U^\alpha)\right) \qquad (2.15)$$

Where $\mathbf{r}$ is the number of particles per unit volume in the mean frame of reference of these particles, $\varepsilon(\mathbf{r})$ is the internal energy density per unit mass of the fluid, u_a is the speed of the fluid in the local frame and λ_1 and λ_2 are Lagrange multipliers that ensure the normalization of u_a and conservation of mass, respectively. Finally, we have that $U_\alpha = e^a_\alpha u_a$, where e^a_α is the Vierbein. From this action, we can see that the independent variables are $g_{\mu\nu}$, $\mathbf{r}$, u_a, λ_1 and λ_2. e^a_α depend of $g_{\mu\nu}$. Therefore, our modified action in a $\tilde{\delta}$ theory is:

$$\begin{aligned} S &= \int d^4x\sqrt{-g}(-\frac{R}{2\kappa} + \mathbf{r}(1+\varepsilon(\mathbf{r})) + \lambda_1(u^a u_a + 1) + \lambda_2 D_\alpha(\mathbf{r}U^\alpha) \\ &\quad + \frac{\kappa_2}{2\kappa}\left(G^{\alpha\beta} - \kappa T^{\alpha\beta}\right)\tilde{g}_{\alpha\beta} + \kappa_2 \tilde{L}_M) \qquad (2.16) \\ \tilde{L}_M &= \tilde{\mathbf{r}}(1+\varepsilon(\mathbf{r}) + \mathbf{r}\varepsilon'(\mathbf{r})) + \tilde{\lambda}_1(u^a u_a + 1) + 2\lambda_1 u^a \tilde{u}_a + \tilde{\lambda}_2 D_\alpha(\mathbf{r}U^\alpha) \\ &\quad + \lambda_2 D_\alpha(\tilde{\mathbf{r}}U^\alpha + \mathbf{r}U^\alpha_T) \qquad (2.17) \end{aligned}$$

with $\tilde{\mathbf{r}} = \tilde{\delta}\mathbf{r}$, $\varepsilon'(\mathbf{r}) = \frac{\partial\varepsilon}{\partial\mathbf{r}}(\mathbf{r})$, $\tilde{u}_a = \tilde{\delta}u_a$, $U^\alpha_T = e^{a\alpha}\tilde{u}_a$, $\tilde{\lambda}_1 = \tilde{\delta}\lambda_1$ and $\tilde{\lambda}_2 = \tilde{\delta}\lambda_2$ are new Lagrange multipliers and:

$$T_{\mu\nu} = -\frac{1}{2}\lambda_{2,\alpha}\mathbf{r}\left(\delta^\alpha_\nu U_\mu + \delta^\alpha_\mu U_\nu\right) - (\mathbf{r}(1+\varepsilon(\mathbf{r})) + \lambda_1(u^a u_a + 1) - \lambda_{2,\alpha}\mathbf{r}U^\alpha)g_{\mu\nu} \qquad (2.18)$$

Where we used (6.48), in the **Appendix C**. Then, we know that:

$$\begin{aligned} \tilde{T}_{\mu\nu} &= -\frac{1}{4}\lambda_{2,\beta}\mathbf{r}\left(\delta^\beta_\nu U^\alpha \tilde{g}_{\mu\alpha} + \delta^\beta_\mu U^\alpha \tilde{g}_{\nu\alpha} + 2g_{\mu\nu}U^\alpha \tilde{g}^\beta_\alpha\right) \\ &\quad - \left(\mathbf{r}(1+\varepsilon(\mathbf{r})) + \lambda_1(u^a u_a + 1) - \lambda_{2,\rho}\mathbf{r}U^\rho\right)\tilde{g}_{\mu\nu} \\ &\quad -\frac{1}{2}\tilde{\lambda}_{2,\alpha}\mathbf{r}\left(\delta^\alpha_\nu U_\mu + \delta^\alpha_\mu U_\nu\right) - \frac{1}{2}\lambda_{2,\alpha}\tilde{\mathbf{r}}\left(\delta^\alpha_\nu U_\mu + \delta^\alpha_\mu U_\nu\right) - \frac{1}{2}\lambda_{2,\alpha}\mathbf{r}\left(\delta^\alpha_\nu U^T_\mu + \delta^\alpha_\mu U^T_\nu\right) \\ &\quad -(\tilde{\mathbf{r}}(1+\varepsilon(\mathbf{r}) + \mathbf{r}\varepsilon'(\mathbf{r})) + \tilde{\lambda}_1(u^a u_a + 1) + 2\lambda_1 u^a \tilde{u}_a - \tilde{\lambda}_{2,\alpha}\mathbf{r}U^\alpha \\ &\quad -\lambda_{2,\alpha}(\tilde{\mathbf{r}}U^\alpha + \mathbf{r}U^\alpha_T))g_{\mu\nu} \qquad (2.19) \end{aligned}$$

Therefore, we have a modified action with ten independent variables: $g_{\mu\nu}$, $\mathbf{r}$, u_a, λ_1, λ_2, $\tilde{g}_{\mu\nu}$, $\tilde{\mathbf{r}}$, $\tilde{u}_a$, $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$. So, we can solve (2.10) and (2.11) using (2.18) and (2.19) to obtain $g_{\mu\nu}$ and $\tilde{g}_{\mu\nu}$. Besides, we have equations of motion to $\mathbf{r}$, u_a, λ_1, λ_2, $\tilde{\mathbf{r}}$, $\tilde{u}_a$, $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$. These equations can be reduced to:

$$u^a u_a + 1 = 0 \tag{2.20}$$

$$D_\alpha(\mathbf{r}U^\alpha) = 0 \tag{2.21}$$

$$2\lambda_1 u^a - \mathbf{r}e^{a\alpha}\lambda_{2,\alpha} = 0 \tag{2.22}$$

$$1 + \varepsilon(\mathbf{r}) + \mathbf{r}\varepsilon'(\mathbf{r}) - U^\alpha \lambda_{2,\alpha} = 0 \tag{2.23}$$

$$u^a \tilde{u}_a = 0 \tag{2.24}$$

$$D_\alpha\left(\tilde{\mathbf{r}}U^\alpha + \mathbf{r}U_T^\alpha - \frac{1}{2}\mathbf{r}\tilde{g}^{\alpha\beta}U_\beta + \frac{1}{2}\mathbf{r}\tilde{g}_\beta^\beta U^\alpha\right) = 0 \tag{2.25}$$

$$2\tilde{\lambda}_1 u^a + 2\lambda_1 \tilde{u}^a - e^{a\alpha}\left(\mathbf{r}\tilde{\lambda}_{2,\alpha} + \tilde{\mathbf{r}}\lambda_{2,\alpha} - \frac{1}{2}\tilde{g}_\alpha^\beta \mathbf{r}\lambda_{2,\beta}\right) = 0 \tag{2.26}$$

$$\tilde{\mathbf{r}}\left(2\varepsilon'(\mathbf{r}) + \mathbf{r}\varepsilon''(\mathbf{r})\right) - U^\alpha\tilde{\lambda}_{2,\alpha} - U_T^\alpha\lambda_{2,\alpha} + \frac{1}{2}U_\beta\tilde{g}^{\alpha\beta}\lambda_{2,\alpha} = 0 \tag{2.27}$$

Now, we can use these equations to simplify (2.18) and (2.19), eliminating the Lagrange multipliers. The equations (2.22) and (2.23) can be rewrite like:

$$\lambda_1 = -\frac{1}{2}\mathbf{r}\left(1 + \varepsilon(\mathbf{r}) + \mathbf{r}\varepsilon'(\mathbf{r})\right) \tag{2.28}$$

$$\lambda_{2,\mu} = -\left(1 + \varepsilon(\mathbf{r}) + \mathbf{r}\varepsilon'(\mathbf{r})\right)U_\mu \tag{2.29}$$

In the same form, (2.26) and (2.27) can be reduce to:

$$\tilde{\lambda}_1 = -\frac{1}{2}\tilde{\mathbf{r}}\left(1 + \varepsilon(\mathbf{r}) + 3\mathbf{r}\varepsilon'(\mathbf{r}) + \mathbf{r}^2\varepsilon''(\mathbf{r})\right) \tag{2.30}$$

$$\tilde{\lambda}_{2,\mu} = -\tilde{\mathbf{r}}\left(2\varepsilon'(\mathbf{r}) + \mathbf{r}\varepsilon''(\mathbf{r})\right)U_\mu - \left(1 + \varepsilon(\mathbf{r}) + \mathbf{r}\varepsilon'(\mathbf{r})\right)\left(U_\mu^T + \frac{1}{2}\tilde{g}_\mu^\beta U_\beta\right) \tag{2.31}$$

So, using these identities, we can reduce the energy-momentum tensors to:

$$T_{\mu\nu} = \mathbf{r}^2\varepsilon'(\mathbf{r})g_{\mu\nu} + \mathbf{r}\left(1 + \varepsilon(\mathbf{r}) + \mathbf{r}\varepsilon'(\mathbf{r})\right)U_\mu U_\nu \tag{2.32}$$

$$\begin{aligned}\tilde{T}_{\mu\nu} = {} & \mathbf{r}^2\varepsilon'(\mathbf{r})\tilde{g}_{\mu\nu} + \mathbf{r}\tilde{\mathbf{r}}\left(2\varepsilon'(\mathbf{r}) + \mathbf{r}\varepsilon''(\mathbf{r})\right)g_{\mu\nu} + \tilde{\mathbf{r}}\left(1 + \varepsilon(\mathbf{r}) + 3\mathbf{r}\varepsilon'(\mathbf{r}) + \mathbf{r}\varepsilon''(\mathbf{r})\right)U_\mu U_\nu \\ & + \mathbf{r}\left(1 + \varepsilon(\mathbf{r}) + \mathbf{r}\varepsilon'(\mathbf{r})\right)\left(\frac{1}{2}\left(U_\nu U^\alpha \tilde{g}_{\mu\alpha} + U_\mu U^\alpha \tilde{g}_{\nu\alpha}\right) + U_\mu^T U_\nu + U_\mu U_\nu^T\right)\end{aligned} \tag{2.33}$$

and survive the equations:

$$U^\alpha U_\alpha + 1 = 0 \tag{2.34}$$

$$D_\alpha(\mathbf{r}U^\alpha) = 0 \tag{2.35}$$

$$(1+\varepsilon(\mathbf{r})+\mathbf{r}\varepsilon'(\mathbf{r}))\,U^\alpha D_\alpha U_\mu + \left(\delta^\alpha_\mu + U^\alpha U_\mu\right)(2\varepsilon'(\mathbf{r})+\mathbf{r}\varepsilon''(\mathbf{r}))\,\partial_\alpha \mathbf{r} = 0 \tag{2.36}$$

$$U^\alpha U^T_\alpha = 0 \tag{2.37}$$

$$D_\alpha\left(\tilde{\mathbf{r}}U^\alpha + \mathbf{r}U^\alpha_T - \frac{1}{2}\mathbf{r}\tilde{g}^{\alpha\beta}U_\beta + \frac{1}{2}\mathbf{r}\tilde{g}^\beta_\beta U^\alpha\right) = 0 \tag{2.38}$$

$$\begin{aligned}
&\tilde{\mathbf{r}}\,(2\varepsilon'(\mathbf{r})+\mathbf{r}\varepsilon''(\mathbf{r}))\,U^\alpha D_\alpha U_\mu + (1+\varepsilon(\mathbf{r})+\mathbf{r}\varepsilon'(\mathbf{r}))\left(U^\alpha_T - \frac{1}{2}\tilde{g}^{\alpha\beta}U_\beta\right) D_\alpha U_\mu \\
&+ (1+\varepsilon(\mathbf{r})+\mathbf{r}\varepsilon'(\mathbf{r}))\,U^\alpha D_\alpha\left(U^T_\mu + \frac{1}{2}\tilde{g}_{\mu\beta}U^\beta\right) + \frac{1}{2}(1+\varepsilon(\mathbf{r})+\mathbf{r}\varepsilon'(\mathbf{r}))\,U^\alpha U^\beta D_\mu \tilde{g}_{\alpha\beta} \\
&+ \left(\left(U^\alpha_T - \frac{1}{2}\tilde{g}^{\alpha\beta}U_\beta\right)U_\mu + U^\alpha\left(U^T_\mu + \frac{1}{2}\tilde{g}_{\mu\beta}U^\beta\right)\right)(2\varepsilon'(\mathbf{r})+\mathbf{r}\varepsilon''(\mathbf{r}))\,\partial_\alpha\mathbf{r} \\
&+ \left(\delta^\alpha_\mu + U^\alpha U_\mu\right)\left(\tilde{\mathbf{r}}\,(3\varepsilon''(\mathbf{r})+\mathbf{r}\varepsilon'''(\mathbf{r}))\,\partial_\alpha\mathbf{r} + (2\varepsilon'(\mathbf{r})+\mathbf{r}\varepsilon''(\mathbf{r}))\,\partial_\alpha\tilde{\mathbf{r}}\right) = 0
\end{aligned} \tag{2.39}$$

These equations are related with the Bianchi identities (2.13) and (2.14). So, we have a complete equations system. Finally, from (2.32) we can identify that $\rho = \mathbf{r}\,(1+\varepsilon(\mathbf{r}))$ and $p(\rho) = \mathbf{r}^2\varepsilon'(\mathbf{r})$. Therefore, the final expressions of the energy-momentum tensors are:

$$T_{\mu\nu} = p(\rho)g_{\mu\nu} + (\rho + p(\rho))\,U_\mu U_\nu \tag{2.40}$$

$$\begin{aligned}
\tilde{T}_{\mu\nu} &= p(\rho)\tilde{g}_{\mu\nu} + \frac{\partial p}{\partial\rho}(\rho)\tilde{\rho}g_{\mu\nu} + \left(\tilde{\rho} + \frac{\partial p}{\partial\rho}(\rho)\tilde{\rho}\right)U_\mu U_\nu \\
&\quad + (\rho + p(\rho))\left(\frac{1}{2}\left(U_\nu U^\alpha \tilde{g}_{\mu\alpha} + U_\mu U^\alpha \tilde{g}_{\nu\alpha}\right) + U^T_\mu U_\nu + U_\mu U^T_\nu\right)
\end{aligned} \tag{2.41}$$

Now, we can use (2.40) and (2.41) to solve (2.10), (2.11), (2.13) and (2.14) in a perfect fluid. In this work, we will see the cosmological and the Non-Relativistic case.

Chapter 3

Test Particle.

In the last chapter, we found the equation of motion to $\tilde{\delta}$ gravity. However, to describe some phenomenology, we need analyze the trajectory of a particle. For this, we need find the coupling of a test particle to the gravitational field. In this chapter, we will find this coupling to massive and massless particle.

3.1 Massive Particles.

We know that, in the standard case, the test particle action is given by:

$$S_0[\dot{x}, g] = -m \int dt \sqrt{-g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu} \tag{3.1}$$

with $\dot{x}^\mu = \frac{dx^\mu}{dt}$. In our model, the modified action is obtained according to (1.16). So, the new test particle action is:

$$S[\dot{x}, g, \tilde{g}] = m \int dt \left(\frac{\bar{g}_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}{\sqrt{-g_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta}} \right) \tag{3.2}$$

where $\bar{g}_{\mu\nu} = g_{\mu\nu} + \frac{\kappa_2}{2}\tilde{g}_{\mu\nu}$. If we very (3.2) with respect to x^μ, we will obtain the equation of motion for a massive test particle. That is:

$$\hat{g}_{\mu\nu}\ddot{x}^\nu + \hat{\Gamma}_{\mu\alpha\beta}\dot{x}^\alpha \dot{x}^\beta = \frac{\kappa_2}{4}\tilde{K}_{,\mu} \tag{3.3}$$

with:

$$\begin{aligned}
\hat{\Gamma}_{\mu\alpha\beta} &= \frac{1}{2}(\hat{g}_{\mu\alpha,\beta} + \hat{g}_{\beta\mu,\alpha} - \hat{g}_{\alpha\beta,\mu}) \\
\hat{g}_{\alpha\beta} &= \left(1 + \frac{\kappa_2}{2}\tilde{K}\right) g_{\alpha\beta} + \kappa_2 \tilde{g}_{\alpha\beta} \\
\tilde{K} &= \tilde{g}_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta
\end{aligned}$$

and we fix $g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = -1$, after choosing t equal to the proper time. In conclusion, the equation of motion of a free massive particle in our modified theory is more complicated that the usual case, because it is not a standard geodesic.

3.2 Massless Particles.

The expression in (3.1) and (3.2) are useless for massless particles, because are null when $m = 0$. To solve this problem, it is usual to start from the action:

$$S_0[\dot{x}, g, v] = \frac{1}{2}\int dt \left(vm^2 - v^{-1}g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu\right) \tag{3.4}$$

where v is a Lagrange multiplier. This action is invariant under reparametrizations:

$$\begin{aligned}
x'^\mu(t') &= x^\mu(t) \\
v'(t')dt' &= v(t)dt \\
t' &= t - \epsilon(t)
\end{aligned} \tag{3.5}$$

and the equation of motion for v is:

$$v = -\frac{\sqrt{-g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}}{m} \tag{3.6}$$

If we substitute (3.6) in (3.4), we recover (3.1). In other words, (3.4) is a good action that include the massless case. To our theory, we must substitute (3.4) in (1.16) to obtain the modified test particle action. That is:

$$S[\dot{x}, g, \tilde{g}, v, \tilde{v}] = \frac{1}{2}\int dt \left(vm^2 - v^{-1}\left(g_{\mu\nu} + \kappa_2\tilde{g}_{\mu\nu}\right)\dot{x}^\mu\dot{x}^\nu + \kappa_2\tilde{v}\left(m^2 + v^{-2}g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu\right)\right) \tag{3.7}$$

This action is invariant under the reparametrization transformations (3.5) plus $\tilde{v}'(t')dt' = \tilde{v}(t)dt$. So, (3.7) is the action that we need to generalize (3.2). Two Lagrange multiplier are unnecessary, so we will eliminate one of them. The equation of motion for $\tilde{v}$ is:

$$\tilde{v} = \frac{m^2 + v^{-2}\left(g_{\mu\nu} + \kappa_2 \tilde{g}_{\mu\nu}\right)\dot{x}^\mu \dot{x}^\nu}{2\kappa_2 v^{-3} g_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta} \tag{3.8}$$

If we now replace (3.8) in (3.7), we obtain the $\tilde{\delta}$ Test Particle Action:

$$S[\dot{x}, g, \tilde{g}, v] = \int dt \left(m^2 v - \frac{\left(g_{\mu\nu} + \kappa_2 \tilde{g}_{\mu\nu}\right)\dot{x}^\mu \dot{x}^\nu}{4v} + \frac{m^2 v^3}{4 g_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta}\left(m^2 + \kappa_2 v^{-2} \tilde{g}_{\mu\nu}\dot{x}^\mu \dot{x}^\nu\right)\right) \tag{3.9}$$

The equation of motion for v is still given by (3.6). If we substitute it in (3.9), we obtain (3.2). So, (3.9) is a good modified action to represent the trajectory of a particle in the presence of a gravitational field, given by g and $\tilde{g}$, for the massive and massless case. Evaluating $m = 0$ in (3.4) and (3.9), they respectively are:

$$S_0^{(m=0)}[\dot{x}, g, v] = -\frac{1}{2}\int dt v^{-1} g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu \tag{3.10}$$

$$S^{(m=0)}[\dot{x}, g, \tilde{g}, v] = -\frac{1}{4}\int dt v^{-1} \mathbf{g}_{\mu\nu}\dot{x}^\mu \dot{x}^\nu \tag{3.11}$$

with $\mathbf{g}_{\mu\nu} = g_{\mu\nu} + \kappa_2 \tilde{g}_{\mu\nu}$. The equation of motion for v implies that, in the usual and modified case, a massless particle will move in a null-geodesic. In the usual case we have $g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = 0$, but in our model the null-geodesic is $\mathbf{g}_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = 0$.

In conclusion, all these mean that, in our theory, the equation of motion of a free massless particle is given by:

$$\begin{aligned} \mathbf{g}_{\mu\nu}\ddot{x}^\nu + \mathbf{\Gamma}_{\mu\alpha\beta}\dot{x}^\alpha \dot{x}^\beta &= 0 \\ \mathbf{g}_{\mu\nu}\dot{x}^\mu \dot{x}^\nu &= 0 \end{aligned} \tag{3.12}$$

with:

$$\mathbf{\Gamma}_{\mu\alpha\beta} = \frac{1}{2}(\mathbf{g}_{\mu\alpha,\beta} + \mathbf{g}_{\beta\mu,\alpha} - \mathbf{g}_{\alpha\beta,\mu})$$

To resume, we have analyze the $\tilde{\delta}$ gravity. We obtained the equation of motion of $g_{\mu\nu}$ and $\tilde{g}_{\mu\nu}$, given by (2.10), (2.11), (2.13) and (2.14), and we know how solve them to a perfect fluid using (2.40) and (2.41). Then, we obtain how a test particle move when it is coupled to $g_{\mu\nu}$ and $\tilde{g}_{\mu\nu}$, given by (3.3) or (3.12) if we have a massive or massless particle respectively.

In the next chapters, we will study some cases to apply $\tilde{\delta}$ gravity. In first place, we will see the Schwarzschild case and apply it to gravitational lensing. Then, we will see the cosmological case to explain the accelerate expansion of the universe with $\tilde{\delta}$ gravity, but without dark energy. Finally, we will study the Non-Relativistic limit to reproduced the Newtonian approximation and introduce an explanation to dark matter.

Chapter 4

Schwarzschild Case.

In this chapter, we will study $\tilde{\delta}$ gravity with a Schwarzschild metric. First, we will calculate $\tilde{g}_{\mu\nu}$ with the correct boundary conditions to solve the differential equations and then we will calculate the photon trajectory to find the light deflection by gravitational lensing.

4.1 Schwarzschild metric.

In first place, we need find the boundary condition. We must remember that, in this case, $g_{\mu\nu} \to \eta_{\mu\nu}$ to $r \to \infty$ and $\tilde{g}_{\mu\nu} = \tilde{\delta} g_{\mu\nu}$. Because $\tilde{\delta}\eta_{\mu\nu} = 0$, it is natural to use that $\tilde{g}_{\mu\nu} \to 0$ to $r \to \infty$.

Now, we can solve the equations of motion to Schwarzschild. In this case, the metric is:

$$g_{\mu\nu}dx^{\mu}dx^{\nu} = -A(r)c^2dt^2 + B(r)dr^2 + r^2(d\theta^2 + \sin^2(\theta)d\phi^2) \tag{4.1}$$

For $\tilde{g}_{\mu\nu}$, we can use a similar expression:

$$\tilde{g}_{\mu\nu}dx^{\mu}dx^{\nu} = -\tilde{A}(r)c^2dt^2 + \tilde{B}(r)dr^2 + \tilde{F}(r)r^2(d\theta^2 + \sin^2(\theta)d\phi^2) \tag{4.2}$$

For simplify the equations, we will solve them outside the matter, this means in the region where $\tilde{T}_{\mu\nu} = T_{\mu\nu} = 0$. The solutions of our equations of motion (2.10) and (2.11) are:

$$
\begin{aligned}
A(r) &= 1-\frac{2\mu}{r} && (4.3)\\
B(r) &= \frac{1}{1-\frac{2\mu}{r}} && (4.4)\\
\tilde{B}(r) &= \frac{r^2(r-2\mu)\tilde{A}'(r)-2\mu r\tilde{A}(r)+r(r-2\mu)(r-\mu)\tilde{F}'(r)+r(r-2\mu)\tilde{F}(r)}{(r-2\mu)^2} && (4.5)
\end{aligned}
$$

Where $\mu = GM$, G is the Newton constant and M is the mass of the gravitational source. And survive the equation:

$$
r\tilde{A}''(r)+2\tilde{A}'(r)-\mu\tilde{F}''(r)=0 \qquad (4.6)
$$

Where $' = \frac{d}{dr}$. In (4.3) and (4.4) we imposed that $A(\infty)=B(\infty)=1$, because we want that $g_{\mu\nu}\to\eta_{\mu\nu}$ to $r\to\infty$. On the other hand, we want that $\tilde{g}_{\mu\nu}\to 0$ to $r\to\infty$, but to find $\tilde{A}(r)$ and $\tilde{F}(r)$ we need an additional equation, fixing the gauge of $\tilde{g}_{\mu\nu}$. For this, we will use the harmonic gauge. A convenient harmonic coordinate system is:

$$
\begin{aligned}
X_1 &= (r-\mu)\sin(\theta)\cos(\phi)\\
X_2 &= (r-\mu)\sin(\theta)\sin(\phi)\\
X_3 &= (r-\mu)\cos(\theta)
\end{aligned}
$$

If we fix the harmonic gauge in these coordinates, we obtain the condition (For more details, see **Appendix D**):

$$
\begin{aligned}
r^2(r-2\mu)\tilde{A}''(r) &+ 4r(r-2\mu)\tilde{A}'(r)-4\mu\tilde{A}(r)\\
&+ r(r-2\mu)(r-\mu)\tilde{F}''(r)+4(r-\mu)^2\tilde{F}'(r)=0 && (4.7)
\end{aligned}
$$

If we solve (4.6) and (4.7) with the condition $\tilde{A}(\infty)=\tilde{B}(\infty)=\tilde{F}(\infty)=0$, we obtain that:

$$
\begin{aligned}
\tilde{A}(r) &= -\frac{2a_0\mu(r-\mu)}{r^2}-a_1\mu\frac{2\mu+(r-\mu)\ln\left(1-\frac{2\mu}{r}\right)}{r^2} && (4.8)\\
\tilde{F}(r) &= \frac{2a_0\mu}{r}-a_1\frac{2\mu+(r-\mu)\ln\left(1-\frac{2\mu}{r}\right)}{r} && (4.9)\\
\tilde{B}(r) &= \frac{2a_0\mu(r-\mu)}{(r-2\mu)^2}-a_1\frac{2\mu(r-2\mu)+(r^2-3\mu r+\mu^2)\ln\left(1-\frac{2\mu}{r}\right)}{(r-2\mu)^2} && (4.10)
\end{aligned}
$$

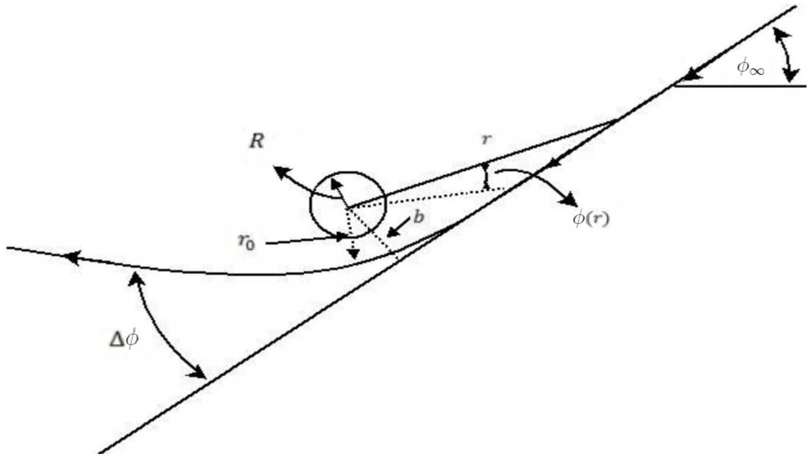

Figure 4.1: Trajectory by gravitational lensing. R is the radius of the star, r_0 is the minimal distance to the star, b is the impact parameter, ϕ_∞ is the incident direction and $\Delta\phi$ is the deflection of light.

where a_0 and a_1 are integrate constants. We must remember that, this solution correspond to the region without matter. In general, this region correspond to $r \gg \mu$. Therefore, it is enough to use the more relevant order. That is:

$$A(r) = 1 - \frac{2\mu}{r} \tag{4.11}$$

$$B(r) = 1 + \frac{2\mu}{r} + O\left(\left(\frac{\mu}{r}\right)^2\right) \tag{4.12}$$

$$\tilde{A}(r) = -\frac{2a_0\mu}{r} + O\left(\left(\frac{\mu}{r}\right)^2\right) \tag{4.13}$$

$$\tilde{F}(r) = \frac{2a_0\mu}{r} + O\left(\left(\frac{\mu}{r}\right)^2\right) \tag{4.14}$$

$$\tilde{B}(r) = \frac{2a_0\mu}{r} + O\left(\left(\frac{\mu}{r}\right)^2\right) \tag{4.15}$$

This is the Newtonian approximation and take up again in **chapter 6**. We will use these expressions in the next section too.

4.2 Gravitational Lensing.

To describe this phenomenon, we need the null geodesic. In our case, this is given by (3.12). To solve these equations, we will consider a coordinate system where $\theta = \frac{\pi}{2}$ such that the trajectory is given by **Figure 4.1**. For more detail see [63].

So, the geodesic equations given by (3.12) are very complicated, but with some work, we can reduce it to:

$$\frac{dt}{du} = \frac{1}{A(r)+\kappa_2\tilde{A}(r)} \tag{4.16}$$

$$\frac{dr}{du} = -\frac{1}{r}\sqrt{\frac{r^2(1+\kappa_2\tilde{F}(r)) - J^2(A(r)+\kappa_2\tilde{A}(r))}{(A(r)+\kappa_2\tilde{A}(r))(B(r)+\kappa_2\tilde{B}(r))(1+\kappa_2\tilde{F}(r))}} \tag{4.17}$$

$$\frac{d\phi}{du} = \frac{J}{r^2(1+\kappa_2\tilde{F}(r))} \tag{4.18}$$

Where u is the trajectory parameter such that $x^\mu = x^\mu(u)$. We have fixed t such that $t \to u$ to $r \to \infty$, using that $A(\infty) = 1$ and $\tilde{A}(\infty) = 0$. J is a constant of motion related to the angular momentum. However, it is convenient to express J in term of r_0. Since r_0 is the minimal radius, we know that $\frac{dr}{du}|_{r=r_0} = 0$. So:

$$J = r_0\sqrt{\frac{1+\kappa_2\tilde{F}(r_0)}{A(r_0)+\kappa_2\tilde{A}(r_0)}} \tag{4.19}$$

From the **Figure 4.1**, we can see that we need $\phi(r)$. To this, we use (4.17) and (4.18) to obtain:

$$\phi(r) - \phi_\infty = \int_r^\infty dr\left(\frac{r_0}{r}\right)\sqrt{\frac{(A(r)+\kappa_2\tilde{A}(r))(B(r)+\kappa_2\tilde{B}(r))}{(1+\kappa_2\tilde{F}(r))(A(r_0)+\kappa_2\tilde{A}(r_0))\left(\frac{r^2(1+\kappa_2\tilde{F}(r))}{1+\kappa_2\tilde{F}(r_0)} - \frac{r_0^2(A(r)+\kappa_2\tilde{A}(r))}{A(r_0)+\kappa_2\tilde{A}(r_0)}\right)}} \tag{4.20}$$

To solve this integral, we will use an approximation. We know that $r \geq r_0 \gg \mu$, therefore we can use (4.11-4.15). So, (4.20) is reduced to:

$$\phi(r) - \phi_\infty \simeq \int_r^\infty \frac{dr}{r}\left(1+\frac{\mu}{r}+\frac{\mu(1+2\kappa_2 a_0)r}{r_0(r+r_0)}\right)\left(\left(\frac{r}{r_0}\right)^2 - 1\right)^{-\frac{1}{2}} \tag{4.21}$$

We want describe a complete trajectory, so the photon start in ϕ_∞ to $\phi(r_0)$ and then go to ϕ_∞. Besides, if the trajectory were a straight line, this would equal just π. All these mean that the deflection of light is:

$$
\begin{aligned}
\Delta\phi &= 2|\phi(r_0) - \phi_\infty| - \pi \\
&\simeq 2\left|\int_{r_0}^{\infty} \frac{dr}{r}\left(1 + \frac{\mu(2\kappa_2 a_0 r^2 + r^2 + r r_0 + r_0^2)}{r_0(r+r_0)r}\right)\left(\left(\frac{r}{r_0}\right)^2 - 1\right)^{-\frac{1}{2}}\right| - \pi \\
&\simeq \frac{4\mu(1+\kappa_2 a_0)}{r_0}
\end{aligned} \tag{4.22}
$$

With usual gravity, we know that $\Delta\phi = \frac{4\mu}{r_0}$. So, in our modified gravity, we have an additional term given by $\frac{4\mu\kappa_2 a_0}{r_0}$. On the other side, we have an experimental value $\Delta\phi_{Exp} = 1.761'' \pm 0.016''$ to the sun [64] and a theoretical value $\Delta\phi_{Theo} = 1.757''$. This means that, to satisfy the experimental value, we need:

$$
\begin{aligned}
\left|\frac{4\mu\kappa_2 a_0}{r_0}\right| &= 1.757''|\kappa_2 a_0| < 0.016'' \\
|\kappa_2 a_0| &< 0.009106
\end{aligned} \tag{4.23}
$$

In this case, we have a small value to $\kappa_2 a_0$. We will look over in **chapter 6** to explain this new term like a dark matter.

Chapter 5

Cosmological Case.

In this work, we will study photons emitted from the supernovas, so we will need the modified null-geodesic given by (3.12). But, it is important to observe that the proper time is defined in terms of massive particles, so that it is necessary to reinterpret the supernova data. So, in this section, we define the measurement of time and distances in the model.

The equation (3.3) preserves the proper time of the particle along the trajectory: Along the trajectory $g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = -1$. So, we must define proper time using the original metric $g_{\mu\nu}$:

$$d\tau = \frac{1}{c}\sqrt{-g_{\mu\nu}dx^\mu dx^\nu} = \sqrt{-g_{00}}dt \tag{5.1}$$

Following [65], we consider the motion of light rays along infinitesimally near trajectories, using (3.12) and (5.1), to get the three-dimensional metric:

$$\begin{aligned} dl^2 &= \gamma_{ij}dx^i dx^j \qquad (5.2) \\ \gamma_{ij} &= \frac{g_{00}}{\mathbf{g}_{00}}\left(\mathbf{g}_{ij} - \frac{\mathbf{g}_{i0}\mathbf{g}_{j0}}{\mathbf{g}_{00}}\right) \end{aligned}$$

Therefore, we measure proper time using the metric $g_{\mu\nu}$, but the space geometry is determined by both tensor fields, $g_{\mu\nu}$ and $\tilde{g}_{\mu\nu}$. These considerations are fundamental to explain the expansion of the Universe with $\tilde{\delta}$ gravity. Now, we will solve the equation of motion to FRW metric.

5.1 FRW metric.

To describe the supernova data, we must use the FRW metric. When a photon emitted from the supernova travels to the Earth, the Universe is expanding. This means that the photon is affected by the cosmological Doppler effect. So the metric $g_{\mu\nu}$ is:

$$g_{\mu\nu}dx^\mu dx^\nu = -c^2dt^2 + R^2(t)\left(dx^2 + dy^2 + dz^2\right) \tag{5.3}$$

Assuming an isotropic and homogeneous Universe, we can use the following ansatz for $\tilde{g}_{\mu\nu}$:

$$\tilde{g}_{\mu\nu}dx^\mu dx^\nu = -3T_d(t)c^2dt^2 + T_d(t)R^2(t)\left(dx^2 + dy^2 + dz^2\right) \tag{5.4}$$

such that, with the change of variable $t \to t(u)$ where $\frac{dt(u)}{du} = R^3(u)$, we have an harmonic gauge. So, the gauge is completely fix (See **Appendix D**).

On the other side, we have a perfect fluid. So, the energy-momentum tensors are given by (2.40) and (2.41). We know that 2.10) and (2.13), with $U_\mu = (c, 0, 0, 0)$, are reduced to:

$$\left(\frac{\dot{R}(t)}{R(t)}\right)^2 = \frac{\kappa c^2}{3}\sum_i \rho_i(t) \tag{5.5}$$

$$\dot{\rho}_i(t) = -\frac{3\dot{R}(t)}{R(t)}(\rho_i(t) + p_i(t)) \tag{5.6}$$

with $\dot{f}(t) = \frac{df}{dt}(t)$. But, to solve (5.5) and (5.6), we need equations of state which relate $\rho_i(t)$ and $p_i(t)$, for which we take $p_i(t) = \omega_i\rho_i(t)$. Since we wish to explain dark energy with $\tilde{\delta}$ gravity, we will assume that in the Universe we only have non relativistic matter (cold dark matter, baryonic matter) and radiation (photons, massless particles). So, we will require two equations of state. For non relativistic matter we use $p_M(t) = 0$ and for radiation $p_R(t) = \frac{1}{3}\rho_R(t)$, where we have assumed that their interaction is null. Replacing in (5.5) and (5.6) and solving them, we find the exact solution:

$$
\begin{aligned}
\rho(X) &= \rho_M(X) + \rho_R(X) \\
&= \frac{3H_0^2\Omega_R}{\kappa c^2 C^4}\frac{X+1}{X^4} && (5.7) \\
p(X) &= \frac{1}{3}\rho_R(t) \\
&= \frac{H_0^2\Omega_R}{\kappa c^2 C^4}\frac{1}{X^4} && (5.8) \\
t(X) &= \frac{2C^2}{3H_0\sqrt{\Omega_R}}\left(\sqrt{X+1}(X-2)+2\right) && (5.9) \\
X &= \frac{R(t)}{R_{eq}} && (5.10)
\end{aligned}
$$

Where $t(X)$ is the time variable, R_{eq} and ρ_{EQ} are the scale factor and density at matter-radiation equality, that is $\rho_M(t_{eq}) = \rho_R(t_{eq})$, $C = \frac{\Omega_R}{\Omega_M}$, and Ω_R and Ω_M are the radiation and non relativistic matter density in the present respectively, with $\Omega_M = 1 - \Omega_R$. We know that $\Omega_R \ll 1$, so $\Omega_M \sim 1$ and $C \ll 1$. We can see that is convenient to use X like our independent variable. By definition $X \gg 1$ describes the non relativistic era and $X \ll 1$ describes the radiation era.

Now, we can solve (2.11) and (2.14) to find $\tilde{g}_{\mu\nu}$. Using (5.7)-(5.9), these equations are reduced to:

$$
\begin{aligned}
U_\mu^T &= 0 && (5.11) \\
\tilde{\rho}_M(X) &= \frac{9H_0^2\Omega_R}{2\kappa c^2 C^4}\frac{(C_1 - T_d(X))}{X^3} && (5.12) \\
\tilde{\rho}_R(X) &= \frac{6H_0^2\Omega_R}{\kappa c^2 C^4}\frac{(C_2 - T_d(X))}{X^4} && (5.13) \\
2X(X+1)T_d'(X) - (3X+2)T_d(X) &= 3C_1 X + 4C_2
\end{aligned}
$$

Now, if we solve (5.14), we obtain that:

$$
\begin{aligned}
T_d(X) &= \frac{3}{2}(2C_2 - C_1)X\left(\sqrt{X+1}\ln\left(\frac{\sqrt{X+1}+1}{\sqrt{X+1}-1}\right) - 2\right) \\
&\quad -2C_2 + C_3 X\sqrt{X+1} && (5.14)
\end{aligned}
$$

Where C_1, C_2 and C_3 are integrate constants.

5.2 Photon Trajectory and Luminosity Distance.

Since we have the cosmological solution of the $\tilde{\delta}$ gravity Action now, we can analyze the trajectory of a supernova photon when it is traveling to the Earth. For this, we use (3.12) in a radial trajectory from r_1 to $r = 0$. So, we have:

$$-(1 + 3\kappa_2 \tilde{T}_d(t))c^2 dt^2 + R^2(t)(1 + \kappa_2 \tilde{T}_d(t))dr^2 = 0$$

In the usual case, we have that $cdt = -R(t)dr$. In the $\tilde{\delta}$ gravity case, we define the modified scale factor:

$$\tilde{R}(t) = R(t)\sqrt{\frac{1 + \kappa_2 \tilde{T}_d(t)}{1 + 3\kappa_2 \tilde{T}_d(t)}} \tag{5.15}$$

such that $cdt = -\tilde{R}(t)dr$ now. With this definition, we obtain that:

$$r_1 = c\int_{t_1}^{t_0} \frac{dt}{\tilde{R}(t)} \tag{5.16}$$

If a second wave crest is emitted at $t = t_1 + \Delta t_1$ from $r = r_1$, it will reach $r = 0$ at $t = t_0 + \Delta t_0$, so:

$$r_1 = c\int_{t_1+\Delta t_1}^{t_0+\Delta t_0} \frac{dt}{\tilde{R}(t)} \tag{5.17}$$

Therefore, for Δt_1, Δt_0 small, which is appropriate for light waves, we get:

$$\frac{\Delta t_0}{\Delta t_1} = \frac{\tilde{R}(t_0)}{\tilde{R}(t_1)} \tag{5.18}$$

Since t is the proper time according to (5.1), we have that

$$\frac{\Delta \nu_1}{\Delta \nu_0} = \frac{\tilde{R}(t_0)}{\tilde{R}(t_1)} \tag{5.19}$$

where ν_0 is the light frequency detected at $r = 0$, corresponding to a source emission at frequency ν_1. So, the redshift is now:

$$1 + z(t_1) = \frac{\tilde{R}(t_0)}{\tilde{R}(t_1)} \tag{5.20}$$

We see that $\tilde{R}(t)$ replaces the usual scale factor $R(t)$ in the calculation of z. This means that we need to redefine the luminosity distance too. For this, let us consider a mirror of radius b that is receiving light from our distant source at r_1. The photons that reach the mirror are inside a cone of half-angle ϵ with origin at the source.

Let us compute ϵ. The path of the light rays is given by $\vec{r}(\rho) = \rho\hat{n} + \vec{r}_1$, where $\rho > 0$ is a parameter and $\hat{n}$ is the direction of the light ray. Since the mirror is in $\vec{r} = 0$, then $\rho = r_1$ and $\hat{n} = -\hat{r}_1 + \vec{\epsilon}$, where ϵ is the angle between $-\vec{r}_1$ and $\hat{n}$ at the source, forming a cone. The proper distance is determined by the 3-dimensional metric (5.2), so we get $b = \tilde{R}(t_0)r_1\epsilon$. Then, the solid angle of the cone is:

$$\begin{aligned}
\Delta\Omega &= \int_0^{2\pi} d\phi \int_0^{\epsilon} \sin(\theta)d\theta = 2\pi(1-\cos(\epsilon)) \\
&= \pi\epsilon^2 = \frac{A}{r_1^2\tilde{R}^2(t_0)}
\end{aligned}$$

where $A = \pi b^2$ is the proper area of the mirror. This means that $\epsilon = \frac{b}{r_1\tilde{R}(t_0)}$. So, the fraction of all isotropically emitted photons that reach the mirror is:

$$\begin{aligned}
f &= \frac{\Delta\Omega}{4\pi} \\
&= \frac{A}{4\pi r_1^2\tilde{R}^2(t_0)}
\end{aligned}$$

We know that the apparent luminosity, l, is the received power per unit mirror area. Power is energy per unit time, so the received power is $P = \frac{h\nu_0}{\Delta t_0}f$, where $h\nu_0$ is the energy corresponding to the received photon, and the total emitted power by the source is $L = \frac{h\nu_1}{\Delta t_1}$, where $h\nu_1$ is the energy corresponding to the emitted photon. Therefore, we have that:

$$\begin{aligned}
P &= \frac{\tilde{R}^2(t_1)}{\tilde{R}^2(t_0)}Lf \\
l &= \frac{P}{A} \\
&= \frac{\tilde{R}^2(t_1)}{\tilde{R}^2(t_0)}\frac{L}{4\pi r_1^2\tilde{R}^2(t_0)}
\end{aligned}$$

where we have used that $\frac{\Delta t_0}{\Delta t_1} = \frac{\nu_1}{\nu_0} = \frac{\tilde{R}(t_0)}{\tilde{R}(t_1)}$. On the other hand, we know that, in a Euclidean space, the luminosity decreases with distance d_L according to $l = \frac{L}{4\pi d_L^2}$. Therefore, using (5.16), the luminosity distance is:

$$
\begin{aligned}
d_L &= \frac{\tilde{R}^2(t_0)}{\tilde{R}(t_1)} r_1 \\
&= c\frac{\tilde{R}^2\,(t_0)}{\tilde{R}(t_1)} \int_{t_1}^{t_0} \frac{dt}{\tilde{R}(t)}
\end{aligned} \tag{5.21}
$$

Moreover, we can use (5.9) to change the t variable for $Y = CX(t) = \frac{R(t)}{R(t_0)}$ (the scale factor normalized to one in the present), where $C = \frac{R_{EQ}}{R(t_0)} = \frac{\Omega_R}{\Omega_M}$. Using (5.14) in (5.15) and define $\tilde{Y} = \frac{\tilde{R}(t)}{R(t_0)}$, we can see that, to $Y \ll 0$, $\tilde{Y} = \sqrt{\frac{1-2k_2C_2}{1-6k_2C_2}} Y + O(Y^2)$. We want $\tilde{Y} = Y + O(Y^2)$, because we expect that $\tilde{\delta}$ gravity explain dark energy and it is irrelevant in the early universe. For this, we use $C_2 = 0$. Therefore, the modified scale factor is:

$$
\tilde{Y}[Y, L_1, L_2, C] = Y\sqrt{\frac{1 - L_1\frac{Y}{3}\sqrt{Y+C} + L_2\frac{Y}{C}\left(\sqrt{\frac{Y}{C}+1}\ln\left(\frac{\sqrt{\frac{Y}{C}+1}+1}{\sqrt{\frac{Y}{C}+1}-1}\right) - 2\right)}{1 - L_1Y\sqrt{Y+C} + 3L_2\frac{Y}{C}\left(\sqrt{\frac{Y}{C}+1}\ln\left(\frac{\sqrt{\frac{Y}{C}+1}+1}{\sqrt{\frac{Y}{C}+1}-1}\right) - 2\right)}} \tag{5.22}
$$

where we used $C_1 = -\frac{2L_2}{3\kappa_2}$ and $C_3 = -\frac{C^{3/2}L_1}{3\kappa_2}$. We chose these constants such that a Big-Rip is produced. We want that because the accelerate expansion of the universe is produced by a polo in (5.22). To study the form of $\tilde{Y}$ in the Big-Rip era, we need use $Y \gg C$. In this case, we have that:

$$
\tilde{Y}(Y, L_1, L_2, C) \simeq Y\sqrt{\frac{3 + 2L_2 - Y^{\frac{3}{2}}L_1}{3(1 + 2L_2 - Y^{\frac{3}{2}}L_1)}} + O(C^{\frac{1}{2}}) \tag{5.23}
$$

It is clear that the Big-Rip is produced when:

$$
Y_{Rip} = \left(\frac{1 + 2L_2}{L_1}\right)^{\frac{2}{3}} \tag{5.24}
$$

To resume, we have that $\tilde{Y} \sim Y$ in the radiation era, that is $Y \ll C$, so the Universe evolves normally in the beginning of the Universe, without differences with the usual gravity. But, when $Y \gg C$, we will have a Big Rip, when the denominator is null. We will give more detail for this when we will study the supernova data.

Now, with all our definitions, the luminosity distance is reduced to:

$$
d_L = c\frac{\sqrt{C}}{H_0\sqrt{\Omega_R}} \frac{\tilde{Y}_0^2(L_1, L_2, C)}{\tilde{Y}(Y, L_1, L_2, C)} \int_Y^1 \frac{Y'dY'}{\tilde{Y}(Y', L_1, L_2, C)\sqrt{Y' + C}} \tag{5.25}
$$

with $\tilde{Y}_0(L_1, L_2, C) = \tilde{Y}(1, L_1, L_2, C)$. This means that the distances will be different now. In the usual case, we have:

$$d_L = \frac{c}{YH_0} \int_Y^1 \frac{dY'}{\sqrt{\Omega_\Lambda Y'^4 + \Omega_M Y' + \Omega_R}} \tag{5.26}$$

where $\Omega_\Lambda = 1 - \Omega_M - \Omega_R$ is the dark energy density in the present. We will use (5.26) to compare both, a Universe with dark energy and our modified gravitation model, with the supernova data.

Finally, we note that (5.9) gives us the time coordinate. In the new notation, it is:

$$t(Y) = \frac{2C^{\frac{1}{2}}}{3H_0\sqrt{\Omega_R}} \left(\sqrt{Y + C}\,(Y - 2C) + 2C^{\frac{3}{2}} \right) \tag{5.27}$$

Therefore, it is possible to obtain a different age of the Universe. A different perception of the distances implies a different perception of time.All these differences arise a consequence of the modified trajectory of photons.

5.3 Analysis and Results.

Before we analyze the data, we will define the parameters to be determined. In the usual gravity, d_L depends upon four parameters: Y, $H_0 = 100h$ km s^{-1} Mpc^{-1}, Ω_M and Ω_R according to (5.26). However, the CMB black body spectrum give us the photons density in the present, Ω_γ, and if we assume that $\Omega_R = \Omega_\gamma + \Omega_\nu = \left(1 + 3\left(\frac{7}{8}\right)\left(\frac{4}{11}\right)^{4/3}\right)\Omega_\gamma$, we obtain $h^2\Omega_R = 4.15 \times 10^{-5}$. Therefore, the parameters in d_L can be reduced to three: Y, h and $h^2\Omega_M$. For the same reasons, in our modified gravity, d_L depends on four parameters: Y, C, $L1$ and L_2, as shown in (5.25). We use $H_0\sqrt{\Omega_R} = 0.644$ km s^{-1} Mpc^{-1}.

The supernova data gives the apparent magnitude as a function of redshift. For this reason, it is useful to use z instead of Y. So, we have:

In the usual gravity:

$$
\begin{aligned}
m(z,h,h^2\Omega_M) &= M + 5\log_{10}\left(\frac{d_L(z,h,h^2\Omega_M)}{10\text{ pc}}\right) && (5.28)\\
d_L(z,h,h^2\Omega_M) &= \frac{c(1+z)\text{ Mpc s}}{100\text{ km}}\int_{\frac{1}{1+z}}^{1}\frac{dY'}{\sqrt{h^2\Omega_\Lambda Y'^4 + h^2\Omega_M Y' + h^2\Omega_R}} && (5.29)
\end{aligned}
$$

With $h^2\Omega_\Lambda = h^2 - h^2\Omega_M - h^2\Omega_R$. On the other side, in our modified gravity:

$$
\begin{aligned}
m(z,L_1,L_2,C) &= M + 5\log_{10}\left(\frac{d_L(z,L_1,L_2,C)}{10\text{ pc}}\right) && (5.30)\\
d_L(z,L_1,L_2,C) &= c(1+z)\frac{\sqrt{C}}{H_0\sqrt{\Omega_R}}\int_0^z\frac{(1+u)Y(u)Y'(u)}{\sqrt{Y(u)+C}}du && (5.31)
\end{aligned}
$$

where m is the apparent magnitude, M is the absolute magnitude, common to all supernova, so it is constant and $Y'(z) = \frac{dY}{dz}(z)$. To find $Y(z)$, we must solve (5.20). That is:

$$
\tilde{Y}(Y(z),L_1,L_2,C) = \frac{\tilde{Y}_0(L_1,L_2,C)}{1+z} \qquad (5.32)
$$

Where $\tilde{Y}(Y(z),L_1,L_2,C)$ is given by (5.22). Therefore, (5.32) is a numerical equation. Now we will introduce the statistical method to fit the data.

We interpret errors in data by the variance σ in a normally distributed random variable. If we are fitting a function $y(x)$ to a set of points (x_i, y_i) with errors $(\sigma_{xi}, \sigma_{yi})$, we must minimize [66]:

$$
\begin{aligned}
\chi^2(\text{per point}) &= \frac{1}{N}\sum_{i=1}^{N}\frac{(y_i - y(x_i))^2}{\sigma_{fi}^2}\\
\sigma_{fi}^2 &= \sigma_{yi}^2 + y'(x_i)^2\sigma_{xi}^2
\end{aligned}
$$

Where N is the number of data points. In our case, we want to fit the data (z_i, m_i) with errors $(\sigma_{zi}, \sigma_{mi})$ to the model:

$$
m(z) = M + 5\log_{10}\left(\frac{d_L(z)}{10pc}\right)
$$

Therefore, we must minimize:

$$\chi^2(\text{per point}) = \frac{1}{N}\sum_{i=1}^{N}\frac{(m_i - m(z_i))^2}{\sigma_{mi}^2 + \left(\frac{dm}{dz}(z_i)\right)^2\sigma_{zi}^2} \tag{5.33}$$

Now, we can proceed to analyze the supernova data given in [67] with $N = 162$ supernovas. In both cases, d_L is given by an exact expression, but we need to use a numeric method to solve the integral and fit the data to determinate the optimum values for the parameters that represent the m v/s z of the supernova data. For this, we used mathematica 7.0 [1]. When we minimized (5.33), we saw that the fit do not depend strongly of L_1 and L_2, but they are important to fix the Big-Rip point given by (5.24). Probably, exist a parameters space to L_1 and L_2, therefore we need fix one of them. Since we do not have a criterion for this, we will keep both parameters and we will fix them with another phenomenons in a future work. So, the parameters that minimize (5.33) are:

In the usual gravity $h = 0.6603$ and $h^2\Omega_M = 0.096$ with $\chi^2(\text{per point}) = 1.033$.

In our modified gravity $L_1 = 0.8095$, $L_2 = 0.2796$ and $C = 2.36 \times 10^{-4}$ with $\chi^2(\text{per point}) = 1.041$.

With these values, we can calculate the age of the Universe. We know that, in the usual case, it is 1.37×10^{10} years, but that in our model it is given by (5.27). Substituting the corresponding values for L_1, L_2, C and taking $Y = \frac{R(t)}{R(t_0)} = 1$, we obtain 1.56×10^{10} years. Finally, we can calculate when the Big-Rip will happen. For this, we use (5.24), giving $Y_{Rip} = 1.93$. Using this in (5.27), we obtain $t_{\text{Big Rip}} = 4.16 \times 10^{10}$ years. Therefore, the Universe has lived less than half of its life.

If we see (5.22), it is clear that in the limit where $C \to 0$, it is not possible obtain a Big-Rip. Then, it is necessary to have $1 \gg C \neq 0$ to obtain an accelerated expansion of the Universe. Therefore a minimal component of radiation explains the supernova data without dark energy. In this way, the accelerated expansion of the Universe, can be understood as geometric effect.

In [53], we did a similar calculus. The procedure is the same, but in this calculus we add $\tilde{\delta}$ matter plus fixing the gauge completely. For this, the result is different, however

[1] To minimize (5.33) we used **FindMinimum**. See the Mathematica 7.0 help for more details.

the explanation of the expansion of the universe is the same.

In the $\tilde{\delta}$ gravity model we can avoid a Big Rip at later time by a mechanism that give masses to all massless particles. Some options are quantum effects (which are finite in this model) or massive photons due to superconductivity [68] which could happen at very low temperatures, which are natural at a later stages of the expansion of the Universe.

Chapter 6

Non-Relativistic case.

Another important case is the Non-Relativistic. In this chapter, we will study the Newtonian and Post-Newtonian limit to verify that our theory do not have strong difference with the usual gravity. On the other side, if we will find a little difference, we could analyze it to find new physics. One possibility is explain the dark matter studying the galaxy's rotation. In this sense, we will expect that $\tilde{\delta}$ matter is dark matter.

6.1 Newtonian limit.

With the equations of motion of **section 2.1**, we can study the Newtonian approximation too. To express this approximation, we must use a metric $g_{\mu\nu}$ given by:

$$g_{\mu\nu} = \begin{pmatrix} -(1+2\phi(x,y,z)\epsilon^2)c^2 & 0 & 0 & 0 \\ 0 & 1-2\phi(x,y,z)\epsilon^2 & 0 & 0 \\ 0 & 0 & 1-2\phi(x,y,z)\epsilon^2 & 0 \\ 0 & 0 & 0 & 1-2\phi(x,y,z)\epsilon^2 \end{pmatrix} \tag{6.1}$$

where $\phi(x,y,z)$ is the gravitational potential and $\epsilon \sim \frac{v}{c}$ is the perturbative parameter. On the other side, to find an expression to $\tilde{g}_{\mu\nu}$, we use the same argument that in the **chapter 4** to Schwarzschild. If $g_{\mu\nu} \to \eta_{\mu\nu}$ to $r \to \infty$, then $\tilde{g}_{\mu\nu} \to 0$. With this in mind, $\tilde{g}_{\mu\nu}$ is given by:

$$\tilde{g}_{\mu\nu} = \begin{pmatrix} -2\tilde{\phi}(x,y,z)\epsilon^2 c^2 & 0 & 0 & 0 \\ 0 & -2\tilde{\phi}(x,y,z)\epsilon^2 & 0 & 0 \\ 0 & 0 & -2\tilde{\phi}(x,y,z)\epsilon^2 & 0 \\ 0 & 0 & 0 & -2\tilde{\phi}(x,y,z)\epsilon^2 \end{pmatrix} \tag{6.2}$$

To solve our equations, we use that $T_{\mu\nu}$ and $\tilde{T}_{\mu\nu}$ are given by (2.40) and (2.41) with $p(\rho) = 0$ and $U_\mu = (\epsilon^{-1}c, 0, 0, 0)$. So, the equations (2.10) and (2.11) are reduced to:

$$U_\mu^T = 0 \tag{6.3}$$

$$\partial^2\phi = \frac{\hat{\kappa}}{2}\rho \tag{6.4}$$

$$\partial^2\tilde{\phi} = \frac{\hat{\kappa}}{2}\tilde{\rho} \tag{6.5}$$

where $\partial^2 = \partial_i\partial_i$ with $\partial_i = \frac{\partial}{\partial x^i}$ and $\hat{\kappa} = \frac{\kappa}{\epsilon^2} \sim O(1)$ is the normalized Newton constant. Besides, (2.13) and (2.14) only say us that ρ and $\tilde{\rho}$ are t-independent. Therefore, we do not have a relation between ρ and $\tilde{\rho}$. We will see, in the next section, that this is due to that the gauge fixing is produced in a postnewtonian approximation.

6.2 Post-Newtonian limit.

If we introduce one order more to the Newtonian limit, the metric is given by:

$$\begin{aligned} g_{\mu\nu}dx^\mu dx^\nu &= -\left(1 + 2\phi\epsilon^2 + 2\left(\phi^2 + \psi\right)\epsilon^4\right)\left(\frac{cdt}{\epsilon}\right)^2 + \left(1 - 2\phi\epsilon^2 - 2\psi\epsilon^4\right)\left(dx^2 + dy^2 + dz^2\right) \\ &+2\epsilon^3\left(\chi_1 dx + \chi_2 dy + \chi_3 dz\right)\left(\frac{cdt}{\epsilon}\right) + \epsilon^4(\xi_{11}dx^2 + \xi_{22}dy^2 + \xi_{33}dz^2 \\ &+2\xi_{12}dxdy + 2\xi_{13}dxdz + 2\xi_{23}dydz) \end{aligned} \tag{6.6}$$

In the same way, $\tilde{g}_{\mu\nu}$ is:

$$\begin{aligned} \tilde{g}_{\mu\nu}dx^\mu dx^\nu &= -2\left(\tilde{\phi}\epsilon^2 + \left(2\phi\tilde{\phi} + \tilde{\psi}\right)\epsilon^4\right)\left(\frac{cdt}{\epsilon}\right)^2 - 2\left(\tilde{\phi}\epsilon^2 + \left(\tilde{\psi} + \tilde{\xi}\right)\epsilon^4\right)\left(dx^2 + dy^2 + dz^2\right) \\ &+2\epsilon^3\left(\tilde{\chi}_1 dx + \tilde{\chi}_2 dy + \tilde{\chi}_3 dz\right)\left(\frac{cdt}{\epsilon}\right) + \epsilon^4(\tilde{\xi}_{11}dx^2 + \tilde{\xi}_{22}dy^2 + \tilde{\xi}_{33}dz^2 \\ &+2\tilde{\xi}_{12}dxdy + 2\tilde{\xi}_{13}dxdz + 2\tilde{\xi}_{23}dydz) \end{aligned} \tag{6.7}$$

All function in (6.6) and (6.7) depend of (t, x, y, z), but $\frac{1}{c}\frac{\partial}{\partial t} \sim \epsilon$. To considerate this, we use that $ct \to \frac{ct}{\epsilon}$ to obtain the equations.

Besides, we have the equations:

$$
\begin{aligned}
4\dot{\phi} + \partial_i \chi_i &= 0 && (6.8)\\
2\phi\partial_i\phi - \dot{\chi}_i - \frac{1}{2}\partial_i \xi_{jj} + \partial_j \xi_{ij} &= 0 && (6.9)\\
4\dot{\tilde{\phi}} + \partial_i \tilde{\chi}_i &= 0 && (6.10)\\
2\phi\partial_i\tilde{\phi} + 2\tilde{\phi}\partial_i\phi - \dot{\tilde{\chi}}_i - \frac{1}{2}\partial_i \tilde{\xi}_{jj} + \partial_j \tilde{\xi}_{ij} &= 0 && (6.11)
\end{aligned}
$$

to fix the harmonic gauge (See **Appendix D**) and $\dot{f} = \frac{1}{c}\frac{\partial f}{\partial t}$. Additionally, we have a perfect fluid. This means that the energy-momentum tensors are given by (2.40) and (2.41) with:

$$
\begin{aligned}
\rho &= \rho^{(0)} + \epsilon^2 \rho^{(2)} && (6.12)\\
\tilde{\rho} &= \tilde{\rho}^{(0)} + \epsilon^2 \tilde{\rho}^{(2)} && (6.13)\\
p(\rho) &= \epsilon^2 p^{(2)}(\rho) && (6.14)\\
U_\mu &= \left(c \left(\frac{1 + \epsilon^2 \left(\phi + \frac{1}{2} U_k^{(1)} U_k^{(1)} \right)}{\epsilon} \right), \epsilon U_i^{(1)} \right) && (6.15)\\
U_\mu^T &= \left(c\epsilon U_k^{T(1)} U_k^{(1)}, \epsilon U_i^{T(1)} \right) && (6.16)
\end{aligned}
$$

With all these, the equations (2.10) and (2.11) are reduced to:

$$
\begin{aligned}
\partial^2\phi &= \frac{\hat\kappa}{2}\rho^{(0)} && (6.17)\\
\partial^2\chi_i &= -2\hat\kappa U_i^{(1)}\rho^{(0)} && (6.18)\\
\partial^2\psi &= \frac{\hat\kappa}{2}\left(2\left(U_k^{(1)}U_k^{(1)}-\phi\right)\rho^{(0)}+\rho^{(2)}+3p^{(2)}(\rho)\right)+\ddot\phi && (6.19)\\
\partial^2\xi_{ij} &= -2\hat\kappa U_i^{(1)}U_j^{(1)}\rho^{(0)}-4(\partial_i\phi)(\partial_j\phi)+2\hat\kappa\left(\left(U_k^{(1)}U_k^{(1)}+\phi\right)\rho^{(0)}+2p^{(2)}(\rho)\right)\delta_{ij}\\
&\quad +4(\partial_k\phi)(\partial_k\phi)\delta_{ij} && (6.20)\\
\partial^2\tilde\phi &= \frac{\hat\kappa}{2}\tilde\rho^{(0)} && (6.21)\\
\partial^2\tilde\chi_i &= -2\hat\kappa\left(U_i^{T(1)}\rho^{(0)}+U_i^{(1)}\tilde\rho^{(0)}\right) && (6.22)\\
\partial^2\tilde\psi &= \hat\kappa\left(\left(2U_k^{(1)}U_k^{T(1)}-\tilde\phi\right)\rho^{(0)}+\left(U_k^{(1)}U_k^{(1)}-\phi+\frac{3}{2}p'^{(2)}(\rho)\right)\tilde\rho^{(0)}+\frac{\tilde\rho^{(2)}}{2}\right)+\ddot{\tilde\phi} && (6.23)\\
\partial^2\xi_{ij} &= -2\hat\kappa\left(\left(U_i^{T(1)}U_j^{(1)}+U_i^{(1)}U_j^{T(1)}\right)\rho^{(0)}+U_i^{(1)}U_j^{(1)}\tilde\rho^{(0)}\right)-4(\partial_i\tilde\phi)(\partial_j\phi)-4(\partial_i\phi)(\partial_j\tilde\phi)\\
&\quad +2\hat\kappa\left(\left(2U_k^{(1)}U_k^{T(1)}+\tilde\phi\right)\rho^{(0)}+\left(U_k^{(1)}U_k^{(1)}+\phi+2p'^{(2)}(\rho)\right)\tilde\rho^{(0)}\right)\delta_{ij}\\
&\quad +8(\partial_k\phi)(\partial_k\tilde\phi)\delta_{ij} && (6.24)
\end{aligned}
$$

Where $p'^{(2)}(\rho)=\frac{\partial p^{(2)}}{\partial\rho}(\rho)$. We can see that the equations (6.17) and (6.21) correspond to (6.4) and (6.5) respectively.

Besides, we have the equations (2.13) and (2.14), but they are null with the gauge equations (6.8-6.11). However, it is useful write them in term of $\rho^{(0)}$, $\rho^{(2)}$, $\tilde\rho^{(0)}$, $\tilde\rho^{(2)}$ and $p^{(2)}$ in the case when $U_i^{(1)}=U_i^{T(1)}=0$. That is:

$$
\begin{aligned}
\dot\rho^{(0)} &= 0\\
\dot\rho^{(0)} &= 0\\
\dot{\tilde\rho}^{(2)} &= 0\\
\dot{\tilde\rho}^{(2)} &= 0\\
\partial_i p^{(2)}(\rho) &= -\rho^{(0)}\partial_i\phi\\
\partial_i\left(p'^{(2)}(\rho)\tilde\rho^{(0)}\right) &= -\rho^{(0)}\partial_i\tilde\phi-\tilde\rho^{(0)}\partial_i\phi && (6.25)
\end{aligned}
$$

These equations give us additional information about $\rho^{(0)}$ that we did not have in the Newton approximation. This information come from the gauge fixing. To see this explicitly, we will analyze the spherical symmetry case. So, the equations in (6.25) say us that all densities are t-independent, therefore they are only depend of r. Besides, we have that:

$$p'^{(2)}\left(\rho(r)\right)\left(\frac{\partial\rho^{(0)}}{\partial r}(r)\right) = -\rho^{(0)}(r)\left(\frac{\partial\phi}{\partial r}(r)\right) \tag{6.26}$$

$$\frac{\partial}{\partial r}\left(p'^{(2)}\left(\rho(r)\right)\tilde{\rho}^{(0)}(r)\right) = -\rho^{(0)}(r)\left(\frac{\partial\tilde{\phi}}{\partial r}(r)\right) - \tilde{\rho}^{(0)}(r)\left(\frac{\partial\phi}{\partial r}(r)\right) \tag{6.27}$$

Where we used that $\left(\frac{\partial p^{(2)}}{\partial r}(r)\right) = p'^{(2)}\left(\rho(r)\right)\left(\frac{\partial\rho^{(0)}}{\partial r}(r)\right)$. Now, if we mix (6.26) and (6.27) we obtain:

$$\tilde{\rho}^{(0)}(r) = \frac{\left(\frac{\partial\rho^{(0)}}{\partial r}(r)\right)}{\left(\frac{\partial\phi}{\partial r}(r)\right)}\left(\tilde{\phi}(r)+\tilde{\phi}_0\right) \tag{6.28}$$

Where $\tilde{\phi}_0$ is an integration constant. This means that we can obtain an expression to $\tilde{\rho}^{(0)}$ if we know $\rho^{(0)}$. So, the Newtonian limit equations to spherical symmetry are reduced to:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi(r)}{\partial r}\right) = \frac{\hat{\kappa}}{2}\rho^{(0)}(r) \tag{6.29}$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\tilde{\phi}(r)}{\partial r}\right) = \frac{\hat{\kappa}}{2}\frac{\left(\frac{\partial\rho^{(0)}}{\partial r}(r)\right)}{\left(\frac{\partial\phi}{\partial r}(r)\right)}\left(\tilde{\phi}(r)+\tilde{\phi}_0\right) \tag{6.30}$$

Therefore, we can obtain $\phi(r)$ and $\tilde{\phi}(r)$ if we know $\rho^{(0)}(r)$, that is the complete Newtonian limit. Now, we can ask us if it is possible explain dark matter with this result. For this, we will study the trajectory of a particle in the next section

6.3 Trajectory of a Particle.

If we have a massive particle, the acceleration is given by (3.3). In the Post-Newtonian limit, we obtain that:

$$\begin{aligned}\frac{1}{c^2}\frac{d^2\vec{x}}{dt^2} &= -\epsilon^2\nabla\left(\phi_N+\left(2\phi_N^2+\psi_N\right)\epsilon^2\right)\\ &\quad+\epsilon^4\left(3\vec{v}\dot{\phi}_N+4\vec{v}\left(\vec{v}\cdot\nabla\phi_N\right)-v^2\nabla\phi_N-\dot{\vec{\chi}}_N+\left(\vec{v}\times\nabla\times\vec{\chi}_N\right)\right)\\ &\quad+\frac{\epsilon^4 k_2^2}{2}\nabla\tilde{\phi}^2+O\left(\epsilon^6\right)\end{aligned} \tag{6.31}$$

Where $\vec{v} = \frac{d\vec{x}}{dt}$, $\phi_N = \phi_N + \kappa_2\tilde{\phi}$ and the same for the others fields. From (6.31), we can deduce a couple of thing. In first place, we can see that $\frac{1}{c^2}\frac{d^2\vec{x}}{dt^2} = -\epsilon^2\nabla\phi_N$ in the Newtonian limit, so ϕ_N is the effective potential. In second place, the acceleration is similar to the usual case if we replace $\phi \to \phi_N$ [63], with the exception of the last term in (6.31). If we analyze the case with spherical symmetry outside the matter, from (6.30) we can see that $\tilde{\phi}^2 \sim r^{-2}$. This means that this term is $\sim -r^{-3}$, therefore is an attractive contribution. A more detailed analysis is required for this term, but in this work we will only do a little analysis of the Newtonian case.

We said that ϕ_N is the effective potential in the Newtonian limit. This means that the effective density is $\rho_{eff} = \rho^{(0)} + \kappa_2\tilde{\rho}^{(0)}$. In spherical symmetry is:

$$\rho_{eff}(r) = \rho^{(0)}(r) + \kappa_2 \frac{\left(\frac{\partial\rho^{(0)}}{\partial r}(r)\right)}{\left(\frac{\partial\phi}{\partial r}(r)\right)}\left(\tilde{\phi}(r) + \tilde{\phi}_0\right) \tag{6.32}$$

Therefore, we have an additional mass given by the second term in (6.32), that could be identify with dark matter. If we compare this result with (4.22), we can see that the deflection of light allow an additional mass given by $M_{add} = \kappa_2 a_0 M$, where M is the mass of sun. If we accept that this mass is dark matter, (4.23) say us that we have $< 1\%$ of dark matter in the solar system scale. On the other side, in a galactic scale, this effect could be even bigger. To verify this, we need use some profile of luminosity for any galaxy, exponential for example, and obtain an expression to the density of luminosity matter, $\rho^{(0)}$, [69]. Finally, we can obtain the effective density using (6.32). Unfortunately, by time problems, we did can complete this calculus to this work.

Conclusions.

We have proposed a modified gravity model with good properties at the quantum level. It is finite on shell in the vacuum and only lives at one loop. It incorporates a new field $\tilde{g}_{\mu\nu}$ that transforms correctly under general coordinate transformation and exhibits a new symmetry: the $\tilde{\delta}$ symmetry. The new action is invariant under these transformations. We call this new gravity model $\tilde{\delta}$ gravity. A quantum field theory analysis of $\tilde{\delta}$ gravity has been developed [51].

In this work, we study the classical effects in a classical level. To this end, we require to set up the following two issues. First, we need to find the equations for $\tilde{\delta}$ gravity. One of them is Einstein's equation, which it gives us $g_{\mu\nu}$, and the other equation is (2.11) to solve for $\tilde{g}_{\mu\nu}$. Second, we need the modified test particle action. This action, (3.9), incorporates the new field $\tilde{g}_{\mu\nu}$. We obtain that a photon, or a massless particle, moves in a null geodesic of $\mathbf{g}_{\mu\nu} = g_{\mu\nu} + \kappa_2 \tilde{g}_{\mu\nu}$ and that a massive particle is governed by the equation of motion (3.3). With all this basic set up, we can study any phenomenon.

In first place, we analyze the Schwarzschild case outside the matter. We found a exact solution to the equations of motion to this case. This solution could be used to study the black hole. To the sun, we can use a Newtonian approximation and found the deflection of light. To explain the experimental data, the correction must be small. This means that the modification of $\tilde{\delta}$ gravity is not important to solar system scale.

In [52] it was shown that $\tilde{\delta}$ gravity predicts an accelerated expansion of the Universe without a cosmological constant or additional scalar fields by using an approximation corresponding for small redshift. In [53], it is developed an exact expression for the cosmological luminosity distance, but we assumed that we do not have $\tilde{\delta}$ matter. We find in the present work the exact solution with $\tilde{\delta}$ matter. For this, it was necessary to fix the gauge to $g_{\mu\nu}$ and $\tilde{g}_{\mu\nu}$. We used an extended harmonic gauge. We verify that $\tilde{\delta}$ gravity do not require dark energy to explain the accelerate expansion of the universe. With this exact expression, we could also study very early phenomenon in the Universe, for example inflation and the CMB power spectrum. This work is in progress.

On the other hand, photons move on a null geodesic of $\mathbf{g}_{\mu\nu} = g_{\mu\nu} + \kappa_2 \tilde{g}_{\mu\nu}$, so we can define a new scale factor $\tilde{R}(t)$. If we assume that the universe only has non relativistic matter and radiation, we can obtain an exact expression for $\tilde{R}(t)$. It is clear in (5.22) that $1 \gg C \neq 0$ is necessary to obtain an accelerated expansion of the Universe. Therefore a minimal component of radiation explains the supernova data without dark energy. In this way, in this model, the accelerated expansion of the Universe, can be understood as geometric effect.

Besides, we calculate the age of the Universe. We find that the Universe has lived a bit more as in GR. This is not a contradiction, but rather a reinterpretation of the observations. This result is a consequence of the new equation of motion for the photons. This model ends in a Big Rip and we calculate when it will happen. The universe almost has lived half of its life. Even though the Big Rip could be seen as a problem, we observe that other cosmological models share this property too [54, 55]. Nevertheless, in our case, we have some way outs from the Big Rip. For example, the appearance of quantum effects or massive photons at times close to the Big Rip, by effects similar to superconductivity [68]. These effects could occur at very low temperatures which are common at the later stages of the evolution of the Universe.

Finally, we studied the Non-Relativistic case. In the Newtonian limit, we obtain a similar expression to the usual case, where we have an effective potential. This potential depend of $\rho^{(0)}$ and $\tilde{\rho}^{(0)}$, where the last one correspond to $\tilde{\delta}$ matter. The Schwarzschild result say us that $\tilde{\delta}$ matter is $< 1\%$ to solar system scale. However, a different result could be find in another scale. For example, to galactic scale. We found a relation between $\rho^{(0)}$ and $\tilde{\rho}^{(0)}$. We can use this relation to study the velocity rotation in a galaxy. For this, a numeric calculus is necessary. This work is in progress.

Appendix A: Analysis of $\tilde{T}_{\mu\nu}$.

The equation (2.9) is:

$$
\begin{aligned}
\tilde{T}^{\mu\nu} &= \frac{1}{\sqrt{-g}}\frac{\delta}{\delta g_{\mu\nu}}\left[\sqrt{-g}\left(T^{\alpha\beta}\tilde{g}_{\alpha\beta}-2\tilde{L}_M\right)\right]+\tilde{g}^{\mu}_{\alpha}T^{\alpha\nu}+\tilde{g}^{\nu}_{\alpha}T^{\alpha\mu}-\frac{1}{2}\tilde{g}^{\alpha}_{\alpha}T^{\mu\nu} \\
&= \frac{1}{\sqrt{-g}}\frac{\delta}{\delta g_{\mu\nu}}\left[\sqrt{-g}T^{\alpha\beta}\tilde{g}_{\alpha\beta}\right]+\tilde{g}^{\mu}_{\alpha}T^{\alpha\nu}+\tilde{g}^{\nu}_{\alpha}T^{\alpha\mu}-\frac{1}{2}\tilde{g}^{\alpha}_{\alpha}T^{\mu\nu}+\mathcal{T}^{\mu\nu}_{(\delta M)} \\
&= \tilde{g}_{\alpha\beta}\frac{\delta T^{\alpha\beta}}{\delta g_{\mu\nu}}+\frac{1}{2}g^{\mu\nu}T^{\alpha\beta}\tilde{g}_{\alpha\beta}+\tilde{g}^{\mu}_{\alpha}T^{\alpha\nu}+\tilde{g}^{\nu}_{\alpha}T^{\alpha\mu}-\frac{1}{2}\tilde{g}^{\alpha}_{\alpha}T^{\mu\nu}+\mathcal{T}^{\mu\nu}_{(\delta M)}
\end{aligned}
$$

Where the equation (2.8) say us that:

$$
\begin{aligned}
\mathcal{T}^{\mu\nu}_{(\delta M)} &= -\frac{2}{\sqrt{-g}}\frac{\delta}{\delta g_{\mu\nu}}\left[\sqrt{-g}\tilde{L}_M\right] \\
&= \tilde{\phi}_I\frac{\partial T^{\mu\nu}}{\partial\phi_I}+(\partial_\alpha\tilde{\phi}_I)\frac{\partial T^{\mu\nu}}{\partial(\partial_\alpha\phi_I)}
\end{aligned}
\tag{6.33}
$$

Now, we use:

$$
\begin{aligned}
\tilde{g}_{\alpha\beta}\frac{\delta T^{\alpha\beta}}{\delta g_{\mu\nu}} &= \tilde{g}_{\alpha\beta}\frac{\delta}{\delta g_{\mu\nu}}\left[g^{\alpha\rho}g^{\beta\lambda}T_{\rho\lambda}\right] \\
&= \tilde{g}^{\alpha\beta}\frac{\delta T_{\alpha\beta}}{\delta g_{\mu\nu}}-\tilde{g}^{\mu}_{\alpha}T^{\nu\alpha}-\tilde{g}^{\nu}_{\alpha}T^{\mu\alpha}
\end{aligned}
$$

So:

$$
\tilde{T}^{\mu\nu}=\tilde{g}^{\alpha\beta}\frac{\delta T_{\alpha\beta}}{\delta g_{\mu\nu}}+\frac{1}{2}g^{\mu\nu}T^{\alpha\beta}\tilde{g}_{\alpha\beta}-\frac{1}{2}\tilde{g}^{\alpha}_{\alpha}T^{\mu\nu}+\mathcal{T}^{\mu\nu}_{(\delta M)}
\tag{6.34}
$$

If we evaluate this identity in the equation of motion of $\tilde{g}_{\mu\nu}$, (2.11), we obtain:

$$
F^{(\mu\nu)(\alpha\beta)\rho\lambda}D_\rho D_\lambda\tilde{g}_{\alpha\beta}+\frac{1}{2}\left(\tilde{g}^{\alpha}_{\alpha}R^{\mu\nu}-\tilde{g}^{\mu\nu}R\right)=\kappa\left(\tilde{g}^{\alpha\beta}\frac{\delta T_{\alpha\beta}}{\delta g_{\mu\nu}}+\mathcal{T}^{\mu\nu}_{(\delta M)}\right)
\tag{6.35}
$$

This equation is obtained in [53, 51] with $\mathcal{T}^{\mu\nu}_{(\delta M)} = 0$. So, (6.34) say us that (2.11) and (6.35) are the same.

On the other side, we know that $\tilde{T}_{\mu\nu} = \tilde{g}_{\alpha\beta}\frac{\delta T_{\mu\nu}}{\delta g_{\alpha\beta}} + \mathcal{T}^{(\delta M)}_{\mu\nu}$. Therefore (6.34) say us:

$$\tilde{g}^{\alpha\beta}\left(\frac{\delta T_{\alpha\beta}}{\delta g_{\mu\nu}}\right) = g^{\mu\rho}g^{\nu\lambda}\tilde{g}_{\alpha\beta}\left(\frac{\delta T_{\rho\lambda}}{\delta g_{\alpha\beta}}\right) - \frac{1}{2}g^{\mu\nu}T^{\alpha\beta}\tilde{g}_{\alpha\beta} + \frac{1}{2}\tilde{g}^{\alpha}_{\alpha}T^{\mu\nu} \tag{6.36}$$

Now, we will verify the identity (6.36) to a perfect fluid. Using (2.18), we obtain that:

$$\begin{aligned}
\frac{\delta T_{\mu\nu}}{\delta g_{\alpha\beta}} &= -\frac{1}{2}\lambda_{2,\rho}\mathbf{r}\left(\delta^{\rho}_{\nu}\left(\frac{\delta U_{\mu}}{\delta g_{\alpha\beta}}\right) + \delta^{\rho}_{\mu}\left(\frac{\delta U_{\nu}}{\delta g_{\alpha\beta}}\right)\right) + g_{\mu\nu}\lambda_{2,\rho}\mathbf{r}\left(\frac{\delta U^{\rho}}{\delta g_{\alpha\beta}}\right) \\
&\quad - (\mathbf{r}(1+\varepsilon(\mathbf{r})) + \lambda_1(u^a u_a + 1) - \lambda_{2,\rho}\mathbf{r}U^{\rho})\left(\frac{\delta g_{\mu\nu}}{\delta g_{\alpha\beta}}\right) \\
&= -\frac{1}{2}\lambda_{2,\rho}\mathbf{r}u_a\left(\delta^{\rho}_{\nu}\left(\frac{\delta e^a_{\mu}}{\delta g_{\alpha\beta}}\right) + \delta^{\rho}_{\mu}\left(\frac{\delta e^a_{\nu}}{\delta g_{\alpha\beta}}\right)\right) + g_{\mu\nu}\lambda_{2,\rho}u_a\mathbf{r}\left(\frac{\delta e^{a\rho}}{\delta g_{\alpha\beta}}\right) \\
&\quad -\frac{1}{2}(\mathbf{r}(1+\varepsilon(\mathbf{r})) + \lambda_1(u^a u_a + 1) - \lambda_{2,\rho}\mathbf{r}U^{\rho})\left(\delta^{\alpha}_{\mu}\delta^{\beta}_{\nu} + \delta^{\beta}_{\mu}\delta^{\alpha}_{\nu}\right) \\
&= -\frac{1}{8}\lambda_{2,\rho}\mathbf{r}\left(\delta^{\rho}_{\nu}\left(\delta^{\alpha}_{\mu}U^{\beta} + \delta^{\beta}_{\mu}U^{\alpha}\right) + \delta^{\rho}_{\mu}\left(\delta^{\alpha}_{\nu}U^{\beta} + \delta^{\beta}_{\nu}U^{\alpha}\right)\right) - \frac{1}{4}g_{\mu\nu}\lambda_{2,\rho}\mathbf{r}\left(U^{\alpha}g^{\rho\beta} + U^{\beta}g^{\rho\alpha}\right) \\
&\quad -\frac{1}{2}(\mathbf{r}(1+\varepsilon(\mathbf{r})) + \lambda_1(u^a u_a + 1) - \lambda_{2,\rho}\mathbf{r}U^{\rho})\left(\delta^{\alpha}_{\mu}\delta^{\beta}_{\nu} + \delta^{\beta}_{\mu}\delta^{\alpha}_{\nu}\right)
\end{aligned} \tag{6.37}$$

So:

$$\begin{aligned}
\left(\frac{\delta T_{\mu\nu}}{\delta g_{\alpha\beta}}\right)\tilde{g}_{\alpha\beta} &= -\frac{1}{4}\lambda_{2,\beta}\mathbf{r}\left(\delta^{\beta}_{\nu}U^{\alpha}\tilde{g}_{\mu\alpha} + \delta^{\beta}_{\mu}U^{\alpha}\tilde{g}_{\nu\alpha} + 2g_{\mu\nu}U^{\alpha}\tilde{g}^{\beta}_{\alpha}\right) \\
&\quad - (\mathbf{r}(1+\varepsilon(\mathbf{r})) + \lambda_1(u^a u_a + 1) - \lambda_{2,\rho}\mathbf{r}U^{\rho})\,\tilde{g}_{\mu\nu}
\end{aligned} \tag{6.38}$$

$$\begin{aligned}
\left(\frac{\delta T_{\alpha\beta}}{\delta g_{\mu\nu}}\right)\tilde{g}^{\alpha\beta} &= -\frac{1}{4}\lambda_{2,\beta}\mathbf{r}\left(U^{\nu}\tilde{g}^{\mu\beta} + U^{\mu}\tilde{g}^{\nu\beta} + \tilde{g}^{\alpha}_{\alpha}\left(U^{\mu}g^{\nu\beta} + U^{\nu}g^{\mu\beta}\right)\right) \\
&\quad - (\mathbf{r}(1+\varepsilon(\mathbf{r})) + \lambda_1(u^a u_a + 1) - \lambda_{2,\rho}\mathbf{r}U^{\rho})\,\tilde{g}^{\mu\nu}
\end{aligned} \tag{6.39}$$

If we replace these expressions in (6.36), we obtain:

$$\tilde{g}^{\alpha\beta}\left(\frac{\delta T_{\alpha\beta}}{\delta g_{\mu\nu}}\right)-g^{\mu\rho}g^{\nu\lambda}\tilde{g}_{\alpha\beta}\left(\frac{\delta T_{\rho\lambda}}{\delta g_{\alpha\beta}}\right)+\frac{1}{2}g^{\mu\nu}T^{\alpha\beta}\tilde{g}_{\alpha\beta}-\frac{1}{2}\tilde{g}^{\alpha}_{\alpha}T^{\mu\nu}=0$$

$$\begin{aligned}
&\rightarrow -\frac{1}{4}\lambda_{2,\beta}\mathbf{r}\left(U^{\nu}\tilde{g}^{\mu\beta}+U^{\mu}\tilde{g}^{\nu\beta}+\tilde{g}^{\alpha}_{\alpha}\left(U^{\mu}g^{\nu\beta}+U^{\nu}g^{\mu\beta}\right)-g^{\nu\beta}U_{\alpha}\tilde{g}^{\mu\alpha}-g^{\mu\beta}U_{\alpha}\tilde{g}^{\nu\alpha}-2g^{\mu\nu}U_{\alpha}\tilde{g}^{\alpha\beta}\right)\\
&-\frac{1}{2}g^{\mu\nu}\left(\lambda_{2,\beta}\mathbf{r}\tilde{g}^{\beta}_{\alpha}U^{\alpha}+\tilde{g}^{\alpha}_{\alpha}(\mathbf{r}(1+\varepsilon(\mathbf{r}))+\lambda_1(u^a u_a+1)-\lambda_{2,\beta}\mathbf{r}U^{\beta})\right)\\
&+\frac{1}{2}\tilde{g}^{\alpha}_{\alpha}\left(\frac{1}{2}\lambda_{2,\beta}\mathbf{r}\left(g^{\nu\beta}U^{\mu}+g^{\mu\beta}U^{\nu}\right)+g^{\mu\nu}(\mathbf{r}(1+\varepsilon(\mathbf{r}))+\lambda_1(u^a u_a+1)-\lambda_{2,\beta}\mathbf{r}U^{\beta})\right)=0
\end{aligned}$$

$$\rightarrow -\frac{1}{4}\lambda_{2,\beta}\mathbf{r}\left(U^{\nu}\tilde{g}^{\mu\beta}+U^{\mu}\tilde{g}^{\nu\beta}-g^{\nu\beta}U_{\alpha}\tilde{g}^{\mu\alpha}-g^{\mu\beta}U_{\alpha}\tilde{g}^{\nu\alpha}\right)=0$$

$$\rightarrow 0=0$$

Where we have used (2.29) in the last line. Now, if we use the equations of motion (2.20), (2.22) and (2.23), (6.38) and (6.39) are reduced to:

$$\begin{aligned}
\left(\frac{\delta T_{\mu\nu}}{\delta g_{\alpha\beta}}\right)\tilde{g}_{\alpha\beta} &= \frac{1}{4}\mathbf{r}\left(1+\varepsilon(\mathbf{r})+\mathbf{r}\varepsilon'(\mathbf{r})\right)\left(U_{\nu}U^{\alpha}\tilde{g}_{\mu\alpha}+U_{\mu}U^{\alpha}\tilde{g}_{\nu\alpha}+2g_{\mu\nu}U^{\alpha}U^{\beta}\tilde{g}_{\alpha\beta}\right)\\
&\quad+\mathbf{r}^2\varepsilon'(\mathbf{r})\tilde{g}_{\mu\nu}\\
&= p(\rho)\tilde{g}_{\mu\nu}+\frac{1}{4}\left(p(\rho)+\rho\right)\left(U_{\nu}U^{\alpha}\tilde{g}_{\mu\alpha}+U_{\mu}U^{\alpha}\tilde{g}_{\nu\alpha}+2g_{\mu\nu}U^{\alpha}U^{\beta}\tilde{g}_{\alpha\beta}\right) \quad (6.40)\\
\left(\frac{\delta T_{\alpha\beta}}{\delta g_{\mu\nu}}\right)\tilde{g}^{\alpha\beta} &= \frac{1}{4}\mathbf{r}\left(1+\varepsilon(\mathbf{r})+\mathbf{r}\varepsilon'(\mathbf{r})\right)\left(U^{\nu}U^{\alpha}\tilde{g}^{\mu}_{\alpha}+U^{\mu}U^{\alpha}\tilde{g}^{\nu}_{\alpha}+2\tilde{g}^{\alpha}_{\alpha}U^{\mu}U^{\nu}\right)\\
&\quad+\mathbf{r}^2\varepsilon'(\mathbf{r})\tilde{g}^{\mu\nu}\\
&= p(\rho)\tilde{g}_{\mu\nu}+\frac{1}{4}\left(p(\rho)+\rho\right)\left(U^{\nu}U^{\alpha}\tilde{g}^{\mu}_{\alpha}+U^{\mu}U^{\alpha}\tilde{g}^{\nu}_{\alpha}+2\tilde{g}^{\alpha}_{\alpha}U^{\mu}U^{\nu}\right) \quad (6.41)
\end{aligned}$$

Where we used that $\rho=\mathbf{r}(1+\varepsilon(\mathbf{r}))$ and $p(\mathbf{r})=\mathbf{r}^2\varepsilon'(\mathbf{r})$. On the other hand, we have that:

$$
\begin{aligned}
\left(\frac{\delta T_{\mu\nu}}{\delta\lambda_1}\right)\tilde{\lambda}_1 &= -\tilde{\lambda}_1(u^a u_a+1) \\
&= 0 && (6.42) \\
\left(\frac{\delta T_{\mu\nu}}{\delta\lambda_2}\right)\tilde{\lambda}_2 &= -\frac{1}{2}\tilde{\lambda}_{2,\rho}\mathbf{r}\left(\delta^\rho_\nu U_\mu+\delta^\rho_\mu U_\nu\right)+\tilde{\lambda}_{2,\rho}\mathbf{r}U^\rho g_{\mu\nu} \\
&= \mathbf{r}\tilde{\mathbf{r}}\left(2\varepsilon'(\mathbf{r})+\mathbf{r}\varepsilon''(\mathbf{r})\right)U_\mu U_\nu \\
&\quad +\frac{1}{2}\mathbf{r}(1+\varepsilon(\mathbf{r})+\mathbf{r}\varepsilon'(\mathbf{r}))\left(U^T_\mu U_\nu+U^T_\nu U_\mu+\frac{1}{2}\tilde{g}^\beta_\nu U_\beta U_\mu+\frac{1}{2}\tilde{g}^\beta_\mu U_\beta U_\nu\right) \\
&\quad +\mathbf{r}\left(\tilde{\mathbf{r}}\left(2\varepsilon'(\mathbf{r})+\mathbf{r}\varepsilon''(\mathbf{r})\right)-\frac{1}{2}\tilde{g}^{\alpha\beta}U_\alpha U_\beta(1+\varepsilon(\mathbf{r})+\mathbf{r}\varepsilon'(\mathbf{r}))\right)g_{\mu\nu} \\
&= \tilde{\mathbf{r}}\frac{\partial p}{\partial\rho}(\rho)U_\mu U_\nu+\frac{1}{2}\left(p(\rho)+\rho\right)\left(U^T_\mu U_\nu+U^T_\nu U_\mu+\frac{1}{2}\tilde{g}^\beta_\nu U_\beta U_\mu+\frac{1}{2}\tilde{g}^\beta_\mu U_\beta U_\nu\right) \\
&\quad +\left(\tilde{\mathbf{r}}\frac{\partial p}{\partial\rho}(\rho)-\frac{1}{2}\left(p(\rho)+\rho\right)\tilde{g}^{\alpha\beta}U_\alpha U_\beta\right)g_{\mu\nu} && (6.43) \\
\left(\frac{\delta T_{\mu\nu}}{\delta u_a}\right)\tilde{u}_a &= -\frac{1}{2}\lambda_{2,\rho}\mathbf{r}\left(\delta^\rho_\nu U^T_\mu+\delta^\rho_\mu U^T_\nu\right)-\left(2\lambda_1 u^a\tilde{u}_a-\lambda_{2,\rho}\mathbf{r}U^\rho_T\right)g_{\mu\nu} \\
&= \frac{1}{2}\mathbf{r}(1+\varepsilon(\mathbf{r})+\mathbf{r}\varepsilon'(\mathbf{r}))\left(U_\nu U^T_\mu+U_\mu U^T_\nu\right) \\
&= \frac{1}{2}\left(p(\rho)+\rho\right)\left(U_\nu U^T_\mu+U_\mu U^T_\nu\right) && (6.44) \\
\left(\frac{\delta T_{\mu\nu}}{\delta\mathbf{r}}\right)\tilde{\mathbf{r}} &= -\frac{1}{2}\lambda_{2,\rho}\tilde{\mathbf{r}}\left(\delta^\rho_\nu U_\mu+\delta^\rho_\mu U_\nu\right)-\tilde{\mathbf{r}}\left(1+\varepsilon(\mathbf{r})+\mathbf{r}\varepsilon'(\mathbf{r})-\lambda_{2,\rho}U^\rho\right)g_{\mu\nu} \\
&= \tilde{\mathbf{r}}(1+\varepsilon(\mathbf{r})+\mathbf{r}\varepsilon'(\mathbf{r}))U_\mu U_\nu \\
&= \tilde{\rho}U_\mu U_\nu && (6.45)
\end{aligned}
$$

Appendix B: Variation of $G_{\mu\nu}$.

In this appendix, we will develop the variation of $G_{\mu\nu}$ with respect $g_{\mu\nu}$. We know that $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$, so:

$$\begin{aligned}
\tilde{\delta}\left[G_{\mu\nu}\right] &= \tilde{\delta}\left[R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right] \\
&= \tilde{\delta}\left[R_{\mu\nu}\right] - \frac{1}{2}\tilde{g}_{\mu\nu}R + \frac{1}{2}g_{\mu\nu}R_{\alpha\beta}\tilde{g}^{\alpha\beta} - \frac{1}{2}g_{\mu\nu}g^{\alpha\beta}\tilde{\delta}\left[R_{\alpha\beta}\right] \\
&= \left(\delta^{\alpha}_{\mu}\delta^{\beta}_{\nu} - \frac{1}{2}g_{\mu\nu}g^{\alpha\beta}\right)\tilde{\delta}\left[R_{\alpha\beta}\right] - \frac{1}{2}\tilde{g}_{\mu\nu}R + \frac{1}{2}g_{\mu\nu}R_{\alpha\beta}\tilde{g}^{\alpha\beta}
\end{aligned}$$

Where we used that $R = R_{\alpha\beta}g^{\alpha\beta}$ and $\tilde{\delta}g^{\mu\nu} = -\tilde{g}^{\mu\nu}$. Now, we use:

$$\begin{aligned}
\tilde{\delta}\left[R_{\alpha\beta}\right] &= D_{\rho}\left(\tilde{\delta}\left[\Gamma^{\rho}_{\alpha\beta}\right]\right) - D_{\alpha}\left(\tilde{\delta}\left[\Gamma^{\rho}_{\rho\beta}\right]\right) \\
\tilde{\delta}\left[\Gamma^{\rho}_{\alpha\beta}\right] &= \frac{1}{2}g^{\alpha\sigma}\left(D_{\beta}\tilde{g}_{\sigma\alpha} + D_{\alpha}\tilde{g}_{\beta\sigma} - D_{\sigma}\tilde{g}_{\alpha\beta}\right)
\end{aligned}$$

to demonstrate that:

$$\begin{aligned}
\left(\delta^{\alpha}_{\mu}\delta^{\beta}_{\nu} - \frac{1}{2}g_{\mu\nu}g^{\alpha\beta}\right)\tilde{\delta}\left[R_{\alpha\beta}\right] &= \frac{1}{4}(\delta^{\alpha}_{\mu}\delta^{\lambda}_{\nu}g^{\rho\beta} - \delta^{\alpha}_{\mu}\delta^{\beta}_{\nu}g^{\rho\lambda} + \delta^{\beta}_{\mu}\delta^{\lambda}_{\nu}g^{\rho\alpha} - \delta^{\beta}_{\mu}\delta^{\alpha}_{\nu}g^{\rho\lambda} \\
&\quad +\delta^{\lambda}_{\mu}\delta^{\alpha}_{\nu}g^{\rho\beta} + \delta^{\lambda}_{\mu}\delta^{\beta}_{\nu}g^{\rho\alpha} - \delta^{\lambda}_{\mu}\delta^{\rho}_{\nu}g^{\alpha\beta} - \delta^{\rho}_{\mu}\delta^{\lambda}_{\nu}g^{\alpha\beta} \\
&\quad +2g_{\mu\nu}g^{\alpha\beta}g^{\rho\lambda} - g_{\mu\nu}g^{\alpha\lambda}g^{\rho\beta} - g_{\mu\nu}g^{\lambda\beta}g^{\rho\alpha})D_{\rho}D_{\lambda}\tilde{g}_{\alpha\beta} \\
&= F^{(\alpha\beta)\rho\lambda}_{(\mu\nu)}D_{\rho}D_{\lambda}\tilde{g}_{\alpha\beta}
\end{aligned}$$

Where $F^{(\mu\nu)(\alpha\beta)\rho\lambda}$ is given by (2.6). Therefore:

$$\tilde{\delta}\left[G_{\mu\nu}\right] = F^{(\alpha\beta)\rho\lambda}_{(\mu\nu)}D_{\rho}D_{\lambda}\tilde{g}_{\alpha\beta} + \frac{1}{2}g_{\mu\nu}R^{\alpha\beta}\tilde{g}_{\alpha\beta} - \frac{1}{2}\tilde{g}_{\mu\nu}R$$

We use this to demonstrate that $(2.11)_{\mu\nu} = \tilde{\delta}\left[(2.10)_{\mu\nu}\right]$ in the **section 2.1**.

Appendix C: Vierbein analysis.

One of our basic identities is:

$$\frac{\delta g_{\alpha\beta}}{\delta g_{\mu\nu}} = \frac{1}{2}\left(\delta^{\mu}_{\alpha}\delta^{\nu}_{\beta} + \delta^{\mu}_{\beta}\delta^{\nu}_{\alpha}\right) \tag{6.46}$$

On the other side, we know that $g_{\mu\nu} = \eta_{ab}e^a_\mu e^b_\nu = e^a_\mu e_{a\nu}$ and $\eta^{ab} = g^{\mu\nu}e^a_\mu e^b_\nu = e^a_\mu e^{b\mu}$, where e^a_μ is the Vierbein. So, (6.46) can be reduced to:

$$\begin{aligned}\eta_{ab}\left(\frac{\delta e^a_\alpha e^b_\beta}{\delta g_{\mu\nu}}\right) &= \frac{1}{2}\left(\delta^{\mu}_{\alpha}\delta^{\nu}_{\beta} + \delta^{\mu}_{\beta}\delta^{\nu}_{\alpha}\right) \\ e_{a\beta}\left(\frac{\delta e^a_\alpha}{\delta g_{\mu\nu}}\right) + e_{a\alpha}\left(\frac{\delta e^a_\beta}{\delta g_{\mu\nu}}\right) &= \frac{1}{2}\left(\delta^{\mu}_{\alpha}\delta^{\nu}_{\beta} + \delta^{\mu}_{\beta}\delta^{\nu}_{\alpha}\right)\end{aligned}$$

$\left(\frac{\delta e^a_\alpha}{\delta g_{\mu\nu}}\right)$ is a three tensor symmetric in $(\mu\nu)$ and a vector in a. For this, we define $\left(\frac{\delta e^a_\alpha}{\delta g_{\mu\nu}}\right) = e^a_\gamma f^{\gamma(\mu\nu)}_\alpha$, where $f^{\beta(\mu\nu)}_\alpha$ is a four tensor symmetric in $(\mu\nu)$. So:

$$f^{(\mu\nu)}_{\beta\alpha} + f^{(\mu\nu)}_{\alpha\beta} = \frac{1}{2}\left(\delta^{\mu}_{\alpha}\delta^{\nu}_{\beta} + \delta^{\mu}_{\beta}\delta^{\nu}_{\alpha}\right) \tag{6.47}$$

Besides, $f^{\alpha\beta(\mu\nu)}$ only must depend of g. Therefore, the most general expression is $f^{(\mu\nu)}_{\alpha\beta} = a\left(\delta^{\mu}_{\alpha}\delta^{\nu}_{\beta} + \delta^{\nu}_{\alpha}\delta^{\mu}_{\beta}\right) + bg^{\mu\nu}g_{\alpha\beta}$. If we evaluate this expression in (6.47), we obtain $a = \frac{1}{4}$ and $b = 0$. This means that:

$$\frac{\delta e^a_\alpha}{\delta g_{\mu\nu}} = \frac{1}{4}\left(\delta^{\mu}_{\alpha}e^{a\nu} + \delta^{\nu}_{\alpha}e^{a\mu}\right) \tag{6.48}$$

From this result, we can conclude that:

$$
\begin{aligned}
\tilde{e}^a_\mu &= \tilde{\delta} e^a_\mu \\
&= \left(\frac{\delta e^a_\mu}{\delta g_{\alpha\beta}}\right)\tilde{g}_{\alpha\beta} \\
&= \frac{1}{4}\left(\delta^\alpha_\mu e^{a\beta} + \delta^\beta_\mu e^{a\alpha}\right)\tilde{g}_{\alpha\beta} \\
&= \frac{1}{2} e^a_\alpha \tilde{g}^\alpha_\mu
\end{aligned}
\tag{6.49}
$$

Now, another identity that we have is:

$$
\frac{\delta \tilde{g}_{\mu\nu}}{\delta g_{\alpha\beta}} = 0 \tag{6.50}
$$

because $g_{\mu\nu}$ and $\tilde{g}_{\mu\nu}$ are the independent variables, therefore e^a_μ and $\tilde{e}^a_\mu$ depend of these variables. Since $\tilde{g}_{\mu\nu} = \tilde{\delta} g_{\mu\nu}$, we have that $\tilde{g}_{\mu\nu} = \eta_{ab}\left(\tilde{e}^a_\mu e^b_\nu + e^a_\mu \tilde{e}^b_\nu\right)$. So, using (6.48) and (6.49), we obtain:

$$
\begin{aligned}
\frac{\delta \tilde{g}_{\mu\nu}}{\delta g_{\alpha\beta}} &= \left(\frac{\delta \tilde{g}_{\mu\nu}}{\delta e^a_\rho}\right)\left(\frac{\delta e^a_\rho}{\delta g_{\alpha\beta}}\right) + \left(\frac{\delta \tilde{g}_{\mu\nu}}{\delta \tilde{e}^a_\rho}\right)\left(\frac{\delta \tilde{e}^a_\rho}{\delta g_{\alpha\beta}}\right) \\
&= \left(\frac{\delta \tilde{g}_{\mu\nu}}{\delta e^a_\rho}\right)\left(\frac{\delta e^a_\rho}{\delta g_{\alpha\beta}}\right) + \frac{1}{2}\left(\frac{\delta \tilde{g}_{\mu\nu}}{\delta \tilde{e}^a_\rho}\right)\left(\frac{\delta e^a_\lambda \tilde{g}^\lambda_\rho}{\delta g_{\alpha\beta}}\right) \\
&= \left(\frac{\delta \tilde{g}_{\mu\nu}}{\delta e^a_\rho}\right)\left(\frac{\delta e^a_\rho}{\delta g_{\alpha\beta}}\right) + \frac{1}{2}\left(\frac{\delta \tilde{g}_{\mu\nu}}{\delta \tilde{e}^a_\rho}\right)\left(\tilde{g}^\lambda_\rho\left(\frac{\delta e^a_\lambda}{\delta g_{\alpha\beta}}\right) + e^a_\lambda \tilde{g}_{\rho\epsilon}\left(\frac{\delta g^{\epsilon\lambda}}{\delta g_{\alpha\beta}}\right)\right) \\
&= \frac{1}{4}\eta_{bc}\left(\tilde{e}^c_\mu \delta^\rho_\nu \delta^b_a + \tilde{e}^b_\nu \delta^\rho_\mu \delta^c_a\right)\left(\delta^\beta_\rho e^{a\alpha} + \delta^\alpha_\rho e^{a\beta}\right) \\
&\quad + \frac{1}{2}\eta_{bc}\left(e^c_\mu \delta^\rho_\nu \delta^b_a + e^b_\nu \delta^\rho_\mu \delta^c_a\right)\left(\frac{1}{4}\tilde{g}^\lambda_\rho\left(\delta^\beta_\lambda e^{a\alpha} + \delta^\alpha_\lambda e^{a\beta}\right) - \frac{1}{2} e^{a\lambda}\tilde{g}^\epsilon_\rho\left(\delta^\alpha_\epsilon \delta^\beta_\lambda + \delta^\beta_\epsilon \delta^\alpha_\lambda\right)\right) \\
&= \frac{1}{4}\left(\tilde{e}^a_\mu \delta^\rho_\nu + \tilde{e}^a_\nu \delta^\rho_\mu\right)\left(\delta^\beta_\rho e^\alpha_a + \delta^\alpha_\rho e^\beta_a\right) - \frac{1}{8}\left(e^a_\mu \delta^\rho_\nu + e^a_\nu \delta^\rho_\mu\right)\left(\tilde{g}^\beta_\rho e^\alpha_a + \tilde{g}^\alpha_\rho e^\beta_a\right) \\
&= \frac{1}{4}\left(\tilde{e}^a_\mu e^\alpha_a \delta^\beta_\nu + \tilde{e}^a_\mu e^\beta_a \delta^\alpha_\nu + \tilde{e}^a_\nu e^\alpha_a \delta^\beta_\mu + \tilde{e}^a_\nu e^\beta_a \delta^\alpha_\mu\right) - \frac{1}{8}\left(\tilde{g}^\beta_\nu \delta^\alpha_\mu + \tilde{g}^\alpha_\nu \delta^\beta_\mu + \tilde{g}^\beta_\mu \delta^\alpha_\nu + \tilde{g}^\alpha_\mu \delta^\beta_\nu\right)
\end{aligned}
$$

Using that $\tilde{e}^a_\mu e^\alpha_a = \frac{1}{2}\tilde{g}^\alpha_\mu$:

$$
\begin{aligned}
\frac{\delta \tilde{g}_{\mu\nu}}{\delta g_{\alpha\beta}} &= \frac{1}{8}\left(\tilde{g}^\alpha_\mu \delta^\beta_\nu + \tilde{g}^\beta_\mu \delta^\alpha_\nu + \tilde{g}^\alpha_\nu \delta^\beta_\mu + \tilde{g}^\beta_\nu \delta^\alpha_\mu\right) - \frac{1}{8}\left(\tilde{g}^\beta_\nu \delta^\alpha_\mu + \tilde{g}^\alpha_\nu \delta^\beta_\mu + \tilde{g}^\beta_\mu \delta^\alpha_\nu + \tilde{g}^\alpha_\mu \delta^\beta_\nu\right) \\
&= 0
\end{aligned}
\tag{6.51}
$$

So, (6.50) has been demonstrated.

Appendix D: Harmonic Gauge.

We know that the Einstein's equations do not fix all degrees of freedom of $g_{\mu\nu}$. This means that, if $g_{\mu\nu}$ is solution, then exist other solution $g'_{\mu\nu}$ given by a general coordinate transformation $x \to x'$. We can eliminate these degrees of freedom by adopting some particular coordinate system, fixing the gauge.

One particularly convenient gauge is given by the harmonic coordinate conditions. That is:

$$\Gamma^{\mu} \equiv g^{\alpha\beta}\Gamma^{\mu}_{\alpha\beta} = 0 \tag{6.52}$$

Under general coordinate transformation, Γ^{μ} transform:

$$\Gamma'^{\mu} = \frac{\partial x'^{\mu}}{\partial x^{\alpha}}\Gamma^{\alpha} - g^{\alpha\beta}\frac{\partial^2 x'^{\mu}}{\partial x^{\alpha}\partial x^{\beta}}$$

Therefore, if Γ^{α} does not vanish, we can define a new coordinate system x'^{μ} where $\Gamma'^{\mu} = 0$. So, it is always possible to choose an harmonic coordinate system. For more detail about harmonic gauge see, for example, [63].

In the same form, we need fix the gauge to $\tilde{g}_{\mu\nu}$. It is natural to choose a gauge given by:

$$\tilde{\delta}\left(\Gamma^{\mu}\right) \equiv g^{\alpha\beta}\tilde{\delta}\left(\Gamma^{\mu}_{\alpha\beta}\right) - \tilde{g}^{\alpha\beta}\Gamma^{\mu}_{\alpha\beta} = 0 \tag{6.53}$$

Where $\tilde{\delta}\left(\Gamma^{\mu}_{\alpha\beta}\right) = \frac{1}{2}g^{\mu\lambda}\left(D_{\beta}\tilde{g}_{\lambda\alpha} + D_{\alpha}\tilde{g}_{\beta\lambda} - D_{\lambda}\tilde{g}_{\alpha\beta}\right)$. So, when we will refer to harmonic gauge, we will use (6.52) and (6.53).

Now, we will study the harmonic gauge to particular cases.

I) Schwarzschild:

In this case, a convenient harmonic coordinate system is (t,X_1,X_2,X_3) with:

$$\begin{aligned} X_1 &= (r-\mu)\sin(\theta)\cos(\phi) \\ X_2 &= (r-\mu)\sin(\theta)\sin(\phi) \\ X_3 &= (r-\mu)\cos(\theta) \end{aligned} \tag{6.54}$$

Where (t,r,θ,ϕ) is the standard coordinate system and $\mu = GM$. So, the metric in harmonic coordinate is:

$$g_{\mu\nu}dx^\mu dx^\nu = -A(r)c^2dt^2 + \left(\frac{r}{r-\mu}\right)^2 d\mathbf{X}^2 + \left(\frac{B(r)}{(r-\mu)^2} - \frac{r^2}{(r-\mu)^4}\right)(\mathbf{X}\cdot d\mathbf{X})^2 \tag{6.55}$$

Where $r = \mu + \sqrt{X_1^2 + X_2^2 + X_3^2}$. It is possible to demonstrate that, if we use (6.54) in (6.55), we obtain (4.1). In the same form, we can write (4.2) in this coordinate system. That is:

$$\tilde{g}_{\mu\nu}dx^\mu dx^\nu = -\tilde{A}(r)c^2dt^2 + \tilde{F}(r)\left(\frac{r}{r-\mu}\right)^2 d\mathbf{X}^2 + \left(\frac{\tilde{B}(r)}{(r-\mu)^2} - \frac{\tilde{F}(r)r^2}{(r-\mu)^4}\right)(\mathbf{X}\cdot d\mathbf{X})^2 \tag{6.56}$$

It is not difficult to see that this system is not convenient to work, so we will fix the gauge in the harmonic coordinate and then we will return to the standard coordinate system. By construction, (6.54) obey (6.52). However, (6.53) say us that we need the condition:

$$r^2(r-2\mu)\tilde{A}''(r) + 4r(r-2\mu)\tilde{A}'(r) - 4\mu\tilde{A}(r) + r(r-2\mu)(r-\mu)\tilde{F}''(r) + 4(r-\mu)^2\tilde{F}'(r) = 0$$

where $' = \frac{d}{dr}$ and we used (4.3-4.5). Therefore, the solution of $\tilde{A}(r)$ and $\tilde{F}(r)$ is given by this condition and (4.6).

II) FRW:

In this case, to find the harmonic coordinate system, we will change the t variable of (5.3) by u. So, the metric is now:

$$g_{\mu\nu}dx^\mu dx^\nu = -T^2(u)c^2du^2 + R^2(u)\left(dx^2 + dy^2 + dz^2\right) \tag{6.57}$$

such that $T(u)=\frac{dt}{du}(u)$. In the same form, (5.4) is changed to:

$$\tilde{g}_{\mu\nu}dx^\mu dx^\nu = -T_c(u)T^2(u)c^2du^2 + T_d(u)R^2(u)\left(dx^2+dy^2+dz^2\right) \tag{6.58}$$

Now, if we fix the harmonic gauge, we obtain that $T(u)=T_0R^3(u)$ from (6.52) and $T_c(u)=3(T_d(u)+T_1)$ from (6.53), where T_0 and T_1 are gauge constants. We use $T_0=1$ and $T_1=0$ to fix the gauge completely. So, with these conditions, the system (u,x,y,z) correspond to harmonic coordinate. Now, we can return to the usual system where $g_{\mu\nu}$ and $\tilde{g}_{\mu\nu}$ are given by:

$$\begin{aligned}
g_{\mu\nu}dx^\mu dx^\nu &= -c^2dt^2 + R^2(t)\left(dx^2+dy^2+dz^2\right)\\
\tilde{g}_{\mu\nu}dx^\mu dx^\nu &= -3T_d(t)c^2dt^2 + T_d(t)R^2(t)\left(dx^2+dy^2+dz^2\right)
\end{aligned}$$

where the gauge is fixed. We will used these expressions on **chapter 5** to solve the cosmological case.

III) Postnewtonian Limit:

The form of $g_{\mu\nu}$ and $\tilde{g}_{\mu\nu}$, given by (6.6) and (6.7) respectively, are the more general expression to covariant tensor of rank two. Therefore, we can choose the functions such that the harmonic gauge is obeyed. For this, we need impose (6.52) and (6.53) in the Post-Newtonian approximation. So, the harmonic gauge is reduced to:

$$\begin{aligned}
4\dot{\phi}+\partial_i\chi_i &= 0\\
2\phi\partial_i\phi-\dot{\chi}_i-\frac{1}{2}\partial_i\xi_{jj}+\partial_j\xi_{ij} &= 0\\
4\dot{\tilde{\phi}}+\partial_i\tilde{\chi}_i &= 0\\
2\phi\partial_i\tilde{\phi}+2\tilde{\phi}\partial_i\phi-\dot{\tilde{\chi}}_i-\frac{1}{2}\partial_i\tilde{\xi}_{jj}+\partial_j\tilde{\xi}_{ij} &= 0
\end{aligned}$$

In all our calculus, we fix the harmonic gauge. So, we use all these expressions to solve (2.10) and (2.11).

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