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# Loop Quantum Gravity and Effective Matter Theories

by

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# Summary

Since Loop Quantum Gravity (LQG) i.e., an approach to quantize gravity, was first formulated in the pioneering work of T. Jacobson and L. Smolin, it has undergone a rapid increase of original ideas leading to profound insights and unexpected connections between: gravity, loops, knots and gauge theory. After two decades of active research in the field, the LQG approach is by now considered viable as well as a promising candidate to quantize General Relativity. The approach is a minimal attempt to combine the ideas of Quantum Mechanics and General Relativity. Minimal in the sense that it sticks with the standard formulation of Quantum Mechanics and General Relativity, but implementing rigourously the diffeomorphism symmetry of General Relativity. The notion of diffeomorphism symmetry by its own leads to a fully background independent and non-perturbative formulation of Quantum Gravity. Partial culmination of these ideas had become crystalized in states of the gravitational field i.e., possible occurrence of 3-geometries with discrete eigenvalues and purely constructed from combinatorial principles.

The impetus in the LQG approach is mainly attributed to the success casting a consistent and formal Kinematical theory that solves the problems of quantum gravity in a great extent. Progress has been made by matching the black hole constraint, and by the recent inspired phenomenological models that put close possible scenarios to test loop quantum gravity effects. These effects have an opportunity to be probed in cosmological events and in particle physics beyond the Standard Model.

Moreover, the Hamiltonian of the Standard Model coupled to gravity, supports a representation based on densely defined operators. The resulting Hamiltonian is anomaly free and completely finite without renormalization. From these Hamilto-

nians we obtain the effective Hamiltonians that contains quantum gravity effects.

More precisely, in this thesis we focus in two objectives, first we present the departing theory from which we obtain the effective models, the Loop Quantum Gravity formalism. Secondly, we derive the Yang Mills and Higgs effective models that contains quantum gravity corrections.

In the introduction we summarize the arguments that supports the idea that a quantum theory of spacetime is important. Certainly, these arguments are not in the category to be considered formal proofs, instead they try to motivate the construction of this theory. The claim that a fundamental theory should not have place for infinities, resume them.

We review in chapter 2 the Hamiltonian formulation of General Relativity and the LQG formalism in which we pay special attention to the subset of kinematical constraints, this roughly includes the spin networks basis and geometric operators spectra.

Last chapters are central part of our investigation, they include a detailed analysis of the method we have developed to obtain Yang Mills and Higgs effective theories. Both effective theories were obtained using semiclassical states picked around a flat three metric and defined to preserve rotational invariance. We compute expectation values of Hamiltonians which describe particle propagation and predict near Planck energy scales the breakdown of Lorentz invariance.

# Chapter 1

# Introduction

## 1.1 Quantum gravity: Origins and motivations

Quantum Gravity is an attempt to amalgamate in a single consistent theory, two of the major revolutionary ideas of contemporary physics, Quantum Mechanics (QM) and General Relativity (GR). The research in this direction has been intense over the past seventy years, attracting a lot of ingenious ideas and the investments of tremendous efforts. However, until date all the proposed models to accomplish this synthesis have shown to be incomplete or inconsistent. The final program of quantizing GR has become extremely elusive and out of our immediate reach; reports and current status can be found in [2, 3]. Nevertheless, one has to recognize that enormous progress has been made in several directions, the most important approaches are among Superstring [10, 11], Non-commutative Geometry [12] and Loop Quantum Gravity (LQG), (for an introduction of LQG, see [4, 5], for the first use of loop variables see [1], and for textbooks see [6, 7] and more recently [8, 9]) which we honestly remark is not minor considering the complexity of the task.

The two theories proposed to unite are GR and QM, both are consistent but also disconnected descriptions of the world, each one separately, provide robust contributions to our knowledge. Let us briefly comment and highlight on their basic aspects.

The framework introduced by Einstein's theory harmoniously allows to accommodate spacetime and matter in a geometrical language. The concepts taking part in the foundational physics of Newton like action-at-distance and absolute observer had been replaced with new notions. In modern terminology, gravitational field is identified with spacetime geometry and absolute observer with general covariance. This is perhaps amongst the most radical changes taking place in physics.

Einstein equation $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}$ is a non linear, second order differential equation for the metric tensor, which synthesize and codes the dynamics of the theory. A large variety of geometric entities in the manifold, such as line elements, Riemann curvature and geodesics are specified through the metric. In that way, a broad range of gravitational phenomena are related to the interplay of matter and geometry, yielding, planetary orbits, gravitational lensing, black holes, etc.

Gravitational interaction is well known to persists over cosmological scales and neglected in high energy phenomena. Therefore, the scenario of gravity becoming crucial for local high energies processes, as suggested by quantum gravity, is quite astonishing. Then, the statement that gravity deals exclusively with the very large seems to be premature yet.

On the other side of our knowledge, the agenda to quantize all the forces has seen to be gloriously achieved in all known matter field theories, culminating in the celebrated $SU(3) \times SU(2) \times U(1)$ Standard Model (SM). Standard Model is a theory of strong and electroweak interactions. It accommodates gluons, W and photons acting as force carriers between quarks and between leptons. All fundamental interactions are described within the SM formalism, except gravity. The data obtained in accelerator experiments, with relativistic particles colliding at energies of the order of $10^2$ Gev, are extremely precise and in high accordance with theoretical predictions, also of course the experiments in condensed matter physics, quantum optics etc, which are ruled with the non relativistic Schrodinger equation. At date, no fundamental interactions of physics deviates from this universally quantum-matter implementation.

However and in spite of the successful implementation of Quantum Mechanics in the matter sector, a deep understanding of the steps that defines the theory is still missing. These fundamental issues, when treating with the gravitational system might turn to be important and not more allowed to be ignored.

Nowadays, it is widely known that the standard methods of functional quantization are sterile when applied to GR; this is the first explanation why gravity has not been quantized yet. Its origin lies on the fact that divergent diagrams occurring in the perturbative serie can not be removed by a renormalization procedure. A natural questions then arises: Are the basic assumptions in GR or QM fully correct?. Several and different approaches, work on the idea that a quantum description of spacetime requires a modification of the two basic theories GR and QM, some more drastic than others (see [13, 14, 15, 16, 17, 18]). Nevertheless, if one insist to apply a perturbative approach in gravity by replacing GR at high energies, just like Fermi theory was replaced by electroweak theory, then the main objection is that metric splitting and background dependence, implicit in perturbation theory, means to lose active diffeomorphism invariance of GR. Another reaction to the non-renormalizability of gravity is support the thesis that quantum corrections have negligible effects on the gravitational interaction which is extremely weak and therefore argue that the unification of GR and QM might be useless, at least from an empirical point of view. We are aware that this is a possibility, but contradicts the next point [ 2 ], and presents a departure from the historical evolution characterized in physics toward a reductionist viewpoint of Nature.

The present status of experiments gives no strong clue, although, some arguments sheds some light on why is important to search for this quantum gravity theory. The most popular claims are related to the role of observables and to the unavoidable inconsistencies resulting from the formulation of semiclassical quantum gravity:

[ 1 ]. A fundamental theory must have finite observables and consequently meaningful predictions. To solve the problem of having divergent observables in a theory the theoretical framework must be extended by replacing some of the basic assumption. An old hope in physics is the idea that quantum gravity could provide the extended framework to remedy the following singularities.

(a) Cosmological objects such as black holes are characterized to have infinite curvature in the origin, $R(0) \to \infty$, *high matter density* regions reflect themselves in that way.

(b) The problem of *initial condition* has not been solved in cosmological models. A description of the origin of the universe might come from the elucidation of some aspects that quantum gravity theory must confront.

(c) Standard Model is saturated with two kind of singularities. One shows up at fixed order when arbitrary high momenta are summed over in the perturbative expansion. The second kind of divergency, worst in nature, comes from the whole series expansion. This last one is not cured even using a renormalization procedure as in the first case. The argument here is that quantum gravity could provide a *natural gravitational regulator* for these infinities, preventing integrals from ultraviolet divergencies. Mainly due to space discreetness which is expected to arise from the quantum gravity theory, therefore prohibiting arbitrary short distance and high momenta availability.

[2]. A classical gravitational field interacting with a quantum system lead to *inconsistencies*. The key line of argument is roughly, after assuming $G_{\mu\nu} = \langle\psi \mid \hat{T}_{\mu\nu} \mid \psi\rangle$, solutions for the metric needs to specify if there has been wavefunction collapse after any measurement on the system. This fact leads to violation of either, uncertainty principle, momentum conservation, signals faster than light, or other unwanted results. [19]

## 1.2 Status and overview: free quantum field theories

In the next subsections, we want to give a brief overview of the axiomatic involved in the quantization of free field theories. The main motivation here, is to point out some misleading generalizations in the methods used in free quantum field theories when they are applied to the gravitational system. We will show two possible interpretations given to free Quantum Field Theories (QFT), the particle and field interpretations. The first interpretation is strongly motivated by the action of the Poincaré group while the second resembles more the general procedures of Quantum Mechanics. Canonical methods to quantize gravity uses the latter which is best suited to give physical meaning to the gravitational field and because the gauge symmetry underlying GR is the diffeomorphism group and not the Lorentz group. GR is a constrained system, thus the tools employed in canonical quantization of gravity are in many aspects different of those handled in traditional QFT. Here we will show those first differences.

### 1.2.1 Fock quantization: particle interpretation

Quantizing a system with infinite degrees of freedom, is commonly known as *second quantization*. In particle physics this is normally realized with the use of creation and annihilation operators that allows to build up the quantum field with all its basic harmonic excitations. And permits to connect by acting on the vacuum, the distinct particle Hilbert subspaces of finite harmonic oscillators. Fock space then is defined by the direct product of these individual Hilbert spaces.

The axiomatization of free QFT are usually given in terms of Wightman axioms [23, 24], which we proceed to summarize

1. **Unitary Representation.**

The existence of a unitary and continuous representation of the Poincaré group $\mathcal{P}$ realized in Hilbert space $\mathcal{H}$: $\mathcal{P} \rightarrow U(a, \Lambda)$, $a$ and $\Lambda$ being typical displacements and rotations in spacetime.

2. **Spectral Condition**

We require for the momentum operator $P^\mu P_\mu \leq m^2; \;\; P^0 \geq 0$

3. **Unique Vacuum**

The trivial representation $U(a, \Lambda) = 1$ correspond to a state which is invariant under all Poincaré transformations called the vacuum $\Omega_0$

$U(a, \Lambda)\ \Omega_0 = \Omega_0$

4. **Covariance**

Consider the smeared field operator $\phi(f) = \int d^4x\ \phi(x)f(x)$. The set of linear combinations of the form $\phi(f_1)...\phi(f_n)\Omega_0$ lie dense in $\mathcal{H}$, hence the vacuum is said to be cyclic. The field satisfy the covariant property

$U(p)\phi(f)U(p)^{-1} = \phi(f \cdot p) \qquad p \in \mathcal{P}$

5. **Microscopic Causality**

Let the supports of $\vec{f}$ and $\vec{g}$ be spacelike separated then the operators $\phi(\vec{f})$ and $\phi(\vec{g})$ must satisfy the relation $[\phi(\vec{f}), \phi(\vec{g})] = 0$.

Recall that in free field quantization, for instance in the case of a scalar field, the bilinear operator $\hat{N}_k = \hat{a}_k^\dagger \hat{a}_k$ which is called number operator, have integers eigenvalues and is a constant of motion. Quantum states which are classified according that integer are mapped by $\hat{a}_k^\dagger$ and $\hat{a}_k$ into states with increased or decreased that integer. The operators $\hat{a}_k^\dagger$ and $\hat{a}_k$ are interpreted as creating and annihilating quanta. Therefore, the notion of quanta being a observer independent quantity, heavily relies on the property of transformation of the operators among themselves. They must transform covariantly under the action of Poincaré group, as can be seen in the expression for the bilinear operator. In that way, the notion of particle in conventional QFT does not depend on the inertial system.

In resume, Fock space is essentially motivated because carries a unitary representation of the Poincaré group, which as we have said, permits to consider an n-particle state observer-independent, and therefore quanta as a absolute notion. The absence of this group in the GR case, is another good reason to consider a different approach than the particle interpretation.

### 1.2.2 Functional quantization: field interpretation

We start with the basic variables, the configuration variables $\phi(x,t)$ and its conjugate momenta $\pi(x,t)$. Both fields, are defined to satisfy the following equal time commutation relation

$$\left[\hat{\phi}(x), \hat{\pi}(y)\right] = i\hbar\delta^{(3)}(x-y) \tag{1.1}$$

Next we choose a representation of the equal time commutations relations, for which we define, respectively, smeared coordinates and momenta $Q$, $P$ of the fields $\phi$ and $\pi$ in terms of the following basis $\{e_1, ..., e_k\}$, $\nu = 1, ..., k$, for example at time $t = 0$, by

$$\hat{Q}_n(e,0) = \int \hat{\phi}_\alpha(x,0)e_\nu(x)d^3x \tag{1.2}$$

$$\hat{P}_n(e,0) = \int \hat{\pi}_\alpha(x,0)e_\nu(x)d^3x, \qquad n \in (\alpha, \nu) \tag{1.3}$$

which could generate temporal evolution through a Hamiltonian $H$, accordingly to

$$\hat{P}_n(x,t) = e^{\frac{i}{\hbar}Ht}\ \hat{P}_n(x,0)\ e^{\frac{-i}{\hbar}Ht} \tag{1.4}$$

$$\hat{Q}_n(x,t) = e^{\frac{i}{\hbar}Ht}\ \hat{Q}_n(x,0)\ e^{\frac{-i}{\hbar}Ht} \tag{1.5}$$

A representation of (1.1) is obtained by choosing state vectors as functionals of the classical configuration space $Q$ and belonging to the Hilbert space $L_2(Q, d\mu(Q))$, where $d\mu(Q)$ is the measure.

$$\hat{Q}_n(x)\Psi\left[\phi(X)\right] = Q_n(x)\Psi\left[\phi(X)\right] \tag{1.6}$$

$$\hat{P}_n(y)\Psi\left[Q(X)\right] = -i\hbar\frac{\delta\Psi\left[Q(X)\right]}{\delta Q_n(y)} \tag{1.7}$$

The Schrodinger equation is

$$H(Q, -i\hbar\frac{\delta}{\delta Q})\Psi\left[Q(X);t\right] = i\hbar\frac{\partial\Psi\left[Q(X);t\right]}{\partial t} \tag{1.8}$$

After a measurement is made of the classical field configuration, the quantity

$$P_B = \int_B |\Psi\left[Q(X);t\right]|^2 d\mu(Q) \tag{1.9}$$

is the probability that the result lie in the set B.

## 1.3 First historical attempts

The quantization program in high energy physics, applies only for *free* fields propagating in a fixed and flat background, as we have seen in the Wightman axiomatization. Therefore, approximative methods as perturbation theory and quantization on curved backgrounds have been developed to treat with interacting fields, which indeed are the ones we experience in real life. The perturbative method in SM relies on the availability to define a unitary operator that describes time evolution and connects free asymptotic quantum states in the far past and future. Haags theorem demonstrates that the interaction picture does not really exists, at least you can write down a perturbative serie that approximates it, but then you need to renormalize the theory. What is called renormalization is the mechanism by which order by order the serie can be made artificially finite, this is surmounted by redefinition of certain parameters in the Lagrangian as masses, charges and fields. This scheme has succeeded in the SM, although the convergence of the whole serie is totally lost and left without control.

The perturbative approach in GR fails because the gravitational field is not renormalizable as we will see in the next subsection. This is one of the reasons why the non-renormalizability of gravity could be considered disappointed. But let us emphasize that even in the case that perturbation techniques were viable in GR we would have to deal in consequence with the same open problem that characterize the SM, the convergence of the whole series.

Let us briefly sketch what happens when trying to implement the perturbative expansion in the gravitational system.

### 1.3.1 Gravity is non-renormalizable

A naive check to know if a theory is renormalizable is by power counting. Consider the dimensionless action $S$ in the units $\hbar = c = 1$. The theory for a certain interaction $g \int \mathcal{L}_{int}$ is renormalizable if the superficial degree of divergence $\triangle$ is not negative, where the mass dimensionality of the coupling constant is $g = M^{-\triangle}$. The reason is that for momenta $K \leq M$ lower than the mass scale of the coupling parameter $g = M^{-\triangle}$ the interaction is suppressed by the term $(\frac{k}{M})^{\triangle} \leq 1$ and therefore we lose predictability, for else we need a mass scale higher than the accessible energies of the system. The mass dimensionality depends on the superficially degree of divergence $\triangle = 4 - d - \sum_f n_f(s_f + 1)$, where $n_f$ is the number of fields of type $f$ and $s_f$, is $0$ , $1/2$ , $1$ , $0$ for scalars and graviton, fermions, massive vector field, and photons respectively.

The first check for gravity, is whether the superficial degree of divergence satisfies the above condition. For which we refer the readers to the calculation developed in [20] using the background field method. To sketch the calculation let us consider the perturbative treatment of the Einstein Hilbert action $\int d^4x\sqrt{g}R$ with the usual split of the metric around a fixed background plus the perturbation, in most cases the background is chosen to be the Minkowskian flat metric, in the following way,

$$g_{\mu\nu}(X) = \eta_{\mu\nu} + \sqrt{G}h_{\mu\nu}(X) \tag{1.10}$$

At first order we obtain an interaction term

$$\sqrt{G}h(\partial h)(\partial h) + ... \tag{1.11}$$

Making the power counting we see that $\triangle = 4 - 2 - 3 = -1$, and GR is not renormalizable neither including supersymmetry.

## 1.4 Main inputs: Concepts and tools in loop quantum gravity

At this point, the first attempt to quantize gravity have failed, when looking for other approaches as the canonical ones, we will have to consider a set of constraint and deal with their solutions (treated in detail in the next chapter). This can be done in two different ways, the first one is solving the constraints at the classical level and then quantizing (phase space reduced methods), and the second is quantizing and then trying to solve the constraints. The latter, which will be revised in the sequel, is named Dirac quantization, and is the one used in the LQG approach.

On the other hand, fair to say, solutions to the Hamiltonian constraint better known as Wheeler DeWitt equation, have never been found in the metric approach (geometrodynamics approach) not even a consistent regularization. Things improve in the LQG approach, following the Dirac quantization program and using Ashtekar variables, some naive solutions can in fact be given, but a complete understanding of the theory, which amounts to solve the Hamiltonian constraint remains an open problem.

Due to the many obstacles when trying to quantize gravity, the discussion to a great extent has been centered on the requirements asked to the quantum gravity theory. Therefore, in the formulation of Quantum Gravity theory one must decide among all the basic structures one wants to preserve. The loop approach tries to implement the quantization program without changing the traditional structures of both QM and GR, and holding further the diffeomorphism symmetry of GR.

The additional requirement of having active diffeomorphism invariance in the theory, might seen at first not enough to produce a substantial change. However, as we would see, a relevant shift in the mathematics involved would take place, with states of the gravitational field, spin network relying ultimately in combinatorial terms, and amazing implications which by now belong to the principal ingredients of the LQG approach, as discreteness of 3-geometries. The notion of space that we have gained in our everyday experience, continuum and smooth has to be considered only an approximation accounting for many basic excitations that are discrete in nature and that should become evident, as we descend to the fundamental levels of space.

Progress has been achieved from the LQG perspective in understanding the non-renormalizability of gravity. Essentially because the separation of the gravitational field into a classical background plus a quantum correction neglects non-perturbative effects, such as discreteness of 3-geometries that contradicts the assumption of having availability of arbitrary short distances.

As we have already mentioned, the irreconcilability of quantum theory and general relativity, is by now, one of the important problems left unsolved in theoretical physics. The problems, had lead us to deal with the interplay of concepts taking part in the final theory. The common sensation, however, and definitely the biggest obstacle that undermines the final program of quantum gravity, which should be always kept in mind, is the shortage confronting experimental data which permits to test the different models and the existent theories in dispute.

### 1.4.1 Diffeomorphism concept in Einstein's theory

The goal in this subsection is review the key notions firmly established by now in the GR formalism, as well as their lesser known implications. More precisely, we want here to emphasize the role of diffeomorphism symmetry and show how this induces the concept of relationalism in physics.

The GR legacy has brought a substantial upgrade in the understanding of Nature. Firstly due to the identification of gravitational field with spacetime geometry. This has changed the old vision that all the things evolve on top of a fixed absolute background structure, the same one that Newton needed to formulate his theory. The rigid background is today considered as a dynamical entity governed through the Einstein's equations. Secondly, the underlying gauge invariance of GR, general covariance or active diffeomorphism invariance implies a spectacular renaissance of the idea of relationism in physics. The notion of relationism has returned with tangible physical meaning and not only as a product of philosophical thoughts. Let us explain this concept in more detail.

The GR formalism is invariant under the group of diffeomorphism $Diff(\mathcal{M})$. They are transformations defined over the manifold $\mathcal{M}$, mapping the spacetime points to the same manifold while the relevant physical quantities are left invariant. This signifies that all the possible locality information regarding individual points in the manifold gets virtually removed by the symmetry, not even to transfer points to a set of equivalent regions, more instead to only one big equivalent region, the whole manifold. This is achieved by means of all the $C^\infty$ transformations. Holding further this idea, it can be realized that space and time must lack of absolute meaning i.e., points can not be referred to some-where or some-when, for else the implicit fact that GR has to be described in a fully relational form [21, 22]. In this sense, relational means that what is capable to give a precise description to things are only through relationships, very different of what happens in Newtonian theory where a system defined by a flat non-dynamical background is such that all the physical effects of other system ultimately relies on this absolute one.

Furthermore the quantum implementation of this symmetry is well captured in the picture of spacetime arising in LQG, in which space is defined by the manner loops are knotted to each other on a single point and subject to change in time. The rationale in GR is different from other physical theories formulated on dependent backgrounds, as the entire SM in which the view is that fields evolve on top of a background. It is meaningful in QED theory to consider a par creation and ask for the direction of the emitted particles, because the direction is supposed to be referred to an external absolute structure. Spacetime objects in GR rests only through relationships, any background structure breaks this symmetry albeit in many cases the theory may still be able to be formulated in a passive invariant fashion.

Diffeomorphism is capable to indicate us the presence of a background, this extra information involves the dynamics of the theory itself. Passive transformations depends on how the theory are mathematically formulated.

### 1.4.2 Dirac quantization

In this subsection we will review the Dirac algorithm for constrained systems which is the one used in the LQG approach. A more detailed discussion can be found in [25].

Consider a classical system with Hamiltonian $H(q_i, p_i)$, with $q_i$ a set of canonical variables with canonical momenta $p_i$. Whenever a system with non independent variables is formulated, a set of constraints is obtained and defined by the relations

$$\phi_a(q_i, p_i) = 0 \qquad i = 1, 2, ...a \tag{1.12}$$

First class constraints are defined to satisfy,

$$\{\phi_a, \phi_b\} = C^d_{ab}\phi_d \tag{1.13}$$

Second class constraints do not satisfy this relation. Although, there are methods to bring second class constraints into first class constraints. After some manipulation involving the lagrange multipliers $\lambda_a$, the time evolution of a function $O(q,p)$ satisfies

$$\dot{O}(q,p) = \{O, H\} + \lambda_a\{O, \phi_a\} \tag{1.14}$$

$O$ is defined to be an observable if $\{O, \phi_a\} = 0$. first class constraints generate gauge transformations connecting physical trajectories in a restricted phase space constraint surface. The Hamiltonian can be written $H' = H + \lambda_a \phi_a$.

Now, let us resume the steps leading to the Dirac quantization of systems with constraints

1. Define an auxiliary Hilbert space $H^{(\text{aux})}$ and pick a polarization such that quantum states are functionals of the configurations variables $\Psi[Q]$

2. Find a representation for the classical algebra into commutators

$$[\hat{P}(x), \hat{Q}(y)] = -i\delta^{(3)}(x-y) \tag{1.15}$$

with self-adjoint operators on a Hilbert space of states

$$\hat{Q} = Q \tag{1.16}$$

$$\hat{P} = -i\frac{\delta}{\delta Q} \tag{1.17}$$

3. Symmetries are represented as constraints, then one has to impose quantum constraints annihilating physical states, narrowing the $H^{(\text{aux})}$ to a physical Hilbert space $H^{(\text{phy})}$

4. We need now a inner product in the physical Hilbert space in order to compute expectation values and make physical predictions

5. Construct a set of observables that could be interpreted

More references for this subsection are found in [26, 27, 28, 29, 30]

# Chapter 2

# The loop quantum gravity formalism

Now that we have discussed the concepts involved in the loop approach, let us turn to the review of the mathematical formulation.

The Hamiltonian formulation of GR was developed in the sixties by Arnowitt, Deser and Misner (ADM) [31]. In rigor the formalism constitutes the starting point of canonical approaches. To describe dynamical evolution, the Hamiltonian is required to depend on a time parameter. Therefore, spacetime manifold $\mathcal{M}$ of 4-dimensions will be arbitrary spilt in 3-dimensional space plus time direction which is unphysical. The explicit covariance of the theory will be spoil, although the constraints will tell us that the theory is invariant under any such choice of coordinatization. Thus, all the concepts analyzed in the previous sections are going to be canalized through the constraints.

The use of complex phase space for GR (Ashtekar or self-dual variables) remarkably leads to a simplification of the algebraic structure of the constraint. Allowing for the first time to found an overcomplete basis of solutions for the Hamiltonian constraint. These solutions called Wilson loops provides the name "loop" and hints on the advantages of a formulation of GR in terms of Ashtekar variables [32].

## 2.1 Hamiltonian Formulation

Consider a global hyperbolic 4-manifold $\mathcal{M}$ with topology $\mathbb{R} \times S$, and $S$ a compact 3-manifold representing space and $t \in \mathbb{R}$ unphysical time. Next, cover $\mathcal{M}$ with a foliation into Cauchy surfaces $\Sigma_t$ and define $t$ as a global time function and $t^\mu$ a time-like vector representing the flow of time, both obeying $t^\mu \nabla_\mu t = 1$.

Let us write an imbedded space metric $q_{\mu\nu}$ as

$$q_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu, \tag{2.1}$$

with $n^\mu$ a normal vector to $\Sigma_t$ and with $n^\mu n^\mu = -1$. The vector field $t^\mu$ can be decomposed in its normal $n^\mu$ and tangential $N^\mu$ components to the surface $\Sigma_t$ in the form,

$$t^\mu = N n^\mu + N^\mu. \tag{2.2}$$

N receives the name of *lapse function*, as it measures the change in proper time while $N^\mu$ is called *shift vector* relating normal displacements. Written in a particular system of local coordinates $(x, t)$ and in terms of the quantities $(q_{\mu\nu}, N, N^i)$, the line element $ds$ reads,

$$ds^2 = -N^2 dt^2 + q_{ab}(dx^a + N^a dt)(dx^b + N^b dt). \tag{2.3}$$

Note that the metric $q_{\mu\nu}$ acts as projector on $\Sigma_t$, therefore verifying $q_{\mu\nu}\, n^\nu = 0$ and $q_{\mu\nu}\, q^{\nu\rho} = q^\rho_\mu$, therefore in what follows $q_{\mu\nu}$ will be considered a space metric written with space indices only.

An important object that describes the velocity of $q_{ab}$ as it moves normally to the surface $\Sigma_t$, is given by the extrinsic curvature $K_{\mu\nu}$:

$$K_{\mu\nu} = q_\mu^{\ \sigma}\ \nabla_\sigma n_\nu, \tag{2.4}$$

or the alternative definition given by,

$$K_{\mu\nu} = \frac{1}{2}\mathcal{L}_{\vec{n}} q_{\mu\nu}, \tag{2.5}$$

where $\mathcal{L}$ is the Lie derivative along the vector $\vec{n}$. With the same considerations as before we will write the extrinsic curvature with spatial indices.

The tensors $q_{ab}$ and $K_{ab}$ are called first and second fundamental forms in $\Sigma_t$ respectively, they behave as Cauchy data for the metric, just like the potential vector $A^a$ and the electric field $E_a$ are Cauchy data for the electromagnetic tensor $F_{\mu\nu}$. Phase space consist in the 3-metric $q_{ab}$ and in the extrinsic curvature $K_{ab}$. They allows to cast the system in canonical form.

After some manipulation the Einstein-Hilbert action $S = \int d^4xR$ written in the variables $q$ and $K$ reduces to

$$S = \int dt d^3x N\sqrt{q}({}^{(3)}R + K_{ab}K^{ab} - K^2), \tag{2.6}$$

with $q$ the 3-determinant of $q_{ab}$, where $\sqrt{g} = N\sqrt{q}$, and using the following notation $K = K^a_a$.

The action is now conveniently expressed in terms of variables that are space functions and which evolves in time, so let us follow the traditional steps in the canonical quantization program. With this purpose, we choose the 3-metric $q_{ab}$ to play the role of position, while its associated conjugate momenta results to be, $\pi^{ab} = \frac{\partial\mathcal{L}}{\partial\dot{q}_{ab}} = \sqrt{q}(K^{ab} - K^2 q^{ab})$. Then, after working out the Hamiltonian density and using the Legendre transform of the Lagriangian density $\mathcal{L}$, we finally arrive to the expression

$$H(q,\pi) = \int_\Sigma d^3x\mathcal{H} = \int_\Sigma d^3x\sqrt{q}(NC + N^aC_a) \tag{2.7}$$

where we have defined

$$C = -{}^{(3)}R + q^{-1}(\pi_{ab}\pi^{ab} - \frac{1}{2}\pi^2) \tag{2.8}$$

and

$$C_a = -2\nabla^b (q^{-1/2}\pi_{ab}), \tag{2.9}$$

the notation is such that $\nabla^a$, is the torsion-free covariant derivative compatible with $q_{ab}$ and $\pi^2 = (\pi^a_a)^2$.

The fact that the Hamiltonian contains no time derivatives with respect to the lapse and the shifts functions, means that $N$ and $N_a$ are Lagrange multipliers, instead than true dynamical variables. It can be shown that,

$$C = 0 \qquad \text{and} \qquad C_a = 0. \tag{2.10}$$

The above equations are instantaneous laws satisfied on shell i.e., on each hypersurface $\Sigma$, analogous to the Gauss constraint in electromagnetism. In electromagnetism the Gauss law constraint, tell us how to control the redundant degree of freedom of the gauge theory $U(1)$, by pointing out, that not any electric field results to be a proper solution. In here we have the same situation, with only certain allowed regions in phase space.

The constraints $C$ and $C^a$ are called Hamiltonian and spatial diffeomorphism respectively, because when we set $\vec{N}$ to zero, the Hamiltonian of GR turns out to be, see eq (2.7),

$$C(N) = \int_\Sigma N C q^{1/2} d^3x. \tag{2.11}$$

Considering the phase space function $J$, it can be shown that $C(N)$ generates infinitesimal gauge transformations

$$\delta_N J = \{C(N), J\}. \tag{2.12}$$

Thus, the Hamiltonian constraint $C(N)$ generates diffeomorphisms in a normal direction that corresponds to the direction of the flow of time. Now, if we set the lapse $N$ to zero,

$$\vec{C}(\vec{N}) = \int_\Sigma N^a C_a q^{1/2} d^3x \tag{2.13}$$

and we arrive to,

$$\delta_{\vec{N}} J = \{\vec{C}(\vec{N}), J\}. \tag{2.14}$$

As before, we see that $C(\vec{N})$ generates diffeomorphism in the $\vec{N}$ direction with displacements tangent to $\Sigma_t$.

The Hamiltonian and diffeomorphism constraint can be shown to verify the Poisson algebra,

$$\{\vec{C}(\vec{N}), \vec{C}(\vec{M})\} = \vec{C}(\mathcal{L}_{\vec{M}} \vec{N}) \tag{2.15}$$

$$\{\vec{C}(\vec{N}), C(M)\} = C(\mathcal{L}_{\vec{N}} \vec{M}) \tag{2.16}$$

$$\{C(N), C(M)\} = \vec{C}(\vec{K}) \tag{2.17}$$

where $K$ is defined to be $K^a = qq^{ab}(N\partial_b M - M\partial_b N)$.

## 2.2 Geometrodynamics quantization

Until this point, GR has been canonically formulated and the gauge character of gravity directly related to the constraints, that in turn play the role of spacetime diffeomorphism generators. The Hamiltonian of GR has seen to vanish on shell, since it's a linear combinations of vanishing constraints. At this stage we could think that the dynamics of the theory is dominated by a trivial Hamiltonian, which is not the case. More precisely, is an indication that the GR Hamiltonian is not a true Hamiltonian, instead generates spacetime diffeomorphisms and therefore tell us that space and time can not be defined as something absolute.

Our next step is to apply the rules of quantization to the gravitational system we have derived, picking for example a polarization for the state vectors in terms of metric variables $\Psi(q)$. Thereafter, we seek for a representation of the symplectic algebra expressed in terms of metric variables, which lead us to the next step in the Dirac quantization program for constrained systems. Promote constraints to operators while requiring physical state vectors to lie in the kernel of the constraints,

$$\hat{C}^{\mu}(\hat{q}, \hat{\pi})\Psi(q) = 0. \tag{2.18}$$

However and unfortunately, the $C^{\mu}$ functions involve terms in which the canonical variables appear non polynomially multiplied, not even analytically on the metric $q_{ab}$, so we must confront operator ordering problems; different orders of writing down equivalent classical formulas for the constraints yield different operators in the quantum level. One can, at least in principle, seek for regularized expressions for the constraints, where operators are to be defined in terms of limits of smeared field variables. Up to date no one has yet demonstrated a consistent choice of canonical variables and operator orderings in which the constraints equations behave in the above sense. This technical issue is the **constraints implementation problem** and is where the program first stalls. Let us go further and analyze the dynamic of the theory. We have said that the Hamiltonian density vanishes on physical states by virtue of equation (2.7) and (2.10). The dynamics of the theory, then, is not dominated by the Hamiltonian, rather describes the states that are invariant under all spacetime diffeomorphism. This is the **time problem** in quantum gravity; this feature is related to the meaning of time as something determined intrinsically by the theory (in terms of geometry), rather than by an external structure. In order to compute expectation values and relevant quantities involved in any measurable phenomena we will require a physical inner product and a real Hilbert space. We must factor out somehow the gauge group of $Diff(\Sigma)$ when one performs an inner product. This resembles the factorization done in the functional integral, and is called the **inner product problem**. These three problems are the major roadblock we face when we try to quantize gravity by means of canonical technics (for a more detailed discussion on these points see [7, 37]).

## 2.3 New canonical framework

Now that we have summarized the main list of problems encountered in the program of geometrodynamics quantization, let us change to a different approach that will lead us close to the basic of the LQG formalism. We will give a chance to the so called first order formulation, which introduce **tetrads** and **spin connections** as new variables in phase space. These new variables introduce extra degrees of freedom in the theory, providing an additional solution to the Einstein equation together with a new subset of constraints related to rotational invariance in tangent space. The new variables allow to describe the system more like a Yang Mills (YM) theory. We will exploit this fact using all the machinery developed to quantize these gauge systems. However constraints will retain their non polynomial character with no substantial progress when we try to implement the quantum version of constraints and with all the subsequent attached problems we have describe before.

Nevertheless, if in addition phase space of GR is extended using self dual variables or the so called Ashtekar variables [32], that is, a complex phase space with self dual connections and tetrads, then the constraints notably simplifies. Wilson loops solutions of the Hamiltonian and Gauss constraint, are in here, the first sign of the loop notion arising.

### 2.3.1 First order variables

The key point of the next step is to shift the description of the system, basically from metric variables to connection ones. First let us define tetrads and spin connections.

A set of basis vectors $\{e_a^I\}$ are called tetrads or vielbeins if the metric of space time looks locally flat, ie.

$$g_{\mu\nu} = e_\mu^I e_\nu^J \eta_{IJ} \qquad \{I, \mu\} = 0, 1, 2, 3 \tag{2.19}$$

We can perform a a Lorentz transformation on the flat indices $\{I\}$ at every point in spacetime, they are called **local Lorentz transformations**, or perform usual

**general coordinate transformations** or **diffeomorphisms** on curved indices $\{\mu\}$. Comparison of the covariant derivative on two different bases permit us to write the following relationship between spin connection $w_\mu{}^I{}_J$ and ordinary connection $\Gamma^\nu_{\mu\lambda}$

$$w_\mu{}^I{}_J = e^I_\nu e^\lambda_J \Gamma^\nu_{\mu\lambda} - e^\lambda_J \partial_\mu e^I_\lambda \tag{2.20}$$

We can think on $e^I_\mu$ and $w_\mu{}^a{}_b$ as a vector and tensor valued one form respectively [36].

### 2.3.2 The Palatini action and self-dual action

With the same assumptions regarding the manifold structure of the previous section, the Palatini action is defined by a curvature 2-form $\Omega^{IJ}_{\mu\nu} = \partial_{[\mu}\omega^{IJ}_{\nu]} + [\omega_\mu, \omega_\nu]^{IJ}$ the connection 1-form $\omega^{IJ}_\mu$ and the tetrad $e^\mu_I$ as,

$$S(e, w) = \int d^4x \; e \; e^a_I e^b_J \Omega^{IJ}_{ab}, \tag{2.21}$$

where $e = \sqrt{-g}$ is the determinant of the tetrad.

After performing the usual spacetime decomposition, we arrive to an additional solution, which is $e^a_I = 0$ and to a new constraint, the Gauss constraint $D^a e^I_a = 0$ [37].

The main idea of the modern formulation is to introduce Ashtekar variables [32] or the **self-dual** Lorentz connection ${}^+A$ keeping the tetradic construction. This extra structure requires complex phase space variables, therefore requires to extend the differential geometric structure of traditional GR to complex GR. Under appropriate reality conditions we can recover the standard theory which makes them totally equivalent. The spacetime decomposition of this new action introduce a fundamental change in the kind of constraints; this in part relies on the fact that the new connection entails information of both variables $q_{ab}$ and $K_{ab}$, like Bargmann or holomorphic variables in the solution of the quantum harmonic oscillator. We define a Lorentz connection by $A^{IJ}_a = -A^{JI}_a$ and the internal Hodge dual of a

Lorentz connection mapping element of the space of connections to itself defined by ${}^*T^{IJ} = \frac{1}{2}\epsilon^{IJ}{}_{KL}T^{KL}$; any Lorentz connection can be write by a sum of self-dual and anti-self dual parts $A = {}^+A + {}^-A$ with ${}^*A = \pm i\ {}^\pm A$. In terms of self and anti self dual parts ${}^\pm A = (A \mp i\ {}^*A)/2$. The other basic field besides the self-dual connection is the tetrad $e^I_\mu$, the proposed self dual action is,

$$S(e, {}^+A) = \int_M d^4x\ e\ e^\mu_I e^\nu_J {}^+F^{IJ}_{\mu\nu}. \tag{2.22}$$

It can be shown that the curvature of a self dual Lorentz connection is self dual $*F = iF$ Where we have defined the internal Hodge dual of the curvature F as $(*F)^{IJ}_{\mu\nu} = \frac{1}{2}\epsilon^{IJ}{}_{KL}F^{KL}_{\mu\nu}$. Now we proceed with the usual decomposition of spacetime; gauge fixing the internal vector $n_I = e^d_I n_d = (1, 0, 0, 0)$ allow to consider only $0I$ components of internal indices.

The Hamiltonian becomes,

$$H = \int_\Sigma \{\frac{1}{2}N\vec{F}_{ab}(\vec{E}^a \times \vec{E}^a) + N^a\vec{F}_{ab} \cdot \vec{E}^b + \Lambda \cdot D_a\vec{E}^a\} \tag{2.23}$$

With $E^a_I = \sqrt{q}e^a_I$ and the self dual connection $A^i_a$. They satisfy the Poisson bracket relation

$$\{A^i_a(x), E^b_j(y)\} = i\delta^b_a\delta^i_j\delta^{(3)}(x - y) \tag{2.24}$$

with new constraints,

$$G^i = D_aE^{ai} \qquad C^a = E^b_iF^i_{ab} \qquad \mathcal{H} = \epsilon^{ij}_k E^a_i E^b_j F^k_{ab} \tag{2.25}$$

and obeying a different constraint algebra [5].

## 2.4 Loop quantization

The use of Ashtekar variables has led to a simplification in the constraints. To proceed we must choose a representation of the relation (2.24) and pick a polarization of the functionals in terms of connection variables for instance, such as

$$\hat{A}^i_a \Psi(A) = A^i_a \Psi(A) \tag{2.26}$$

$$\hat{E}^a_i \Psi(A) = \frac{\delta}{\delta A^i_a} \Psi(A) \tag{2.27}$$

and then promote the constraints to operators equations. First we will consider a order for the constraints, with the triads to the right (2.25),

$$\hat{G}^i = D_a \frac{\delta}{\delta A^i_a} \qquad C_a = F^i_{ab} \frac{\delta}{\delta A^i_a} \qquad H = \epsilon^{ijk} F^i_{ab} \frac{\delta}{\delta A^j_a} \frac{\delta}{\delta A^k_b} \tag{2.28}$$

Gauss constraint require that states be gauge invariant functionals states of the connection $A$, therefore states can be described by the loop states $\Psi_\gamma(A) = \prod_i Tr(h(A, \gamma_i))$ as the basis states for quantum gravity. They allow us to control the diffeomorphism constraint and are solutions of the Hamiltonian constraint. However one of the difficulties of these objects is that they form a overcomplete basis.

It can be shown that the Gauss and the Diffeomorphism constraint generates gauge transformations and diffeomorphisms on the wavefunctionals $\Psi(A)$. A first inspection suggest to consider Wilson loops $W(A) = Tr\left(\mathcal{P}exp \oint A_a dx^a\right)$ which are well known to be gauge invariant functionals under transformations of the connection $A$, and therefore automatically solutions of the Gauss constraints. We would continue to require these states to be annihilated by the Diffeomorphism constraints and then continue with the Hamiltonian constraints, with the hope to finish at the end with a genuine Hilbert space, with physical states belonging to it.

### 2.4.1 Holonomies

Let a curve $\gamma$ be defined as a continuous, piecewise smooth map from the interval $[0, 1]$ into the 3-manifold $M$,

$$\gamma : [0, 1] \longrightarrow M \tag{2.29}$$

$$s \longmapsto \{\gamma^a(s)\}\ ,\ a = 1, 2, 3\ . \tag{2.30}$$

The holonomy or parallel propagator $h[A,\gamma]$, of the connection $A$ along the curve $\gamma$ is defined by

$$
\begin{aligned}
h[A,\gamma](s) &\in SU(2)\ , && (2.31)\\
h[A,\gamma](0) &= \mathbb{1}\ , && (2.32)\\
\frac{d}{ds}\, h[A,\gamma](s) &+ A_a\big(\gamma(s)\big)\,\dot{\gamma}^a(s)\,U[A,\gamma](s) = 0\ , && (2.33)
\end{aligned}
$$

where $\dot{\gamma}(s) := \frac{d\gamma(s)}{ds}$ is the tangent to the curve. The formal solution of (2.33) is given in terms of the series expansion

$$
\mathcal{P}\exp\int_0^1 ds\, A\big(\gamma(s)\big) = \sum_{n=0}^{\infty}\int_0^1 ds_1\int_0^{s_1} ds_2\cdots\int_0^{s_{n-1}} ds_n\, A\big(\gamma(s_n)\big)\cdots A\big(\gamma(s_1)\big)\ . \qquad (2.34)
$$

$$
h[A,\gamma](s) = \mathcal{P}\exp\int_\gamma ds\,\dot{\gamma}^a\,A_a^i\big(\gamma(s)\big)\,\tau_i \equiv \mathcal{P}\exp\int_\gamma A\ , \qquad (2.35)
$$

for any matrix-valued function $A\big(\gamma(s)\big)$ which is defined along $\gamma$.

Here $\mathcal{P}$ denotes path ordering, i.e. the parameters $s_i$ are ordered with respect to their moduli from the left to the right, or more explicitely $s_1 \le s_2 \le \dots$ .

### 2.4.2 Spin Networks

A graph $\Gamma_n = \{\gamma_1, \dots, \gamma_n\}$ is a finite collection of $n$ piecewise smooth curves or edges $\gamma_i$, $i = 1, \dots, n$, respectively, embedded in the 3-manifold $M$, that meet only at their endpoints.

A spin network is a generalization of a graph, namely a colored graph. More precisely, by definition a spin network is a triple $s = (\Gamma, \vec{j}, \vec{N})$, where to each link $\gamma_i$ we assign a non-trivial irreducible representation of $SU(2)$ which is labelled by the numbers $\vec{j} = \{j_i\}$. Let $\mathcal{H}_{j_1}, \dots, \mathcal{H}_{j_k}$ be the Hilbert spaces of the representations associated to the $k$ links. Consider a node $p$ where the $k$ links meet, and associate to it the Hilbert space $\mathcal{H}_p = \mathcal{H}_{j_1} \bigotimes \dots\dots \bigotimes \mathcal{H}_{j_k}$. Fix an orthonormal basis element

$N_p$ in $H_p$. An element $N_p$ of the basis is called a coloring of the node $p$. A spin network state can be defined by taking holonomies at each link associated to the $j$ representation and contracting it with the elements of the basis in $\mathcal{H}_p$ where links meet. The spin networks states $\Psi(A)$ can be shown to satisfy the orthonormal condition

$$< \Psi_S \mid \Psi_{S'} >= \delta_{\Gamma,\Gamma'}\delta_{j,j'}\delta_{S,S'} \tag{2.36}$$

### 2.4.3 Area Operator

The spectrum computation of geometric operators in the loop representation were done taking advantage of their non locality properties [78]. Using the complete basis of spin network states is possible to calculate operators spectrums that are observables in the Dirac sense.

A surface $\Sigma$ is a 2-dimensional submanifold embedded in $M$. The associated embedding is given by the map $X : \Sigma \to M$ and is characterized by local coordinates $x^a, a = 1, 2, 3$ on $M$ and coordinates on the surface $\sigma^\mu = (\sigma^1, \sigma^2), \mu, \nu = 1, 2$.

$$\Sigma : (\sigma^1, \sigma^2) \mapsto x^a(\sigma^1, \sigma^2) \tag{2.37}$$

The pullback metric $g^\Sigma$ and the normal $n_a$ on $\Sigma$ are given by

$$g^\Sigma_{\mu\nu} = \frac{\partial x^a}{\partial \sigma^\mu}\frac{\partial x^b}{\partial \sigma^\nu} g_{ab} \qquad \text{and} \qquad n_a = \frac{1}{2}\epsilon^{\mu\nu}\epsilon_{abc}\frac{\partial x^b(\vec{\sigma})}{\partial \sigma^\mu}\frac{\partial x^c(\vec{\sigma})}{\partial \sigma^\nu} \tag{2.38}$$

The area is then

$$\begin{aligned} A[\Sigma] &= \int_\Sigma d^2\sigma \sqrt{\text{det} g^\Sigma} = \int_\Sigma d^2x \sqrt{\frac{1}{2!}\epsilon^{\mu_1\mu_2}\epsilon^{\nu_1\nu_2} g^\Sigma_{\mu_1\nu_1} g^\Sigma_{\mu_2\nu_2}} \\ &= \int_\Sigma d^2x \sqrt{n_a n_b E^{ai} E^{bi}} \end{aligned} \tag{2.39}$$

After regularizing the area $A(\Sigma)$ the area operator $\hat{A}(\Sigma)$ is given by

$$\hat{A}(\Sigma) = \lim_{\varepsilon\to\infty} \sum_{n(\varepsilon)} \sqrt{\hat{E}^i(\Sigma_n)\hat{E}^i(\Sigma_n)} \tag{2.40}$$

where we have define the smeared operator

$$\hat{E}^i(\Sigma) \quad = \quad -i\hbar G \int_\Sigma d^2\sigma n_a(\vec{\sigma}) \frac{\delta}{\delta A_a^i(x(\vec{\sigma}))} \tag{2.41}$$

Finally, the eigenvalues corresponding to the area operator are given by the expression

$$\mathfrak{A}(\Sigma) = 8\pi\beta\hbar G \sum_l \sqrt{j_l(j_l+1)}\ . \tag{2.42}$$

Where $j_l$ labels de $SU(2)$ representation associated to the link $l$ crossing the surface $\Sigma$ .

### 2.4.4 Volume Operator

Let us consider the volume of a three dimensional region $R$, this is given by

$$\int_R d^3x \sqrt{\det g} \tag{2.43}$$

$g$ is the three-dimensional space metric. In the same way as in the case of the area operator, a regularization for the volume operator is needed.

We will not show the technical steps in the derivation of the volume operator but merely give its final expression, which is

$$V = \frac{1}{4} l_p^3 \sum_i \sqrt{a_i b_i c_i + a_i b_i + b_i c_i} \tag{2.44}$$

where $p$ , $q$ and $r$ are the colors of the three adjacent link of the node $i$. And where we have defined

$$p = a_i + b_i \quad q = b_i + c_i \quad r = c_i + a_i \tag{2.45}$$

# Chapter 3

# Yang Mills effective model

Let us go now on to the construction of the Yang Mills effective theory when diffeomorphism invariance, geometric operators, spin networks, and what we have study in the previous chapters are extended or included to matter dynamics. The concrete implementation given in here is due to the calculations of our work which forms part of the results of this thesis.

First attempts to include matter in the framework of loop quantum gravity, were done in the pioneering work [48, 49]. For an approach based on the Kinematical framework for diffeomorphism invariant theories of connections, see [50]. The posterior breakthrough came by generalizing to diffeomorphism invariant quantum field theories, including, besides connections, also fermions and Higgs fields. And facing directly the task of including matter fields in the Kinematical scheme with well defined adjoint relations and using the volume operator to solve order ambiguities and finiteness of the theory [51]. In that way a consistent representation with holonomies-like excitations of quantum fields, was successfully implemented to describe all the sectors of the Standard Model.

In this chapter we concentrate in the process of generalizing the loop quantum gravity inspired model described in [56, 57] to Yang-Mills fields, in order to obtain the non-Abelian generalization of the corrections previously found for the dynamics of photons. Namely, corrections to standard matter dynamics are obtained by means of calculating non-Abelian holonomies, either of gravitational or Yang-Mills type, around triangular paths. The basic tool in our analysis is the holonomy along a straight line segment which path order property we consistently keep to all orders in our expansion.

The work is organized as follows: we start working on the regularization of the Yang Mills Hamiltonian using the Thiemann procedure [51]. We summarizes the results for the Abelian expansion of holonomies in subsection 3.4.1, which we generalize [53]. In the last step we calculate the expectation value of the Yang Mills Hamiltonian with matter fields expanded around vertices which allows to construct finally the Yang Mills effective theory.

## 3.1 Electric sector

We will follow the original work of Thiemann and its basics ingredients for the regularization of the Yang Mills field [51]. The regularization procedure heavily depends on the action of the volume operator on spin networks. The volume operator annihilates states unless they act on a vertex of the graph, which permits to syntonize the two triangulations arising in the regularization. Moreover, the procedure is such that a large class of Hamiltonians of weight one which are diffeomorphism covariant and are coupled to gravity, can be turned into densely defined and anomaly-free operators on a formal defined Hilbert space $\mathcal{H}$.

In addition, let us mention that our expressions for the regularized Hamiltonians are slightly different from the original ones. Mainly because matter fields in our approximation are not considered full quantum states, instead they are treated in an approximation where they are largely parameterized by unknown terms. This treatment is more well adapted to the semiclassical approximations we are interested in, instead than the investigation of exact states for gravity plus matter, which lies beyond our scope.

The Yang Mills Hamiltonian is composed by an electric and magnetic part smeared on a surface $\Sigma$ over a space function $N(x)$ as

$$H_{YM}(N) = \int_{\Sigma} d^3x N(x) \frac{q_{ab}}{2Q^2\sqrt{\det q}} (E^a_{\underline{I}} E^b_{\underline{I}} + B^a_{\underline{I}} B^b_{\underline{I}}), \tag{3.1}$$

the notation introduced is such that the underlines indices denotes an arbitrary compact gauge group $G$, for instance, the gauge group of the Standard Model (SM) and where $a, b, \ldots$ denotes spatial indices. We are assuming the usual electromagnetic tensor in terms of magnetic and electric variables as $F^{ab}_{\underline{I}} = \epsilon^{abc} B^{\underline{I}}_c$ and $F^{0a}_{\underline{I}} = E^a_{\underline{I}}$

Focussing first in the electric part, the identity $1/\kappa\{A^i_a, V\} = 2\ \mathrm{sgn}(\det e^j_b) e^i_a$ allows to rewrite the above expression as

$$
\begin{aligned}
H^E = \frac{1}{2\kappa Q^2} \lim_{\epsilon \to 0} \int_\Sigma & d^3x N(x) \{A^i_a(x), \sqrt{V(x,\epsilon)}\} E^a_{\underline{I}}(x) \\
\times \int_\Sigma & d^3y \chi_\epsilon(x,y) \{A^i_b(y), \sqrt{V(y,\epsilon)}\} E^b_{\underline{I}}(y)
\end{aligned}
\tag{3.2}
$$

where $\epsilon$ is a small number and $\chi_\epsilon(x,y) = \prod_{a=1}^3 \theta(\epsilon/2 - |x^a - y^a|)$ is defined as the characteristic function of a cube of volume $\epsilon^3$ centered at $x$. In addition let $V(x,\epsilon) := \int d^3y \chi_\epsilon(x,y) \sqrt{\det(q)}$ be the volume of the box as measured by $q_{ab}$.

This coordinatization procedure will spoil the explicit diffeomorphism covariance, which however, will be recovered once the regulator is removed in the next steps. We note that the trick works as long as we keep the density weight of the constraint to be one, since then a natural balance between point splitting and powers of $\sqrt{\det q}$ in the denominator will permit us to eliminate the divergent factor $1/\epsilon^3$ [51].

Let us define

$$
\begin{aligned}
\Theta^i_{\underline{I}}[f] &:= \int d^3x\ f(x) E^a_{\underline{I}}(x) \left\{ A^i_a(x), \sqrt{V(x,\epsilon)} \right\} \\
&= \sum_\Delta \int_\Delta d^3x\ f(x) E^a_{\underline{I}}(x) \left\{ A^i_a(x), \sqrt{V(x,\epsilon)} \right\} \\
\Theta^i_{\underline{I}}[f] &=: \sum_\Delta \Theta^i_{\Delta \underline{I}}[f]
\end{aligned}
\tag{3.3}
$$

and the covariant flux of $E^a_{\underline{I}}$ through the two-surface $S$ as,

$$
\Phi^E_{\underline{I}}(S) := \mathrm{tr} \left[ \tau_{\underline{I}} h_e \left( \int_S h_{\rho(p)} E^a(p) h^{-1}_{\rho(p)} \epsilon_{abc} ds^{bc}(p) \right) h_e^{-1} \right] \tag{3.4}
$$

where $\rho(p)$ is the path from the vertex $v$ to the point $p$ lying in the two-surface $S$ and $h_{\rho(p)}$ the holonomy associated to the connection of the gauge field $\underline{A}^a_I$.

Note that

$$
\begin{aligned}
tr\left(\tau_i h_{s_L}\left\{h^{-1}_{s_L}, \sqrt{V(x,\epsilon)}\right\}\right) &= tr\left(\tau_i\tau_m \int_0^1 dt\ \dot{s}_L^{-1a}(t)\left\{A^m_a(s_L^{-1}(t)), \sqrt{V(x,\epsilon)}\right\}\right) + \ldots \\
&= -\frac{\delta_{im}}{2}\int_0^1 dt\ \dot{s}_L^{-1a}(t)\left\{A^m_a(s_L^{-1}(t)), \sqrt{V(x,\epsilon)}\right\} + \ldots \\
&\approx -\frac{1}{2}s^a_L(1)\left\{A^i_a(s_L^{-1}(0)), \sqrt{V(x,\epsilon)}\right\} \qquad (3.5)
\end{aligned}
$$

therefore, for small tetrahedra $\Phi^E_{\underline{I}}(F_{JK}) \approx \frac{1}{2}\epsilon_{abc}\, s^b_J(\Delta)\, s^c_K(\Delta)\, E^a_{\underline{I}}$ , it follows

$$
\begin{aligned}
& f(v)\,\epsilon^{JKL}\Phi^E_{\underline{I}}(F_{JK})\,\mathrm{tr}\left(\tau^i\ h_{s_L(\Delta)}\left\{h^{-1}_{s_L(\Delta)},\ \sqrt{V(v(\Delta),\epsilon)}\right\}\right) \\
\approx\ & -\frac{1}{4}f(v)\epsilon^{JKL}\epsilon_{abc}s^b_J(\Delta)s^c_K(\Delta)E^a_{\underline{I}}s^d_L(\Delta)\left\{A^i_d(s_l^{-1}(0)), \sqrt{V(x,\epsilon)}\right\} \\
=\ & -\frac{3!}{2}f(v)vol(\Delta)E^a_{\underline{I}}\left\{A^i_a(s_l^{-1}(0)), \sqrt{V(x,\epsilon)}\right\} \\
=\ & -\frac{3!}{2}\int_\Delta f\, e_{\underline{I}}\wedge\left\{A^i(x), \sqrt{V(x,\epsilon)}\right\}\ . \qquad (3.6)
\end{aligned}
$$

We have then

$$
\Theta^i_{\Delta\underline{I}}[f] = -\frac{2}{3!}\ f(v)\ \epsilon^{JKL}\Phi^E_{\underline{I}}(F_{JK})\mathrm{tr}\left(\tau^i\ h_{s_L(\Delta)}\left\{h^{-1}_{s_L(\Delta)},\ \sqrt{V(v(\Delta),\epsilon)}\right\}\right), \qquad (3.7)
$$

where $s_J(\Delta), s_K(\Delta), s_L(\Delta)$ denotes the edges of the tetrahedra $\Delta$ having $v$ as common vertex, and $F_{JK}$ the surface parallel to the face determined by $s_J(\Delta), s_K(\Delta)$ which is transverse to $s_L(\Delta)$.

Hence

$$
H^E[N] = \frac{1}{2\kappa^2 Q^2}\lim_{\epsilon\longrightarrow 0}\sum_{\Delta\Delta'}\Theta^i_{\Delta\underline{I}}[N]\Theta^i_{\Delta'\underline{I}}[\chi]. \qquad (3.8)
$$

Next we promote $\underline{E}^a$ and $V(x,\epsilon)$ to quantum operators and adapts the triangulation to the embedded graph $\gamma$ that corresponds to the state acted upon.

This is done with the prescription that at each vertex $v$ of $\gamma$ having the triplet of edges $e, e', e''$ a tetrahedron is defined with basepoint at the vertex $v(\Delta) = v$ and segments $s_I(\Delta)$, $I = 1, 2, 3$, corresponding to $s(e), s(e'), s(e'')$ [51]. Let us denote the arcs connecting the end points of $s_I(\Delta)$ and $s_J(\Delta)$ by $a_{IJ}(\Delta)$, so that a loop $\alpha_{IJ} := s_I \circ a_{IJ} \circ s_J^{-1}$ can be formed.

The action of the regulated operator hereby obtained gets concentrated in the vertices of the graph, in essence due to the action of the volume operator which annihilates a state unless the region defined by the $\epsilon$-box contains a vertex, which in successive steps we tend to zero.

We now rearrange the electric Hamiltonian using the following expressions

$$\begin{aligned}\hat{\Theta}^i_{\Delta \underline{I}}[N] &= -\frac{2}{3!}\frac{1}{i\hbar}\ N(v(\Delta))\ \epsilon^{JKL}\hat{\Phi}^E_{\underline{I}}(F_{JK}) \\ &\times \mathrm{tr}\left(\tau^i\ h_{s_L(\Delta)}\left[h^{-1}_{s_L(\Delta)},\ \sqrt{\hat{V}(v(\Delta),\epsilon)}\right]\right)\end{aligned} \tag{3.9}$$

and

$$\begin{aligned}\hat{\Theta}^i_{\Delta' \underline{I}}[\chi] &= -\frac{2}{3!}\frac{1}{i\hbar}\ \chi_\epsilon(v(\Delta), v(\Delta'))\ \epsilon^{MNP}\hat{\Phi}^E_{\underline{I}}(F'_{MN}) \\ &\times \mathrm{tr}\left(\tau^i\ h_{s_P(\Delta')}\left[h^{-1}_{s_P(\Delta')},\ \sqrt{\hat{V}(v(\Delta'),\epsilon)}\right]\right),\end{aligned} \tag{3.10}$$

which after replacing in (3.8) results in

$$\begin{aligned} H^E[N] &= -\frac{1}{\hbar^2 2\kappa^2 Q^2}\sum_{v\in V(\gamma)} N(v)\left(\frac{2}{3!}\frac{8}{E(v)}\right)^2 \sum_{v(\Delta)=v(\Delta')=v} \times \\ &\times \mathrm{tr}\left(\tau^i\ h_{s_L(\Delta)}\left[h^{-1}_{s_L(\Delta)},\ \sqrt{\hat{V}(v(\Delta),\epsilon)}\right]\right)\ \epsilon^{JKL}\hat{\Phi}^E_{\underline{I}}(F_{JK}) \times \\ &\times \mathrm{tr}\left(\tau^i\ h_{s_P(\Delta')}\left[h^{-1}_{s_P(\Delta')},\ \sqrt{\hat{V}(v(\Delta'),\epsilon)}\right]\right)\ \epsilon^{MNP}\hat{\Phi}^E_{\underline{I}}(F'_{MN}). \end{aligned} \tag{3.11}$$

We define the valence $n(v)$ of the vertex $v$, which produce the contribution $E(v) = n(v)(n(v)-1)(n(v)-2)/3!$ of the adapted triangulation in each vertex of $\gamma$. Moreover, we have considered that as $\epsilon \to 0$, $v(\Delta) = v(\Delta')$ is the only contributions left over in the sum.

## 3.2 Magnetic sector

Let us continue with the magnetic part of (3.1), using the same regularization scheme. We start concentrating in the expression for the holonomy of the $G$ connection $A$, which will be used to rewrite the magnetic constraint in the following.

$$\underline{h}_{\alpha_{IJ}} = \mathcal{P}\exp\left(\oint_{\alpha_{IJ}} \underline{A}_a(\vec{x}(s))\,\frac{dx^a}{ds}\,ds\right) \tag{3.12}$$

for small tetrahedra it reduces to, see Eq(3.62)

$$\mathrm{tr}(\tau_{\underline{I}}\underline{h}_{\alpha_{IJ}}) \approx -i\frac{1}{2}\epsilon_{abc}s_J^{\,b}(1)s_K^{\,c}(1)B^a_{\underline{I}}(v(\Delta)) \tag{3.13}$$

And with the use of

$$\begin{aligned}
& f(v)\epsilon^{JKL}\mathrm{tr}(\tau_{\underline{I}}\underline{h}_{\alpha_{JK}})\mathrm{tr}\left(\tau_i h_{s_L(\Delta)}\left\{h^{-1}_{s_L(\Delta)}, \sqrt{V(x,\epsilon)}\right\}\right) \\
\approx\ & i\frac{1}{4}\epsilon^{JKL}\epsilon_{abc}s_J^{\,b}s_K^{\,c}s_L^{\,d}f(v)B^a_{\underline{I}}(v)\left\{A^i_d(v), \sqrt{V(x,\epsilon)}\right\} \\
=\ & i\frac{1}{2}vol(s_J,s_K,s_L)\;\delta^d_a\;f(v)\;B^a_{\underline{I}}(v)\left\{A^i_d(v), \sqrt{V(x,\epsilon)}\right\} \\
=\ & i\frac{3!}{2}vol(\Delta)f(v)B^a_{\underline{I}}(v)\left\{A^i_a(v), \sqrt{V(x,\epsilon)}\right\} \\
=\ & i\frac{3!}{2}\int_\Delta f(x)\;B_{\underline{I}}(x)\wedge\left\{A^i(x), \sqrt{V(x,\epsilon)}\right\},
\end{aligned} \tag{3.14}$$

we can write

$$H^B[N] = \frac{1}{2\kappa^2Q^2}\lim_{\epsilon\longrightarrow 0}\sum_{\Delta\Delta'}\Xi^i_{\Delta\underline{I}}[N]\;\Xi^i_{\Delta'\underline{I}}[\chi], \tag{3.15}$$

where

$$\Xi^i_{\Delta\underline{I}}[f] := i\frac{2}{3!}f(v)\epsilon^{JKL}\mathrm{tr}(\tau_{\underline{I}}\underline{h}_{\alpha_{JK}})\mathrm{tr}\left(\tau_i h_{s_L(\Delta)}\left\{h^{-1}_{s_L(\Delta)}, \sqrt{V(x,\epsilon)}\right\}\right). \tag{3.16}$$

The quantum counterparts of the above expressions are

$$\hat{\Xi}^i_{\Delta\underline{I}}[f] := i\frac{2}{3!}\frac{1}{i\hbar}\, f(v)\, \epsilon^{JKL}\mathrm{tr}(\tau_{\underline{I}}\underline{h}_{\alpha_{JK}})\mathrm{tr}\left(\tau_i h_{s_L(\Delta)}\left[h^{-1}_{s_L(\Delta)}, \sqrt{\hat{V}(x,\epsilon)}\right]\right).$$

And the regularized magnetic piece of the Hamiltonian constraint is

$$\begin{aligned} H^B[N] &= +\frac{1}{\hbar^2 2\kappa^2 Q^2}\sum_{v\in V(\gamma)} N(v)\left(\frac{2}{3!}\frac{8}{E(v)}\right)^2 \sum_{v(\Delta)=v(\Delta')=v} \times \\ &\times\ \epsilon^{JKL}\,\mathrm{tr}\left(\tau_i\, h_{s_L(\Delta)}\left[h^{-1}_{s_L(\Delta)}, \sqrt{\hat{V}_v}\right]\right)\,\mathrm{tr}(\tau_{\underline{I}}\underline{h}_{\alpha_{JK}}) \times \\ &\times\ \epsilon^{MNP}\,\mathrm{tr}\left(\tau_i\, h_{s_P(\Delta')}\left[h^{-1}_{s_P(\Delta')}, \sqrt{\hat{V}_v}\right]\right)\,\mathrm{tr}(\tau_{\underline{I}}\underline{h}_{\alpha_{JK}}). \end{aligned} \tag{3.17}$$

## 3.3 The total regularized Hamiltonian

From (3.11) and (3.17), the total Hamiltonian can be written as

$$\hat{H}_{\mathrm{Yang-Mills}}[N] = \frac{1}{\hbar^2 2\kappa^2 Q^2}\sum_{v\in V(\gamma)} N(v)\left(\frac{2}{3!}\frac{8}{E(v)}\right)^2 \sum_{v(\Delta)=v(\Delta')=v} \mathrm{tr}\left(\tau_i\, h_{s_L(\Delta)}\left[h^{-1}_{s_L(\Delta)}, \sqrt{\hat{V}_v}\right]\right)$$
$$\mathrm{tr}\left(\tau_i h_{s_P(\Delta')}\left[h^{-1}_{s_P(\Delta')}, \sqrt{\hat{V}_v}\right]\right)\epsilon^{JKL}\epsilon^{MNP}\left[\mathrm{tr}(\tau_{\underline{I}}\underline{h}_{\alpha_{JK}})\mathrm{tr}(\tau_{\underline{I}}\underline{h}_{\alpha_{MN}}) - \hat{\Phi}^E_{\underline{I}}(F_{JK})\hat{\Phi}^E_{\underline{I}}(F'_{MN})\right]. \tag{3.18}$$

Before proceeding a comment on the general structure of the above regularized Hamiltonian is in order to fix some ideas. So far we have obtained a well first quantized operator anomaly free and finite which includes kinematical gravitational degrees of freedom coupled to matter dynamics. The underlying invariant group being $SU(2)$ and $G$, with holonomies excitations has appeared well suited to describe the theory [62, 63, 64, 65, 66, 67] .

The algebraic structure is such that a global gravitational factor is included in the $SU(2)$ trace, each one acting along one edge of the graph. The basic matter entities that regularize the electromagnetic part are the magnetic holonomy along a triangular path and the electric flux smeared in a face surface spanned by the tetrahedra of the triangulation .

Let us recall that according to Thiemann's conventions, the flat space case reduces to

$$H_{\mathrm{Yang-Mills}} = \int d^3x \frac{1}{2\,Q^2}\left(E^a_{\underline{I}} E^a_{\underline{I}} + B^a_{\underline{I}}\, B^a_{\underline{I}}\right), \tag{3.19}$$

where $Q$ is the electromagnetic coupling constant. The units are such that the gravitational connection $A^i_a$ has dimensions of $1/L$ (inverse Length) and the Newton's constant $\kappa$ has dimensions of $L/M$ (Length over Mass). Also we have that $[E^{\underline{I}}_a/Q^2] = M/L^3$. Taking the dimensions of the electromagnetic potential $A^{\underline{I}}_a$ to be $1/L$, according to the corresponding normalization of the Wilson loop, we conclude that $[E^{\underline{I}}_a] = [B^{\underline{I}}_a] = 1/L^2$ and $[Q^2] = 1/(M\,L)$. In our case we also have $[\hbar] = M\,L$, which in fact leads to $\alpha_{EM} = Q^2\,\hbar$ to be the dimensionless fine-structure constant, as defined by Thiemann [51].

## 3.4 Holonomies expansion

The method to obtain the quantum gravity induced corrections to the magnetic part of the Yang-Mills Lagrangian requires, see the expression (3.18), the calculation of the object

$$T_\rho = tr\left(\mathbf{G}_\rho\, \mathbf{h}_{\alpha_{IJ}}\right), \tag{3.20}$$

where $\mathbf{G}_\rho$ are the generators of the corresponding Lie algebra and $\mathbf{h}_{\alpha_{ij}(\Delta)}$ is the holonomy of the Yang-Mills connection $\mathbf{A}_a = A^\rho_a\,\mathbf{G}_\rho$ in the triangle $\alpha_{IJ}$, with vertex $v$, defined by the vectors $\vec{s}_I$ and $\vec{s}_J$, arising from the vertex $v$ Fig1.

Our main task will be to construct an expansion of $T_\rho$ in powers of the segments $s^a_I$, $s^b_J$.

To be more precise, we have

$$\mathbf{h}_{\alpha_{IJ}} = \mathcal{P}\,\exp\left(\oint_{\alpha_{IJ}} \mathbf{A}_a(\vec{x}(s))\,\frac{dx^a}{ds}\,ds\right), \tag{3.21}$$

where $\mathcal{P}$ is a path-ordered product specified in the subsection 2.4. As shown in Fig.3.1, the closed path $\alpha_{IJ}$, parameterized by $\vec{x}(s)$, is defined in the following way:

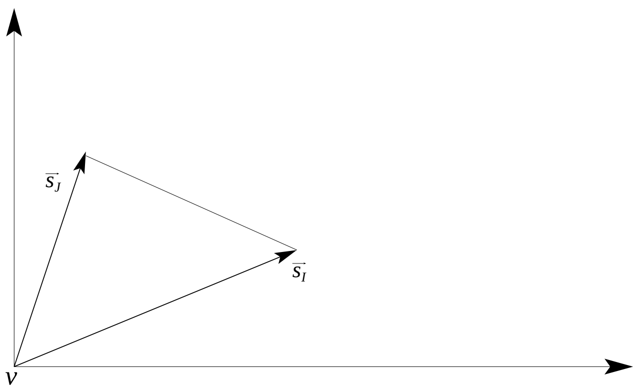


Figure 3.1: Triangle $\alpha_{IJ}$ with vertex $v$

we start from the vertex $v$ following a straight line in the direction and length of $\vec{s}_I$, then follow another straight line in the direction and length of $\vec{s}_J - \vec{s}_I$, and finally returning to $v$ following $-\vec{s}_J$. From the definition of the holonomy, we have the transformation property

$$\mathbf{h}_{\alpha_{IJ}} \to \mathbf{U}(v)\,\mathbf{h}_{\alpha_{IJ}}\,\mathbf{U}(v)^{-1}, \tag{3.22}$$

under a gauge transformation of the connection, where $\mathbf{U}(v)$ is a group element valued on the vertex $v$. In other words, $\mathbf{h}_{\alpha_{IJ}}$ transforms covariantly under the group.

### 3.4.1 The Abelian case

The corresponding calculation was performed in [57] and here we summarize the results in order to have the correct expressions to which the non-Abelian result must reduce when taking the commuting limit. In this case Eq.(3.20) reduces to

$$T = exp(\Phi_{IJ}) - 1, \tag{3.23}$$

where $\Phi_{IJ}$ is the magnetic flux through the area of the triangle, given by

$$\begin{aligned}\Phi^B(F_{IJ}) &= \oint_{\alpha_{IJ}} dt\ \dot{s}^a(t) A_a(t) \\ &= \int_{\vec{v}}^{\vec{v}+\vec{s}_I} A_a\, dx^a + \int_{\vec{v}+\vec{s}_I}^{\vec{v}+\vec{s}_J} A_a\, dx^a + \int_{\vec{v}+\vec{s}_J}^{\vec{v}} A_a\, dx^a, \end{aligned}\tag{3.24}$$

where the connection $A_a(\vec{x}(s))$ is now a commuting object.

The basic building block in (3.24) is

$$\begin{aligned}\int_{\vec{v}_1}^{\vec{v}_2} A_a(\vec{x})\, dx^a &= \int_0^1 A_a(\vec{v}_1 + t\,(\vec{v}_2 - \vec{v}_1))\,(\vec{v}_2 - \vec{v}_1)^a\, dt \\ &= \int_0^1 A_a(\vec{v}_1 + t\,\vec{\Delta})\,\Delta^a\, dt \\ &= \left(1 + \frac{1}{2!}\Delta^b\partial_b + \frac{1}{3!}(\Delta^b\partial_b)^2 + \dots\right)\Delta^a A_a(v), \end{aligned} \tag{3.25}$$

with $\Delta^a = (\vec{v}_2 - \vec{v}_1)^a$. The infinite series in parenthesis is

$$F(x) = 1 + \frac{1}{2!}x + \frac{1}{3!}x^2 + \frac{1}{4!}x^3 + \cdots = \frac{e^x - 1}{x}, \tag{3.26}$$

yielding

$$\int_{\vec{v}_1}^{\vec{v}_2} A_a(\vec{x})\, dx^a = F(\Delta^a\,\partial_a)\,(\Delta^a\, A_a(\vec{v}_1))\,. \tag{3.27}$$

In the following we employ the notation $\Delta^a\, V_a = \vec{\Delta}\cdot\vec{V}$. Using the above result in the three integrals appearing in (3.24) and after some algebra, we obtain

$$\begin{aligned}\Phi^B(F_{IJ}) &= F_1(\vec{s}_I\cdot\nabla, \vec{s}_J\cdot\nabla)\; s^a_J\, s^b_I\;(\partial_a\, A_b(\vec{v}) - \partial_b\, A_a(\vec{v})) \\ &= F_1(\vec{s}_I\cdot\nabla, \vec{s}_J\cdot\nabla)\; s^a_J\, s^b_I\,\epsilon_{abc}B^c(v),\end{aligned} \tag{3.28}$$

where the gradient acts upon the coordinates of $\vec{v}$. The function $F_1$ is

$$F_1(x,y) = \frac{y(\mathrm{e}^x - 1) - x(\mathrm{e}^y - 1)}{x\,y\,(y-x)} = -\sum_{n=1}^{\infty}\frac{1}{(n+1)!}\,\frac{x^n - y^n}{x-y}. \tag{3.29}$$

Let us emphazise that $F_1(x,y)$ is just a power series in the variables $x$ and $y$. Expanding in powers of the segments $s^a_I$ we obtain

$$\begin{aligned}\Phi^B(F_{IJ}) &= \left(1\; + \frac{1}{3}(s^c_I + s^c_J)\,\partial_c\; + \frac{1}{12}(s^c_I\, s^d_I + s^c_I\, s^d_J + s^c_J\, s^d_J)\,\partial_c\,\partial_d + ...\right)\times \\ &\quad\times\frac{1}{2}s^a_I s^b_J\epsilon_{abc}B^c(v).\end{aligned} \tag{3.30}$$

Notice that the combination

$$\frac{1}{2}s^a_I s^b_J\epsilon_{abc} = \mathcal{A}\; n_c, \tag{3.31}$$

is just the oriented area of the triangle with vertex $v$ and sides $s_I^c$, $s_J^c$, joining at this vertex, having value $\mathcal{A}$ and unit normal vector $n_c$.

To conclude we have to calculate

$$\left(e^{\Phi^B(F_{IJ}(\Delta))}-1\right) \;=\; \sum_{n=2}^{\infty}\frac{1}{n!}(\Phi^B(F_{IJ}))^n=\sum_{n=2}^{\infty}M_{nIJ(\Delta)}, \tag{3.32}$$

where the subindex $n$ labels the corresponding power in the vectors $s^a$. The results are

$$M_{2IJ(\Delta)} \;:=\; s_I^a s_J^b \frac{1}{2!}F_{ab}, \tag{3.33}$$

$$M_{3IJ(\Delta)} \;:=\; s_I^a s_J^b \frac{1}{3!}(x_I+x_J)F_{ab}, \tag{3.34}$$

$$M_{4IJ(\Delta)} \;:=\; s_I^a s_J^b \frac{1}{4!}(x_I^2+x_Ix_J+x_J^2)F_{ab}+s_I^a s_J^b s_I^c s_J^d \frac{1}{8}F_{ab}F_{cd}, \tag{3.35}$$

$$M_{5IJ(\Delta)} \;:=\; s_I^a s_J^b s_I^c s_J^d \left[\frac{1}{4\cdot 3!}(x_I+x_J)F_{ab}F_{cd}+\frac{1}{4\cdot 3!}F_{ab}(x_I+x_J)F_{cd}\right] \tag{3.36}$$

up to fifth order. We are using the notation $x_I=\vec{s}_I\cdot\nabla=s_I^a\,\partial_a$.

We expect that the non-Abelian generalization of the quantities (3.85), (4.52), (3.35), (3.36) is produced by the replacement

$$A_a \to \mathbf{A}_a = A_a^\rho\, G_\rho, \quad \partial_a \to \mathbf{D}_a=\partial_a-[\mathbf{A}_a,\ ] \tag{3.37}$$

$$F_{ab}\to \mathbf{F}_{ab}=\partial_a\mathbf{A}_b-\partial_b\mathbf{A}_a-[\mathbf{A}_a,\mathbf{A}_b] \tag{3.38}$$

Nevertheless, at this level there are potential ordering ambiguities which will be resolved in the next subsections.

### 3.4.2 The non-Abelian case

In a similar way to the Abelian case we separate the calculation of the holonomy $\mathbf{h}_{\alpha_{IJ}}$ in three basic pieces through the straight lines along the sides of the triangle $\alpha_{IJ}$. We have

$$\mathbf{h}_{\alpha_{IJ}}=P(e^{L_3})P(e^{L_2})P(e^{L_1})\equiv U_3\,U_2\,U_1, \tag{3.39}$$

where

$$L_1 = \int_0^1 dt\, \mathbf{A}_a(\vec{v} + t\,\vec{s}_I)\, s_I^a \tag{3.40}$$

$$L_2 = \int_0^1 dt\, \mathbf{A}_a(\vec{v} + \vec{s}_I + t\,(\vec{s}_J - \vec{s}_I))\,(s_J^a - s_I^a) \tag{3.41}$$

$$L_3 = \int_0^1 dt\, \mathbf{A}_a(\vec{v} + \vec{s}_J - t\,\vec{s}_J)\,(-s_J^a) \tag{3.42}$$

Here we have parameterized each segment with $0 \leq t \leq 1$.

Let us consider in detail the contribution

$$U_1 = P(e^{L_1}), \qquad L_1 = \int_0^1 dt\, \mathbf{A}_a(\vec{v} + t\,\vec{s}_I)\, s_I^a, \tag{3.43}$$

with $\vec{s}_I = \{s_I^a\}$.

Using the definition

$$\begin{aligned} U_1 \;&=\; 1 + \int_0^1 dt \mathbf{A}_a(\vec{v} + t\vec{s}_I) s_I^a + \int_0^1 dt \int_0^t dt' \mathbf{A}_a(\vec{v} + t\vec{s}_I) \mathbf{A}_b(\vec{v} + t'\vec{s}_I) s_I^a s_I^b \\ &+ \int_0^1 dt \int_0^t dt' \int_0^{t'} dt'' \mathbf{A}_a \mathbf{A}_b \mathbf{A}_c s_I^a s_I^b s_I^c + \ldots, \end{aligned} \tag{3.44}$$

for the path ordering, we arrive at the following expression

$$\begin{aligned} U_1 \;&=\; 1 + I_1(x) \mathbf{A}_a(v) s_I^a + I_2(x, \bar{x}) \mathbf{A}_a(v) \bar{\mathbf{A}}_b(v) s_I^a s_I^b \\ &+ I_3(x, \bar{x}, \bar{\bar{x}}) \mathbf{A}_a(v) \bar{\mathbf{A}}_b(v) \bar{\bar{\mathbf{A}}}_c(v) s_I^a s_I^b s_I^c + \ldots \end{aligned} \tag{3.45}$$

Here we are using the conventions

$$x = s_I^c \partial_c \qquad \bar{x} = s_I^c \bar{\partial}_c \qquad \bar{\bar{x}} = s_I^c \bar{\bar{\partial}}_c \tag{3.46}$$

$$I_1(x) = F(x), \qquad I_2(x, \bar{x}) = \frac{F(x + \bar{x}) - F(x)}{\bar{x}} \tag{3.47}$$

$$I_3(x, \bar{x}, \bar{\bar{x}}) = \frac{1}{\bar{\bar{x}}} \left[ \frac{1}{\bar{x} + \bar{\bar{x}}} (F(x + \bar{x} + \bar{\bar{x}}) - F(x)) - \frac{1}{\bar{x}} (F(x + \bar{x}) - F(x)) \right] \tag{3.48}$$

with $F(x)$ given by Eq.(3.26). The notation in Eq. (3.45) is that each operator $x, \bar{x}, \bar{\bar{x}}$ acts only in the corresponding field $A, \bar{A}, \bar{\bar{A}}$ respectively. We write

$$U_1 = \sum_N U_1^{(N)}, \tag{3.49}$$

where the superindex $N$ indicates the powers of $s^a_I$ contained in each term. A detailed calculation produces

$$U_1^{(1)} = s^a_I \mathbf{A}_a, \qquad U_1^{(2)} = \frac{1}{2}(x s^a_I \mathbf{A}_a + s^a_I s^b_I \mathbf{A}_a \mathbf{A}_b) \tag{3.50}$$

$$U_1^{(3)} = \frac{1}{3!}(x^2 s^a_I \mathbf{A}_a + (\bar{x} + 2x) s^a_I s^b_I \mathbf{A}_a \bar{\mathbf{A}}_b + s^a_I s^b_I s^c_I \mathbf{A}_a \mathbf{A}_b \mathbf{A}_c) \tag{3.51}$$

$$\begin{aligned} U_1^{(4)} &= \frac{1}{4!}\Big[x^3 s^a_I \mathbf{A}_a + (3x^2 + 3x\bar{x} + \bar{x}^2) s^a_I s^b_I \mathbf{A}_a \bar{\mathbf{A}}_b + (3x + 2\bar{x} + \bar{\bar{x}}) s^a_I s^b_I s^c_I \mathbf{A}_a \bar{\mathbf{A}}_b \bar{\bar{\mathbf{A}}}_c \\ &\qquad + s^a_I s^b_I s^c_I s^d_I \mathbf{A}_a \mathbf{A}_b \mathbf{A}_c \mathbf{A}_d\Big] \end{aligned} \tag{3.52}$$

Now we put the remaining pieces together in order to calculate $\mathbf{h}_{\alpha_{IJ}} = U_3 U_2 U_1$. Using the notation

$$y = s^a_J \, \partial_a \tag{3.53}$$

and starting from the basic structure (3.45) we obtain, mutatis mutandis,

$$U_2^{(1)} = (s^a_J - s^a_I)\mathbf{A}_a, \tag{3.54}$$

$$U_2^{(2)} = \frac{1}{2}[(x+y)(s^a_J - s^a_I)\mathbf{A}_a + (s^a_J - s^a_I)(s^b_J - s^b_I)\mathbf{A}_a\mathbf{A}_b], \tag{3.55}$$

$$\begin{aligned} U_2^{(3)} &= \frac{1}{3!}[(x^2 + y^2 + xy)(s^a_J - s^a_I)\mathbf{A}_a + (x + 2y + \bar{y} + 2\bar{x})(s^a_J - s^a_I)(s^b_J - s^b_I)\mathbf{A}_a\bar{\mathbf{A}}_b \\ &\quad + (s^a_J - s^a_I)(s^b_J - s^b_I)(s^c_J - s^c_I)\mathbf{A}_a\mathbf{A}_b\mathbf{A}_c)], \end{aligned} \tag{3.56}$$

$$\begin{aligned} U_2^{(4)} &= \frac{1}{4!}[(x^3 + y^3 + x^2 y + x y^2)(s^a_J - s^a_I)\mathbf{A}_a + (x\bar{y} + 3x\bar{x} + x^2 + 2xy + 2\bar{x}\bar{y} + 3\bar{x}^2 + 3y^2 + \bar{y}^2 + \\ &\quad + 3y\bar{y} + 5\bar{x}y)(s^a_J - s^a_I)(s^b_J - s^b_I)\mathbf{A}_a\bar{\mathbf{A}}_b + (x + 2\bar{x} + 3\bar{\bar{x}} + 2\bar{y} + \bar{\bar{y}} + 3y)(s^a_J - s^a_I) \\ &\quad \times (s^b_J - s^b_I)(s^c_J - s^c_I)\mathbf{A}_a\bar{\mathbf{A}}_b\bar{\bar{\mathbf{A}}}_c + (s^a_J - s^a_I)(s^b_J - s^b_I)(s^c_J - s^c_I)(s^d_J - s^d_I)\mathbf{A}_a\mathbf{A}_b\mathbf{A}_c\mathbf{A}_d] \end{aligned} \tag{3.57}$$

for $U_2$, together with

$$U_3^{(1)} = -s^a_J \mathbf{A}_a, \tag{3.58}$$

$$U_3^{(2)} = \frac{1}{2}(-y s^a_J \mathbf{A}_a + s^a_J s^b_J \mathbf{A}_a \mathbf{A}_b), \tag{3.59}$$

$$U_3^{(3)} = \frac{1}{3!}(-y^2 s^a_J \mathbf{A}_a + (2\bar{y} + y) s^a_J s^b_J \mathbf{A}_a \bar{\mathbf{A}}_b - s^a_J s^b_J s^c_J \mathbf{A}_a \mathbf{A}_b \mathbf{A}_c), \tag{3.60}$$

$$\begin{aligned} U_3^{(4)} &= \frac{1}{4!}\Big[-y^3 s^a_J \mathbf{A}_a + (3\bar{y}^2 + 3y\bar{y} + y^2) s^a_J s^b_J \mathbf{A}_a \bar{\mathbf{A}}_b - (3\bar{\bar{y}} + 2\bar{y} + y) s^a_J s^b_J s^c_J \mathbf{A}_a \bar{\mathbf{A}}_b \bar{\bar{\mathbf{A}}}_c \\ &\qquad + s^a_J s^b_J s^c_J s^d_J \mathbf{A}_a \mathbf{A}_b \mathbf{A}_c \mathbf{A}_d\Big], \end{aligned} \tag{3.61}$$

for $U_3$. Let us emphasize that in all the expressions above for $U_1$, $U_2$ and $U_3$, the connection is evaluated at the vertex $v$. The bars only serve to indicate the position in which the corresponding derivative acts.

Next we write the contributions to the holonomy in powers of the segments. We obtain

$$\mathbf{h}^{(2)}_{\alpha_{IJ}} = \frac{1}{2} s_I^a s_J^b \, \mathbf{F}_{ab}, \tag{3.62}$$

$$\mathbf{h}^{(3)}_{\alpha_{IJ}} = \frac{1}{3!}(s_I^c + s_J^c) s_I^a s_J^b \mathbf{D}_c \mathbf{F}_{ab}, \tag{3.63}$$

$$\mathbf{h}^{(4)}_{\alpha_{IJ}} = \frac{1}{4!}(s_I^c s_I^d + s_I^c s_J^d + s_J^c s_J^d) s_I^a s_J^b \mathbf{D}_c \mathbf{D}_d \mathbf{F}_{ab} + \frac{1}{8} s_I^a s_J^b s_I^c s_J^d \mathbf{F}_{ab} \mathbf{F}_{cd}. \tag{3.64}$$

Eq. (3.64) resolves the ordering ambiguity which apparently arises in covariantizing the first term in the RHS of Eq.(3.35). Nevertheless, as we subsequently show there is no such ambiguity at this order. Let us consider the combination

$$\begin{aligned} s_I^a s_J^b s_I^c s_J^d \left(\mathbf{D}_c \mathbf{D}_d - \mathbf{D}_d \mathbf{D}_c\right) \mathbf{F}_{ab} &= s_I^a s_J^b s_I^c s_J^d \left[\mathbf{D}_c, \mathbf{D}_d\right] \mathbf{F}_{ab} \\ &= -s_I^a s_J^b s_I^c s_J^d \left[\mathbf{F}_{cd}, \mathbf{F}_{ab}\right] = [\mathbf{F}, \mathbf{F}] = 0 \end{aligned} \tag{3.65}$$

where we have used the notation $\mathbf{F} = s_I^a s_J^b \mathbf{F}_{ab}$ together with the property

$$\left[\mathbf{D}_c, \mathbf{D}_d\right] \mathbf{G} = - \left[\mathbf{F}_{cd}, \mathbf{G}\right] \tag{3.66}$$

valid for any object $\mathbf{G}$ in the adjoint representation.

The results (3.62), (3.63) and (3.64), which we have obtained by direct calculation, constitute in fact the unique gauge covariant generalization of the corresponding Abelian expressions (3.85), (4.52) and (3.35). This provides a strong support to our method of calculation.

## 3.5 Electric flux expansion

Now we turn to the electric piece of the regularization. The basic covariant object involved in the expression (3.18) is

$$\mathcal{F} \;=\; \mathbf{h}_{e+\rho(p)}\;\mathbf{E}(p)\;\mathbf{h}^{-1}_{e+\rho(p)} \tag{3.67}$$

In order to simplify the calculation, we will consider the path $\rho(p)$ connecting the vertex $v$ to the point $p$ to be a straight line $x$.

Recalling from the expression (3.45), we can write

$$\mathbf{h}_{v+s_I} = 1 + \Big[I_1(x)\mathbf{A} + I_2(x,\bar{x})\mathbf{A}\bar{\mathbf{A}} + I_3(x,\bar{x},\bar{\bar{x}})\mathbf{A}\bar{\mathbf{A}}\bar{\bar{\mathbf{A}}} + \dots\Big](v) \tag{3.68}$$

where the notation is such that $\mathbf{A} = x^a\mathbf{A}_a$ and the same definitions as before.

Considering the inverse operator $\mathbf{h}^{-1}_{v+x}$, defined as the operator along the inverse path, which amounts the replacement $\mathbf{A} \to -\mathbf{A}$ and the cyclic permutation that takes into consideration a proper ordering of $x'$ segments, it follows

$$\mathbf{h}^{-1}_{v+x} = [1 - I_1(x)\mathbf{A} + I_2(\bar{x},x)\mathbf{A}\bar{\mathbf{A}} - I_3(\bar{\bar{x}},x,\bar{x})\mathbf{A}\bar{\mathbf{A}}\bar{\bar{\mathbf{A}}} + \dots](v) \tag{3.69}$$

Taking into account the fact that $\mathbf{h}_{v+x}\mathbf{h}^{-1}_{v+x} = \mathbb{1}$ and the explicit form of the $I$'s expressions, we get

$$I_2(x,\bar{x}) + I_2(\bar{x},x) = I_1(x)I_1(\bar{x}) \tag{3.70}$$

and

$$I_1(\bar{\bar{x}})I_2(\bar{x},x) - I_1(x)I_2(\bar{x},\bar{\bar{x}}) = I_3(\bar{\bar{x}},\bar{x},x) - I_3(x,\bar{x},\bar{\bar{x}}) \tag{3.71}$$

and the permutations $(x,\bar{x}) \to (\bar{x},x)$. Together with

$$I_3(x,\bar{\bar{x}},\bar{x}) + I_3(\bar{\bar{x}},\bar{x},x) + I_3(\bar{\bar{x}},x,\bar{x}) = I_1(x)I_2(\bar{\bar{x}},\bar{x}) \tag{3.72}$$

and the permutations $(x,\bar{\bar{x}}) \to (\bar{\bar{x}},x)$.

After a tricky manipulation we find the the compact equation with a clear tendency,

$$\begin{aligned}\mathcal{F} &= \mathbf{h}_{v+x}\mathbf{E}(v+x)\mathbf{h}^{-1}_{v+x} = \mathbf{E} - I_1(x)[A,\,\mathbf{E}] + I_2(x,\bar{x})[\,\bar{A},\,[\,A,\,\mathbf{E}]\,] + \\ &\quad -I_3(x,\bar{x},\bar{\bar{x}})\left[\bar{\bar{A}},[\,\bar{A},\,[\,A,\,\mathbf{E}\,]\,]\right] + \ldots \end{aligned} \tag{3.73}$$

Which results to be

$$\mathcal{F} \approx \left\{ \mathbf{E} + D\mathbf{E} + \frac{1}{2!}D^2\mathbf{E} + \frac{1}{3!}D^3\mathbf{E} + \ldots \right\}(v) \tag{3.74}$$

Where again the notation is $D = x^a D_a$, and $D_a$ the covariant derivative

## 3.6 Semiclassical analysis: A general scheme

The corresponding modifications to the Yang Mills theory requires now to resolve the last technical and conceptual problem. For the sake of clarity, let us resume what has been done and to describe the calculations which would take place next.

The Yang Mills Hamiltonian has been regularized, with little deviation from the original work [51]. It is now written in terms of the volume operator and in terms of matter fields operators ready for the semiclassical approximation. The volume operator has arise as a natural regulator of QFT acting exclusively on vertex points of the graph introduced by the state.

The problem we are left in now, is to the search for good semiclassical states that could approximate a flat geometry. A flat space is the geometry we are requiring for our effective theory to live on. There has been extensive deal of attention in the problem of semiclassical loop gravity [69, 70, 71, 72, 73, 74, 75, 76, 77], in particular because the problem is tightly interlaced with one of the open problems in the LQG approach. The problem is how to recover the classical geometry solely from quantum properties of space. The analogy is close of the one faced when trying to reobtain the classical properties of liquids from the quantum molecular theory.

In the calculations of expectations values of regularized Hamiltonian, we will choose semiclassical states picked around a three metric with the properties described in [57]. However, these are not the correct semiclassical states neither the so called weave states, since they not fulfill peakedness properties around the connection, which relies on the more fundamental problem described above.

Taking into account the expression for the regularized Hamiltonian, let us separate the operator in gravitational and matter components as,

$$\hat{H}[N] = \sum_{v,\,\Delta,\,\Delta'} \epsilon^{JKL}\epsilon^{MNP}\hat{w}_{i\,L\,\Delta}(e)\hat{w}_{i\,P\,\Delta}(e')\hat{H}^{\underline{I}}_{\Delta JK}(e)\hat{H}^{\underline{I}}_{\Delta' MN}(e') \tag{3.75}$$

where

$$\hat{w}_{i\,L\,\Delta}(e) = \operatorname{tr}\left(\tau_i\, h_{e_L(\Delta)}\left[h^{-1}_{e_L(\Delta)}, \sqrt{\hat{V}_v}\right]\right) \tag{3.76}$$

The magnetic contribution have the structure

$$\begin{aligned}\hat{H}^{\underline{I}}_{\Delta JK}(e) &= \epsilon_{abc}e^a_J e^b_K \times \left[\frac{1}{2!}\hat{B}^c_{\underline{I}} + \frac{1}{3!}(s^d_J + s^d_K)D_d\hat{B}^c_{\underline{I}} + \frac{1}{4!}(s^f_J s^d_J + s^f_J s^d_K + s^f_K s^d_K)D_f D_d \hat{B}^c_{\underline{I}}\right.\\ &\left. + \frac{1}{8}s^f_J s^p_K \epsilon^{fpd}\epsilon^{\underline{I}\,\underline{L}\,\underline{M}}\hat{B}^d_{\underline{L}}\hat{B}^c_{\underline{M}} + \dots\right](v)\end{aligned} \tag{3.77}$$

The electric part is analogous but considering only at linear order in the field, also with modifications in some coefficients calculated in the appendix A.

To proceed with the approximation we think of space as made up of boxes, each centered at a given point $\vec{x}$ and with volume $\mathcal{L}^3 \approx d^3\,x$. Each box is defined to contain a large number of vertices of the semiclassical state ($\mathcal{L} >> \ell_P$), but infinitesimal in the scale where the space can be regarded as continuous. Also, we assume that the magnetic operators are slowly varying inside the box ($\ell_P << \lambda_D$), in such a way that for all the vertices inside the box one can write

$$\langle W,\, \underline{\vec{E}},\, \underline{\vec{B}}| \dots \hat{B}^a_{\underline{I}}(v) \dots |W,\, \underline{\vec{E}},\, \underline{\vec{B}}\rangle = \mu B^a_{\underline{I}}(\vec{x}). \tag{3.78}$$

Here $B$ is the classical magnetic field at the center of the box and $\mu$ is a dimensionless constant to be determined in such a way that we recover the standard classical result (3.19) in the zeroth order approximation. In the next subsection we show that

$$\mu = \left(\frac{\mathcal{L}}{\ell_P}\right)^{1+\Upsilon}, \tag{3.79}$$

with $\Upsilon$ being a parameter defining the leading order contribution of the gravitational connection to the expectation value. Applying to (3.75) the procedure just described leads to

$$
\begin{aligned}
&\mathrm{H} = \sum_{\mathrm{Box}} N(\vec{x}) \Bigg[ B^{I}_{\underline{a_1}}(\vec{x}) B^{I}_{\underline{b_1}}(\vec{x}) \sum_{v\in\mathrm{Box}} \epsilon^{JKL}\epsilon^{MNP} < W,\, \underline{\vec{E}},\, \underline{\vec{B}}|\, \hat{Q}_{\Delta L}(e)\hat{Q}_{\Delta' P}(e')\times \\
&\times \epsilon^{a_1 a_2 a_3} \times \epsilon^{b_1 b_2 b_3} \times e^{J}_{a_2} e^{K}_{a_3} e'^{M}_{b_2} e'^{N}_{b_3} |W,\, \underline{\vec{E}},\, \underline{\vec{B}} > + B^{I}_{\underline{a_1}}(\vec{x}) D_p B^{I}_{\underline{b_1}}(\vec{x}) \sum_{v\in\mathrm{Box}} \epsilon^{JKL} \times \epsilon^{MNP} \\
&\times < W,\, \underline{\vec{E}},\, \underline{\vec{B}}|\, \hat{Q}_{\Delta L}(e)\hat{Q}_{\Delta' P}(e') \epsilon^{a_1 a_2 a_3} \times \epsilon^{b_1 b_2 b_3} e^{J}_{a_1} e^{K}_{a_2} e'^{M}_{b_2} e'^{N}_{b_3} (e'^{M}_{p} + e'^{N}_{p}) |W,\, \underline{\vec{E}},\, \underline{\vec{B}} > + \\
&D_q B^{I}_{\underline{a_1}}(\vec{x}) D_p B^{I}_{\underline{b_1}}(\vec{x}) \sum_{v\in\mathrm{Box}} \epsilon^{JKL}\epsilon^{MNP} \times \\
&\times < W,\, \underline{\vec{E}},\, \underline{\vec{B}}|\, \hat{Q}_{\Delta L}(e)\hat{Q}_{\Delta' P}(e') e^{J}_{a_2} e^{K}_{a_3} (e^{J}_{q} + e^{K}_{q}) e'^{M}_{b_2} e'^{N}_{b_3} (e'^{M}_{p} + e'^{N}_{p}) |W,\, \underline{\vec{E}},\, \underline{\vec{B}} > \\
&+ \ldots ]
\end{aligned}
\tag{3.80}
$$

The general structure is

$$
\begin{aligned}
\mathrm{H}^{B} &= \sum_{\mathrm{Box}} N(\vec{x}) B^{I}_{\underline{r_1}}(\vec{x}) \ldots B^{I}_{\underline{r_n}}(\vec{x}) \left( D^{a_1} \ldots D^{a_m} B^{I}_{\underline{r}}(\vec{x}) \right) \; d^3x \; R_{a_1 \ldots a_m}{}^{r r_1 \ldots r_n}(\vec{x}) \\
\mathrm{H}^{B} &= \int d^3x \; N(\vec{x}) B^{I}_{\underline{r_1}}(\vec{x}) \ldots \; B^{I}_{\underline{r_n}}(\vec{x}) \; \left( D^{a_1} \ldots D^{a_m} \; B^{I}{}_{\underline{r}}(\vec{x}) \right) \; R_{a_1 \ldots a_m}{}^{r r_1 \ldots r_n}(\vec{x})
\end{aligned}
\tag{3.81}
$$

R is constructed from flat space tensors like $\delta_{ab}$, $\epsilon_{abc}$. In this way we are demanding covariance under rotations at the scale $\mathcal{L}$.

When averaging inside each box, the scaling of the expectation values of the gravitational operators is estimated according to

$$
\langle W,\, \underline{\vec{E}},\, \underline{\vec{B}}\,| \ldots A_{ia} \ldots |W,\, \underline{\vec{E}},\, \underline{\vec{B}}\,\rangle \approx \ldots \frac{1}{\mathcal{L}} \left( \frac{\ell_P}{\mathcal{L}} \right)^{\Upsilon} \ldots, \tag{3.82}
$$

$$
\langle W,\, \underline{\vec{E}},\, \underline{\vec{B}}\,| \ldots \sqrt{V_v} \ldots |W,\, \underline{\vec{E}},\, \underline{\vec{B}}\,\rangle \approx \ldots \ell_P^{3/2} \ldots, \tag{3.83}
$$

## 3.7 Yang Mills effective Hamiltonian

In constructing the effective theory we will put all the elements of the previous sections altogether, and note that the following calculation of elements $R$ has been already calculated in [57].

Let us now continue with the calculation of the contribution to (3.17) due to the magnetic flux by writing the expansion

$$\begin{aligned} \mathrm{tr}(\tau_{\underline{I}}\underline{\mathbf{h}}_{\alpha_{JK}}) &= \mathrm{tr}\left(\tau_{\underline{I}}\sum_{n=2}^{\infty}\underline{\mathbf{h}}^{(n)}{}_{\alpha_{JK}}\right) = M^{\underline{I}}_{2JK(\Delta)} + M^{\underline{I}}_{3JK(\Delta)} + M^{\underline{I}}_{4JK(\Delta)} \\ &\quad +\mathcal{O}\left(s^5\underline{F}^4\right) \end{aligned} \tag{3.84}$$

where

$$M^{\underline{I}}_{2IJ(\Delta)} := s^a_I s^b_J \frac{i}{2!} F^{\underline{I}}_{ab} \tag{3.85}$$

$$M^{\underline{I}}_{3IJ(\Delta)} := s^a_I s^b_J \frac{i}{3!}(x_I + x_J) F^{\underline{I}}_{ab} - s^a_I s^b_J s^c_I s^d_J \epsilon^{\underline{I}\,\underline{L}\,\underline{M}} \frac{1}{8} F^{\underline{L}}_{ab} F^{\underline{M}}_{cd} \tag{3.86}$$

$$M^{\underline{I}}_{4JK(\Delta)} := s^a_K s^b_J \frac{i}{4!}(x_J^2 + x_J x_K + x_K^2) F^{\underline{I}}_{ab} \tag{3.87}$$

and $B^{\underline{I}}_c = \epsilon^{abc} F^{\underline{I}}_{ab}$

according to the previous analysis. We are using the notation $x_I = \vec{s}_I \cdot \vec{D} = s^a_I\, D_a$. Let us remark that, contrary to the electric case, the magnetic contribution will incorporate non-linear terms due to the expansion of the exponential in powers of $\vec{B}$. This implies that the exact duality symmetry of Maxwell equations in vacuum will be lost due to quantum gravity corrections.

Next let us consider the gravitational contributions to (3.18), arising from the gravitational part of (3.76), which we expand as

$$\hat{w}_{i\,L\Delta} = s^a_L w_{ia} + s^a_L s^b_L w_{iab} + s^a_L s^b_L s^c_L w_{iabc} + \mathcal{O}(s^4 w), \tag{3.88}$$

with

$$w_{ia} = \frac{1}{2}[A_{ia}, \sqrt{V}], \quad w_{iab} = \frac{1}{8}\epsilon_{ijk}[A_{ja}, [A_{kb}, \sqrt{V}]], \quad w_{iabc} = -\frac{1}{48}[A_{ja}, [A_{jb}, [A_{ic}, \sqrt{V}]]] \tag{3.89}$$

The scaling properties of the above gravitational operators under the semiclassical expectation value are

$$\langle W\,\underline{\vec{E}}\,\underline{\vec{B}}|\ldots w_{i\,a_1\ldots a_n}\ldots|W\,\underline{\vec{E}}\,\underline{\vec{B}}\rangle \to \frac{{\ell_P}^{3/2}}{\mathcal{L}^n}\left(\frac{\ell_P}{\mathcal{L}}\right)^{n\Upsilon}. \tag{3.90}$$

For the product $\hat{w}_{i\,L\Delta}\hat{w}_{i\,P\Delta'}$ we need only

$$\hat{w}_{i\,L\Delta}\hat{w}_{i\,P\Delta'} \;=\; U_{2LP}+U_{3LP}+U_{4LP}+\mathcal{O}\left(s^5 w^2\right) \tag{3.91}$$

with

$$\begin{aligned} U_{2LP} &= s_L^a s_P'^d w_{ia}w_{id}, \\ U_{3LP} &= s_L^a s_P'^d s_P'^e w_{ia}w_{ide} + s_L^a s_L^b s_P'^d w_{iab}w_{id}, \\ U_{4LP} &= s_L^a s_P'^d s_P'^e s_P'^f w_{ia}w_{idef} + s_L^a s_L^b s_P'^d s_P'^e w_{iab}w_{ide} + s_L^a s_L^b s_L^c s_P'^d w_{iabc}w_{id}. \end{aligned} \tag{3.92}$$

Here all the $w$'s are evaluated at a common vertex $v$.

At this level it is convenient to state the result (no sum over $L$)

$$\begin{aligned} s_L^a\,s_L^b\,w_{iab} &= \frac{1}{8}\,s_L^a\,s_L^b\,\epsilon_{ijk}\,[A_{ja},\,[A_{kb},\,\sqrt{V}]] \\ &= \frac{1}{8}\,s_L^a\,s_L^b\,\epsilon_{ijk}\,\left(A_{ja}A_{kb}\sqrt{V}-A_{ja}\sqrt{V}A_{kb}-A_{kb}\sqrt{V}A_{ja}+\sqrt{V}A_{kb}A_{ja}\right), \\ &= 0, \end{aligned} \tag{3.93}$$

which holds due to symmetry properties. This leads to

$$U_{3LP}=0. \tag{3.94}$$

After taking the expectation value the terms contributing to order $\ell_P^2$

$$T \;=\; T_0+T_1+T_2+\mathcal{O}(\to \ell_P^3) \tag{3.95}$$

$$T_0 \;=\; -\frac{1}{2\,Q^2}\frac{1}{\ell_P^4}\,\epsilon^{JKL}\,\epsilon^{MNP}\left[U_{2LP}\,M_{1JK}^I\,M'^I_{1MN}\right], \tag{3.96}$$

$$T_1 \;=\; -\frac{1}{2\,Q^2}\frac{1}{\ell_P^4}\,\epsilon^{JKL}\,\epsilon^{MNP}U_{2LP}\left[\,M_{1JK}^I\,M'^I_{2MN}+M_{2JK}^I\,M'^I_{1MN}\right], \tag{3.97}$$

$$\begin{aligned} T_2 \;=\; -\frac{1}{2\,Q^2}\frac{1}{\ell_P^4}\,\epsilon^{JKL}\,\epsilon^{MNP}\Big[\,&U_{2LP}\,(M_{1JK}^I\,M'^I_{3MN}+M_{3JK}^I\,M'^I_{1MN}+M_{2JK}\,M'_{2MN})+ \\ &+U_{4LP}\,M_{1JK}^I\,M'^I_{1MN}\Big]. \end{aligned} \tag{3.98}$$

Now we are ready to calculate the different contributions to the magnetic sector of the Hamiltonian (3.18). The task is to calculate the $R$ pieces which are symmetric in the indices $r_1, r_2, \ldots, r_n$. The first contribution to $T_o$ is

$$\begin{aligned} R_0{}^{r_1 r_2} &= \sum_{v \in \text{Box}} \frac{1}{2Q^2} \left( \frac{2}{3!} \frac{8}{E(v)} \right)^2 \sum_{v(\Delta)=v(\Delta')=v} \frac{\mu^2}{4\,\ell_P^7}\, \epsilon^{r_1}{}_{ab}\, \epsilon^{r_2}{}_{uv}\, \epsilon^{JKL}\, \epsilon^{MNP} \times \\ &\times\, s^a_K\, s^b_J\, s^c_L\, s'^v_M\, s'^u_N\, s'^d_P\, \langle W\, \underline{\vec{E}}\, \underline{\vec{B}} | w_{ic}\, w_{id}\, | W\, \underline{\vec{E}}\, \underline{\vec{B}} \rangle. \end{aligned} \tag{3.99}$$

And with the relations

$$\epsilon^{KJL}\, s^a_K\, s^b_J\, s^c_L = det(s)\, \epsilon^{abc}, \quad det(s) = det(s^a_K), \qquad \epsilon^{abp}\epsilon_{abq} = 2\delta^p_q. \tag{3.100}$$

(3.99) can be rewriten in the simpler form

$$\begin{aligned} R_0{}^{r_1 r_2} &= \sum_{v \in \text{Box}} \frac{1}{2\,Q^2} \left( \frac{2}{3!} \frac{8}{E(v)} \right)^2 \sum_{v(\Delta)=v(\Delta')=v} \frac{\mu^2}{\ell_P^7} \times \\ &\times\, det(s)\ det(s')\ \langle W,\, \underline{\vec{E}},\, \underline{\vec{B}} | \frac{1}{2} \{ w_i{}^{r_1},\, w_i{}^{r_2} \}\, | W,\, \underline{\vec{E}},\, \underline{\vec{B}} \rangle. \end{aligned} \tag{3.101}$$

The above equation implies

$$R_0{}^{r_1 r_2} = \frac{1}{2\,Q^2} \frac{\mu^2}{\ell_P^7}\, \ell_P^6\, \frac{\ell_P^3}{\mathcal{L}^2} \left( \frac{\ell_P}{\mathcal{L}} \right)^{2\Upsilon} \delta^{r_1 r_2} = \frac{1}{2\,Q^2} \delta^{r_1 r_2}, \tag{3.102}$$

which reproduces the zeroth-order magnetic contribution (3.19) with the choice

$$\mu = \left( \frac{\mathcal{L}}{\ell_P} \right)^{1+\Upsilon}. \tag{3.103}$$

The contribution to the $T's$ have been calculated in the previous work [57]. Let us just write them

The correction arising from $T_1$ is

$$R_{11}{}^{a_1 r r_1} = \kappa_8\, \frac{\mu^2}{Q^2\, \ell_P^7}\, \ell_P^7\, \frac{\ell_P^3}{\mathcal{L}^2} \left( \frac{\ell_P}{\mathcal{L}} \right)^{2\Upsilon} \epsilon^{a_1 r r_1}, \tag{3.104}$$

which produces a parity-violating term in the magnetic sector of the effective Hamiltonian.

The next contribution arises from $T_2$ and can be separated into three pieces

$$\begin{aligned}
\mathrm{H}^B_{21} &= \sum_{v\in V(\gamma)} \frac{1}{2\,Q^2}\left(\frac{2}{3!}\frac{8}{E(v)}\right)^2 \ell_P^3 \sum_{v(\Delta)=v(\Delta')=v} -\frac{1}{\ell_P^7}\,\epsilon^{JKL}\,\epsilon^{MNP}\times \\
&\quad \times\langle W,\,\underline{\vec{E}},\,\underline{\vec{B}}|\,U_{2LP}\,(M^{I}_{1JK}\,M'^{I}_{3MN}+M^{I}_{3JK}\,M'^{I}_{1MN})\,|W,\,\underline{\vec{E}},\,\underline{\vec{B}}\rangle, \qquad (3.105)
\end{aligned}$$

$$\begin{aligned}
\mathrm{H}^B_{22} &= \sum_{v\in V(\gamma)} \frac{1}{2\,Q^2}\left(\frac{2}{3!}\frac{8}{E(v)}\right)^2 \ell_P^3 \sum_{v(\Delta)=v(\Delta')=v} -\frac{1}{\ell_P^7}\,\epsilon^{JKL}\,\epsilon^{MNP}\times \\
&\quad \times\langle W,\,\underline{\vec{E}}\,\underline{\vec{B}}|\,U_{2LP}\,M^{I}_{2JK}\,M'^{I}_{2MN}\,|W,\,\underline{\vec{E}},\,\underline{\vec{B}}\rangle, \qquad (3.106)
\end{aligned}$$

$$\begin{aligned}
\mathrm{H}^B_{23} &= \sum_{v\in V(\gamma)} \frac{1}{2\,Q^2}\left(\frac{2}{3!}\frac{8}{E(v)}\right)^2 \ell_P^3 \sum_{v(\Delta)=v(\Delta')=v} -\frac{1}{\ell_P^7}\,\epsilon^{JKL}\,\epsilon^{MNP}\times \\
&\quad \times\langle W,\,\underline{\vec{E}},\,\underline{\vec{B}}|\,U_{4LP}\,M^{I}_{1JK}\,M'^{I}_{1MN}\,|W,\,\underline{\vec{E}},\,\underline{\vec{B}}\rangle. \qquad (3.107)
\end{aligned}$$

Let us start discussing $\mathrm{H}^B_{21}$. After some algebra we obtain

$$\begin{aligned}
\mathrm{H}^B_{21} &= \sum_{v\in V(\gamma)} \frac{1}{2\,Q^2}\left(\frac{2}{3!}\frac{8}{E(v)}\right)^2 \ell_P^3 \sum_{v(\Delta)=v(\Delta')=v} -\frac{i}{\ell_P^7}\,\epsilon^{JKL}\,\epsilon^{MNP}\times \\
&\quad \times\; s^a_L s'^d_P\, s'^x_N s'^q_M\,\langle W,\,\underline{\vec{E}},\,\underline{\vec{B}}|\,w_{ia}w_{id}\left(s^r_K s^t_J\frac{i}{4!}(x_J^2+x_Jx_K+x_K^2)F^I_{rt}\right)F^I_{xq}\,|W,\,\underline{\vec{E}},\,\underline{\vec{B}}\rangle, \\
&\qquad (3.108)
\end{aligned}$$

which naturally splits into the following pieces

$$\begin{aligned}
\mathrm{H}^B_{211} &= \sum_{v\in V(\gamma)} \frac{1}{2\,Q^2}\left(\frac{2}{3!}\frac{8}{E(v)}\right)^2 \ell_P^3 \sum_{v(\Delta)=v(\Delta')=v} -\frac{i}{\ell_P^7}\,\epsilon^{JKL}\,\epsilon^{MNP}\times \\
&\quad \times\, s^a_L s'^d_P\, s'^x_N s'^q_M\,\langle W,\,\underline{\vec{E}},\,\underline{\vec{B}}|\,w_{ia}w_{id}\left(s^r_K s^t_J\frac{i}{4!}(x_J^2+x_Jx_K+x_K^2)F^I_{rt}\right)F^I_{xq}\,|W,\,\underline{\vec{E}},\,\underline{\vec{B}}\rangle, \\
&\qquad (3.109)
\end{aligned}$$

The corresponding $R$ tensors are

$$R_{211}{}^{a_1 a_2 r r_1} = \sum_{v\in\text{Box}} \frac{1}{2}\left(\frac{2}{3!}\frac{8}{E(v)}\right)^2 \sum_{v(\Delta)=v(\Delta')=v} \frac{2\,\mu^2}{4!\,Q^2\ell_P^7}\ det(s')det(s)$$
$$\epsilon_{NKJ}\left(s^{-1}\right)^{rN}\ \epsilon^{JKL}\ s_L^a\ (2s_J^{a_1}\ s_J^{a_2} + s_J^{a_1}\ s_K^{a_2})\,\langle W,\,\underline{\vec{E}},\,\underline{\vec{B}}|\,w_{ia}w_i{}^{r_1}|W,\,\underline{\vec{E}},\,\underline{\vec{B}}\rangle,$$
$$R_{211}{}^{a_1 a_2 r r_1} = \frac{\mu^2}{Q^2\ell_P^7}\,\ell_P^8\,\frac{\ell_P^3}{\mathcal{L}^2}\left(\frac{\ell_P}{\mathcal{L}}\right)^{2\Upsilon}\left(\kappa_6\,\delta^{a_1 a_2}\,\delta^{r r_1} + \kappa_7\,(\delta^{a_1 r}\,\delta^{a_2 r_1} + \delta^{a_2 r}\,\delta^{a_1 r_1})\right)\!. \quad (3.110)$$

Now we continue with the correction arising from (3.106), which reduces to

$$\text{H}_{22}^B = \sum_{v\in\text{Box}} \frac{1}{2\,Q^2}\left(\frac{2}{3!}\frac{8}{E(v)}\right)^2 \ell_P^3 \sum_{v(\Delta)=v(\Delta')=v} \frac{4}{(3!)^2\,\ell_P^7}\,\epsilon^{JKL}\,\epsilon^{MNP}\times$$
$$\times s_L^a s_P'^d\,\langle W,\,\underline{\vec{E}},\,\underline{\vec{B}}|\,w_{ia}w_{id}\,\left(s_K^u s_J^y\,x_J \underline{F_{uy}^I}\right)\,\left(s_N'^r s_M'^s\,x'_M \underline{F_{rs}^I}\right)\,|W,\,\underline{\vec{E}},\,\underline{\vec{B}}\rangle, \quad (3.111)$$

Then

$$R_{22}{}^{a_1 a_2 r r_1} = \sum_{v\in\text{Box}} \frac{1}{2\,Q^2}\left(\frac{2}{3!}\frac{8}{E(v)}\right)^2 \sum_{v(\Delta)=v(\Delta')=v} \frac{4\,\mu^2}{(3!)^2\,\ell_P^7}\ det(s)det(s')\times$$
$$\times\left(-\epsilon_{YMN}\epsilon^{PMN}s'^d_P\ s'^{a_2}_M\right)\ \left(\epsilon_{XJK}\epsilon^{LJK}\ s_L^a\ \ s_J^{a_1}\right)\ \left(s^{-1}\right)^{Xr_1}\ \left(s'^{-1}\right)^{Yr}$$
$$\times\langle W,\,\underline{\vec{E}},\,\underline{\vec{B}}|\,w_{ia}w_{id}\,|W,\,\underline{\vec{E}},\,\underline{\vec{B}}\rangle. \quad (3.112)$$

where we have used the relation

$$\epsilon_{mbc}\ s_J^b\,s_L^c = det(s)\,\epsilon_{KJL}\left(s^{-1}\right)_m^K, \quad (3.113)$$

we find

$$R_{22}{}^{a_1 a_2 r r_1} = \frac{\mu^2}{Q^2\ell_P^7}\,\ell_P^8\,\frac{\ell_P^3}{\mathcal{L}^2}\left(\frac{\ell_P}{\mathcal{L}}\right)^{2\Upsilon}\left(\kappa_9\,\delta^{a_1 a_2}\,\delta^{r r_1} + \kappa_{10}\,(\delta^{a_1 r}\,\delta^{a_2 r_1} + \delta^{a_2 r}\,\delta^{a_1 r_1})\right). \quad (3.114)$$

This contribution is of the same kind as the one given by $R_{211}{}^{a_1 a_2 r r_1}$.
Finally we are left with

$$\begin{aligned}
\mathrm{H}^B_{23} &= \sum_{\mathrm{Box}(\vec{x})} \underline{B}_{r_1}(\vec{x})\,\underline{B}_r(\vec{x}) \sum_{v\in\mathrm{Box}(\vec{x})} \frac{1}{2\,Q^2}\left(\frac{2}{3!}\frac{8}{E(v)}\right)^2 \ell_P^3 \sum_{v(\Delta)=v(\Delta')=v} \frac{\mu^2}{4\,\ell_P^7} \times \\
&\times \epsilon^{JKL}\,\epsilon^{MNP}\, s^u_K s^v_J\, s'^x_N s'^y_M \left(s^d_L s'^a_P s'^b_P s'^c_P\right) \epsilon_{uv}{}^{r_1}\,\epsilon_{xy}{}^{z}\langle W,\,\underline{\vec{E}},\,\underline{\vec{B}}|\,\{w_{id},\,w_{iabc}\}\,|W,\,\underline{\vec{E}},\,\underline{\vec{B}}\rangle,
\end{aligned} \tag{3.115}$$

which leads to

$$\begin{aligned}
R_{23}{}^{rr_1} &= \sum_{v\in\mathrm{Box}(\vec{x})} \frac{1}{2\,Q^2}\left(\frac{2}{3!}\frac{8}{E(v)}\right)^2 \sum_{v(\Delta)=v(\Delta')=v} -\frac{\mu^2}{\ell_P^7}\,\det(s')\,det(s) \\
&\times\ \left(s'^{-1}\right)^{rP}\left(s'^a_{\ P}\, s'^b_{\ P}\, s'^c_{\ P}\right)\,\langle W,\,\underline{\vec{E}},\,\underline{\vec{B}}|\,\{w_i^{r_1},\,w_{iabc}\}\,|W,\,\underline{\vec{E}},\,\underline{\vec{B}}\rangle.
\end{aligned} \tag{3.116}$$

Taking the symmetric part, we have

$$R_{23}{}^{rr_1} = \kappa_{11}\,\frac{\mu^2}{Q^2\ell_P^7}\,\ell_P^8\,\frac{\ell_P^3}{\mathcal{L}^4}\left(\frac{\ell_P}{\mathcal{L}}\right)^{4\Upsilon}\delta^{r\,r_1}. \tag{3.117}$$

Adding all previous contributions, we obtain the magnetic sector of the effective Hamiltonian, up to order $\ell_P^2$,

$$\begin{aligned}
\mathrm{H}^B &= \frac{1}{Q^2}\mathrm{tr}\int d^3(\vec{x})\left[\left(1+\theta_7\left(\frac{\ell_P}{\mathcal{L}}\right)^{2+2\Upsilon}\right)\frac{1}{2}\vec{B}^2 + \ell_P^2\left(\theta_2\,B^a D_a D_b B^b + \theta_3\,B^a\,D^2 B^a\right) + \right. \\
&\left. +\theta_8\ell_P\vec{B}\cdot(D\times\vec{B}) + \dots\right].
\end{aligned} \tag{3.118}$$

The numbers $\theta_i$ are linear combinations of the corresponding $\kappa_j$ appearing in the tensors $R$. The correspondences are

$$\kappa_7,\kappa_{10}\to\theta_2,\quad \kappa_6,\kappa_9\to\theta_3,\theta_4,\quad \kappa_{11}\to\theta_7,\quad ,\kappa_8\to\theta_8. \tag{3.119}$$

As pointed out and showed in the appendix (3.18), the electric sector of the effective Hamiltonian can be obtained by changing $\vec{B}$ into $\vec{E}$ and keeping up to quadratic contribution of the corresponding expression for the magnetic sector. In this way the complete electromagnetic effective Hamiltonian becomes up to order $\ell_P^2$, $B^2$, and with $(\vec{D} \times \vec{B}_{\underline{I}})^a = \epsilon^{abc} D_b B_c^{\underline{I}}$ and $D^2 = D_a D_a$

$$
\begin{aligned}
\mathrm{H}^{YM} &= \frac{1}{Q^2} \mathrm{Tr} \int d^3\vec{x} \left[ \left( 1 + \theta_7 \left( \frac{\ell_P}{\mathcal{L}} \right)^{2+2\Upsilon} \right) \frac{1}{2} \left( \underline{\vec{B}}^2 + \underline{\vec{E}}^2 \right) + \theta_3\, \ell_P^2 \left( \underline{\vec{B}}\, D^2 \underline{\vec{B}} + \underline{\vec{E}}\, D^2 \underline{\vec{E}} \right) + \right. \\
&\quad + \theta_2\, \ell_P^2 \left( \underline{E}^a D_a D_b \underline{E}^b + \frac{1}{2} \vec{E} \cdot (\vec{B} \times \vec{E}) + \underline{B}^a D_a D_b \underline{B}^b \right) + \\
&\quad \left. \theta_8 \ell_P \left( \underline{\vec{B}} \cdot (\vec{D} \times \underline{\vec{B}}) + \underline{\vec{E}} \cdot (\vec{D} \times \underline{\vec{E}}) \right) + \ldots \right] \qquad (3.120)
\end{aligned}
$$

# Chapter 4

# Higgs effective model

In the following chapter the goal we pursue is to obtain the corresponding modifications to the Higgs field, which will be carried using the same techniques developed in the Yang Mills case. The Higgs field remains the last unexplored sector of the SM [56, 57].

In the derivation of the Higgs effective theory we will shift from the usual variables to new ones, which are the smeared in space momentum and the adimensional point holonomy variable. The main reason in doing so, is that the new variables allows to implement correctly the adjointness relations, which are not trivially implemented for diffeomorphism invariant field theories (a complete discussion can be found in [85]

## 4.1 Higgs regularization

The Higgs regularization will be carried in the same terms that led to the Yang Mills effective theory.

The total Higgs Hamiltonian in curved space can be written as

$$H_{Higgs}(N) = \frac{1}{2\kappa}\int_\Sigma d^3xN(x)\left(\frac{p^Ip^I}{\sqrt{\det(q)}} + \sqrt{\det(q)}\left[q^{ab}\mathcal{D}_a\phi_I\mathcal{D}_b\phi_I + \frac{P(\phi\phi)}{\hbar\kappa}\right]\right) \quad (4.1)$$

It will be convenient to separate each contribution and analize them independently. From the Hamiltonian it can be distinguish the usual kinematical, derivative and potential pieces

$$H_{kin}(N) = \frac{1}{2\kappa}\int_\Sigma d^3xN(x)\frac{p^Ip^I}{\sqrt{\det(q)}} \quad (4.2)$$

$$H_{der}(N) = \frac{1}{2\kappa}\int_\Sigma d^3xN(x)\sqrt{\det(q)}q^{ab}\mathcal{D}_a\phi_I\mathcal{D}_b\phi_I \quad (4.3)$$

$$H_{pot}(N) = \frac{1}{2\hbar\kappa^2}\int_\Sigma d^3xN(x)P(\phi\phi) \quad (4.4)$$

Let us consider the following regulated *four-fold* point-splitting of the kinematical term and the techniques developed in the previous chapter

$$\begin{aligned}
&H^\epsilon_{kin}(N)\\
= \; & \frac{1}{2\kappa}\lim_{\epsilon\to 0}\int_\Sigma d^3xN(x)p^I(x)\int_\Sigma d^3y\ \chi_\epsilon(x,y)p^I(y)\int_\Sigma d^3u\chi_\epsilon(u,x)(\frac{\det(e^i_a)}{[\sqrt{V(u,\epsilon)}]^3})(u)\\
& \int_\Sigma d^3v\chi_\epsilon(v,y)(\frac{\det(e^i_a)}{[\sqrt{V(v,\epsilon)}]^3})(v)\\
= \; & \frac{1}{2\kappa}\frac{1}{3^2\kappa^6}\lim_{\epsilon\to 0}\int_\Sigma d^3xN(x)p^I(x)\int_\Sigma d^3y\chi_\epsilon(x,y)p^I(y)\times\\
\times \; & \int_\Sigma \chi_\epsilon(u,x)\mathrm{tr}(\{A(u),\sqrt{V(u,\epsilon)}\}\wedge\{A(u),\sqrt{V(u,\epsilon)}\}\wedge\{A(u),\sqrt{V(u,\epsilon)}\})\times\\
\times \; & \int_\Sigma \chi_\epsilon(v,y)\mathrm{tr}(\{A(v),\sqrt{V(v,\epsilon)}\}\wedge\{A(v),\sqrt{V(v,\epsilon)}\}\wedge\{A(v),\sqrt{V(v,\epsilon)}\})
\end{aligned} \quad (4.5)$$

Recall that $\int d^3x \det(e^i_a) = \frac{1}{3!}\int \epsilon_{ijk} e^i \wedge e^j \wedge e^k = -\frac{1}{3}\int \mathrm{tr}(e \wedge e \wedge e)$.
Let us define

$$\begin{aligned}
Q[\chi] &:= \int_\Sigma \chi_\epsilon(u,x)\mathrm{tr}\left(\{A(u),\sqrt{V(u,\epsilon)}\} \wedge \{A(u),\sqrt{V(u,\epsilon)}\} \wedge \{A(u),\sqrt{V(u,\epsilon)}\}\right) \\
&= \sum_\Delta \int_\Delta \chi_\epsilon(u,x)\mathrm{tr}\left(\{A(u),\sqrt{V(u,\epsilon)}\} \wedge \{A(u),\sqrt{V(u,\epsilon)}\} \wedge \{A(u),\sqrt{V(u,\epsilon)}\}\right) \\
Q[\chi] &= \sum_\Delta Q_\Delta[\chi]. \qquad (4.6)
\end{aligned}$$

Where we have defined

$$Q_\Delta[\chi] = \int_\Delta \chi_\epsilon(u,x)\mathrm{tr}\left(\{A(u),\sqrt{V(u,\epsilon)}\} \wedge \{A(u),\sqrt{V(u,\epsilon)}\} \wedge \{A(u),\sqrt{V(u,\epsilon)}\}\right) \quad (4.7)$$

And let us define the gauge invariant object $p_I(B_e)$ by,

$$p_I(B_e) := tr\left[\tau_I \underline{h}_e \left(\int_{B_e} \underline{h}_{\rho(r)} p(r) \underline{h}^{-1}_{\rho(r)} d^3x(r)\right) \underline{h}_e^{-1}\right] \quad (4.8)$$

Where $\underline{h}_{\rho(r)}$ is the parallel propagator from the vertex $v$ to the point $r$ in the region $B_e$

Now consider the three vectors $s^a_J(\Delta)s^b_K(\Delta)s^c_L(\Delta)$ that span the tetrahedra $\Delta$ with base in $e = v$, we have

$$\begin{aligned}
p^I(B_v) &= \frac{1}{3!}\epsilon^{JKL}\epsilon^{abc}s^a_J s^b_K s^c_L p^I(v) + \\
&+\frac{1}{4!}\epsilon^{JKL}\epsilon^{abc}s^a_J s^b_K s^c_L (s^a_J + 2s^a_I + s^a_K) D_a p(v) + O(s^5) \qquad (4.9)
\end{aligned}$$

Therefore for a small tetrahedra $\Delta$

$$f(v)p^I(B_v) = \frac{1}{3!}f(v)\epsilon^{JKL}\epsilon^{abc}s^a_J s^b_K s^c_L p^I(v) + D_a p(v) = \int_{B_v} d^3x f(x) p^I(x) \quad (4.10)$$

And considering

$$\begin{aligned}
&\int_\Delta \mathrm{tr}(\{A(x), \sqrt{\hat{V}(x,\epsilon)}\} \wedge \{A(x), \sqrt{\hat{V}(x,\epsilon)}\} \wedge \{A(x), \sqrt{\hat{V}(x,\epsilon)}\}) \\
\approx\ & \frac{1}{3!}\epsilon^{IJK}\mathrm{tr}(h_{s_I(\Delta)}\{h^{-1}_{s_I(\Delta)}, \sqrt{\hat{V}(v(\Delta),\epsilon)}\}\mathrm{tr}(h_{s_J(\Delta)}\{h^{-1}_{s_J(\Delta)}, \sqrt{\hat{V}(v(\Delta),\epsilon)}\} \times \\
\times\ & \mathrm{tr}(h_{s_K(\Delta)}\{h^{-1}_{s_K(\Delta)}, \sqrt{\hat{V}(v(\Delta),\epsilon)}\}) \qquad (4.11)
\end{aligned}$$

We have then for the familiar triangulation,

$$\begin{aligned}
Q_\Delta = & \frac{1}{3!}\epsilon^{IJK}\mathrm{tr}(h_{s_I(\Delta)}\{h^{-1}_{s_I(\Delta)}, \sqrt{V(v(\Delta),\epsilon)}\}\mathrm{tr}(h_{s_J(\Delta)}\{h^{-1}_{s_J(\Delta)}, \sqrt{V(v(\Delta),\epsilon)}\} \times \\
\times\ & \mathrm{tr}(h_{s_K(\Delta)}\{h^{-1}_{s_K(\Delta)}, \sqrt{V(v(\Delta),\epsilon)}\}) \qquad (4.12)
\end{aligned}$$

And we rewrite

$$\begin{aligned}
H^\epsilon_{kin}(N) &= \frac{1}{2\kappa^7 3^2}\lim_{\epsilon\to 0}\sum_{pqrs\in V(\gamma)} N(p)p^I(B_p)p^I(B_q)\chi(p,q) \times \\
&\times \frac{8}{E(r)}\chi(r,p)\sum_{v(\Delta)=r} Q_\Delta \frac{8}{E(s)}\chi(s,q)\sum_{v(\Delta')=s} Q_{\Delta'} \qquad (4.13)
\end{aligned}$$

Now we replace the classical objects by their quantum counterpart and we take the limit. Now we just take $\epsilon$ to zero, realize that only terms with $v = p = q = r = s$ contribute and find that,

$$\begin{aligned}
\hat{H}_{kin}[N] &= \frac{1}{2\kappa^7 3^2 (i\hbar)^6}\sum_{v\in V(\gamma)} N(v)\left(\frac{8}{E(v)}\right)^2 \hat{p}^I(B_v)\hat{p}^I(B_v) \times \\
&\times \sum_{v(\Delta)=v(\Delta')=v} \epsilon^{IJK}\epsilon^{LMN}\mathrm{tr}(h_{s_I(\Delta)}[h^{-1}_{s_I(\Delta)}, \sqrt{\hat{V}(v(\Delta),\epsilon)}]\mathrm{tr}(h_{s_J(\Delta)}[h^{-1}_{s_J(\Delta)}, \sqrt{\hat{V}(v(\Delta),\epsilon)}] \times \\
&\times\ \mathrm{tr}(h_{s_K(\Delta)}[h^{-1}_{s_K(\Delta)}, \sqrt{\hat{V}(v(\Delta),\epsilon)}])\mathrm{tr}(h_{s_L(\Delta')}[h^{-1}_{s_L(\Delta')}, \sqrt{\hat{V}(v(\Delta'),\epsilon)}] \times \\
&\times\ \mathrm{tr}(h_{s_M(\Delta')}[h^{-1}_{s_M(\Delta')}, \sqrt{\hat{V}(v(\Delta'),\epsilon)}])\mathrm{tr}(h_{s_N(\Delta')}[h^{-1}_{s_N(\Delta')}, \sqrt{\hat{V}(v(\Delta'),\epsilon)}] \qquad (4.14)
\end{aligned}$$

now because only tetrahedra based at vertices $\gamma$ contribute in the sum $\int_\sigma = \sum_\Delta \int_\Delta$

$$\begin{aligned}
\hat{H}_{kin}[N] &= \frac{1}{2\kappa^7 3^2 (i\hbar)^6} \sum_{v\in V(\gamma)} N(v) \left(\frac{8}{E(v)}\right)^2 \hat{p}^I(B_v)\hat{p}^I(B_v) \times \\
&\times \sum_{v(\Delta)=v(\Delta')=v} \epsilon^{IJK}\epsilon^{LMN} \mathrm{tr}(h_{s_I(\Delta)}[h^{-1}_{s_I(\Delta)}, \sqrt{\hat{V}_v}] \mathrm{tr}(h_{s_J(\Delta)}[h^{-1}_{s_J(\Delta)}, \sqrt{\hat{V}_v}] \times \\
&\times \ \mathrm{tr}(h_{s_K(\Delta)}[h^{-1}_{s_K(\Delta)}, \sqrt{\hat{V}_v}]) \mathrm{tr}(h_{s_L(\Delta')}[h^{-1}_{s_L(\Delta')}, \sqrt{\hat{V}_v}] \times \\
&\times \ \mathrm{tr}(h_{s_M(\Delta')}[h^{-1}_{s_M(\Delta')}, \sqrt{\hat{V}_v}]) \mathrm{tr}(h_{s_N(\Delta')}[h^{-1}_{s_N(\Delta')}, \sqrt{\hat{V}_v}] \qquad (4.15)
\end{aligned}$$

where $B$ is a compact region in $\Sigma$, and $E(v)$ is the usual function $E(v) = n(v)(n(v)-1)(n(v)-2)$ where $n(v)$ is the valence of the vertex $v$.

Next we turn to the term containing the derivatives of the scalar field.

Then considering eq(4.3)

$$H_{der}[N] = \frac{1}{2\kappa} \int_\Sigma d^3x N(x) \sqrt{\det(q)} q^{ab} \mathcal{D}_a\phi_I \mathcal{D}_b\phi_I \qquad (4.16)$$

And with

$$q^{ab}\sqrt{\det(q)} = \frac{E^a_i E^b_i}{\sqrt{\det(q)}} \qquad \text{and} \qquad E^a_i = \epsilon^{acd}\epsilon_{ijk}\frac{e^j_c e^k_d}{2}$$

We write and regulate (again we could have chosen to replace only one of the $E^a_i$ by the term quadratic in $e^i_a$ and still would arrive at a well-defined result at the price of losing symmetry of the expression).

$$\begin{aligned}
& H_{der}[N] \\
=\ & \frac{1}{2\kappa} \lim_{\epsilon\to 0} \int d^3x \int d^3y N(x)\chi_\epsilon(x,y)\epsilon^{ijk}\epsilon^{imn}\epsilon^{abc}\frac{(\mathcal{D}_a\phi_I e^j_b e^k_c)(x)}{\sqrt{V(x,\epsilon)}}\epsilon^{bef}\frac{(\mathcal{D}_b\phi_I e^m_e e^n_f)(y)}{\sqrt{V(y,\epsilon)}} \\
=\ & \frac{1}{2\kappa^5}(\frac{2}{3})^4 \lim_{\epsilon\to 0} \int N(x)\epsilon^{ijk}\mathcal{D}\phi_I(x) \wedge \{A^j(x), V(x,\epsilon)^{3/4}\} \wedge \{A^k(x), V(x,\epsilon)^{3/4}\} \times \\
\times\ & \int \chi_\epsilon(x,y)\epsilon^{imn}\mathcal{D}\phi_I(y) \wedge \{A^m(x), V(y,\epsilon)^{3/4}\} \wedge \{A^n(y), V(y,\epsilon)^{3/4}\}\,. \qquad (4.17)
\end{aligned}$$

It is clear where we are driving at. We replace Poisson brackets by commutators times $1/i\hbar$ and $V$ by its operator version. Furthermore we introduce the already familiar triangulation of $\Sigma$ and have, using that with $v = s(0)$ for some path $s$

Let us define the quantity

$$
\begin{aligned}
\Gamma[f] &= \int_\Sigma f(x)\mathcal{D}\phi_I(x) \wedge \{A^j(x), V(x,\epsilon)^{3/4}\} \wedge \{A^k(x), V(x,\epsilon)^{3/4}\} \\
&= \sum_\Delta \int_\Delta f(x)\mathcal{D}\phi_I(x) \wedge \{A^j(x), V(x,\epsilon)^{3/4}\} \wedge \{A^k(x), V(x,\epsilon)^{3/4}\} \quad (4.18)
\end{aligned}
$$

Then

$$
H_{der}[N] = \frac{1}{2\kappa^5}(\frac{2}{3})^4 \lim_{\epsilon\to 0} \sum_{\Delta\Delta'} \Gamma_\Delta[N]\Gamma_{\Delta'}[\chi] \; . \quad (4.19)
$$

Now the functional derivative of $\Phi(v) = e^{\phi_I \tau_I}$ with respect to $\phi_I(x)$ turns out to be meaningless without a regularization of $\Phi(v)$ as well.

In [85] a regularization which takes the interpretation of $\Phi(v)$ as the nontrivial limit of a holonomy $\underline{h}_e$ as $e$ shrinks to $e$ serious has been used to take into account a consistent implementation of the adjoint relations. We will use this variables explained in that work.

We see that

$$
\begin{aligned}
& \underline{h}_s(0,\delta t)\Phi(s(\delta t))\underline{h}_s(0,\delta t)^{-1} - \Phi(v) \\
=& \exp(\underline{h}_s(0,\delta t)\phi(s(\delta t))\underline{h}_s(0,\delta t)^{-1}) - \Phi(v) \\
=& \exp([1 + \delta t\dot{s}^a(0)\underline{A}_a][\phi(v) + \delta t\dot{s}^a(0)\partial_a\phi(v)][1 - \delta t\dot{s}^a(0)\underline{A}_a] + o((\delta t)^2)) - \Phi(v) \\
=& \exp(\delta t\dot{s}^a(0)(\partial_a\phi(v) + [\underline{A}_a, \phi(v)]) + o((\delta t)^2)) - U(v) = \delta t\dot{s}^a(0)\mathcal{D}_a\phi(v) + o((\delta t)^2),
\end{aligned}
\quad (4.20)
$$

And using the expression derived in the previous chapter, we arrive to

$$
\begin{aligned}
& 6\int_\Delta f(x)\mathcal{D}\phi_I(x)\wedge\{A^j(x),V(x,\epsilon)^{3/4}\}\wedge\{A^k(x),V(x,\epsilon)^{3/4}\} \\
\approx\; & -\frac{4}{d}\epsilon^{mnp}\; f(v(\Delta))\;\mathrm{tr}(\underline{\tau}_I[\underline{h}_{v+s_M}\Phi(s_M(\Delta))\underline{h}_{v+s_M}^{-1}-\Phi(v(\Delta))]) \times \\
\times\; & \mathrm{tr}(\tau_j h_{s_N(\Delta)}\{h_{s_N(\Delta)}^{-1},V(v(\Delta),\epsilon)^{3/4}\})\mathrm{tr}(\tau_k h_{s_P(\Delta)}\{h_{s_P(\Delta)}^{-1},V(v(\Delta),\epsilon)^{3/4}\}) \qquad (4.21)
\end{aligned}
$$

and with $\mathrm{tr}(\tau_i\tau_j)=-\delta_{ij}/2, \mathrm{tr}(\underline{\tau}_I\underline{\tau}_J)=-d\delta_{IJ}$, $d$ the dimension of the fundamental representation of $G$ that

$$
\begin{aligned}
& \hat{H}_{der}[N]=\frac{1}{2\kappa^5\hbar^4}(\frac{2}{3})^4(\frac{2}{3d})^2\lim_{\epsilon\to 0}\sum_{v,v'\in V(\gamma)} N(v)\chi_\epsilon(v,v')\epsilon^{ijk}\epsilon^{ilm}\times \\
\times\; & \sum_{v(\Delta)=v}\frac{8}{E(v)}\epsilon^{NPQ}\mathrm{tr}(\underline{\tau}_I[\underline{h}_{v+s_N}\Phi_\Delta(s_N)\underline{h}_{v+s_N}^{-1}-\Phi(v)])\times \\
\times\; & \mathrm{tr}(\tau_j h_{s_P(\Delta)}[h_{s_P(\Delta)}^{-1},\hat{V}_v^{3/4}])\mathrm{tr}(\tau_k h_{s_Q(\Delta)}[h_{s_Q(\Delta)}^{-1},\hat{V}_v^{3/4}])\times \\
\times\; & \sum_{v(\Delta')=v}\frac{8}{E(v)}\epsilon^{RST}\mathrm{tr}(\underline{\tau}_I[\underline{h}_{v+s_R}\Phi_{\Delta'}(s_R)\underline{h}_{v+s_R}^{-1}-\Phi(v)])\times \\
\times\; & \mathrm{tr}(\tau_l h_{s_S(\Delta')}[h_{s_S(\Delta')}^{-1},\hat{V}_v^{3/4}])\mathrm{tr}(\tau_m h_{s_T(\Delta')}[h_{s_T(\Delta')}^{-1},\hat{V}_v^{3/4}]) \qquad (4.22)
\end{aligned}
$$

since only tetrahedra with vertices as basepoints contribute. Thus we find in the limit $\epsilon\to 0$

$$
\begin{aligned}
& \hat{H}_{der}[N]=\frac{1}{2\kappa^5\hbar^4}(\frac{2}{3})^4(\frac{2}{3d})^2\sum_{v\in V(\gamma)} N(v)\epsilon^{ijk}\epsilon^{ilm}\times \\
\times\; & \sum_{v(\Delta)=v(\Delta')=v}\frac{8}{E(v)}\epsilon^{NPQ}\mathrm{tr}(\underline{\tau}_I[\underline{h}_{v+s_N}\Phi_\Delta(s_N)\underline{h}_{v+s_N}^{-1}-\Phi(v)])\times \\
\times\; & \mathrm{tr}(\tau_j h_{s_P(\Delta)}[h_{s_P(\Delta)}^{-1},\hat{V}_v^{3/4}])\mathrm{tr}(\tau_k h_{s_Q(\Delta)}[h_{s_Q(\Delta)}^{-1},\hat{V}_v^{3/4}])\times \\
\times\; & \epsilon^{RST}\mathrm{tr}(\underline{\tau}_I[\underline{h}_{v+s_R}\Phi_{\Delta'}(s_R)\underline{h}_{v+s_R}^{-1}-\Phi(v)])\times \\
\times\; & \mathrm{tr}(\tau_l h_{s_S(\Delta')}[h_{s_S(\Delta')}^{-1},\hat{V}_v^{3/4}])\mathrm{tr}(\tau_m h_{s_T(\Delta')}[h_{s_T(\Delta')}^{-1},\hat{V}_v^{3/4}]) \qquad (4.23)
\end{aligned}
$$

Again, despite its complicated appearance, (4.23) defines a densely defined operator.

Finally the potential term, is trivial to quantize. Notice that certain functions of $\phi_I(v)\phi_I(v)$ can be recovered from polynomials of the functions $[\mathrm{tr}(U(v)^n)]^m$ where $m, n$ are non-negative integers. Thus we may *define* for instance a mass term through

$$\phi_i(v)^2 := [\arccos(\frac{\mathrm{tr}(\Phi(v))}{2})]^2$$

we find

$$\hat{H}_{pot}(N) = \frac{m_p}{\ell_p^3} \sum_{v \in V(\gamma)} N_v P(\phi_I \phi_I)(v) \hat{V}_v \tag{4.24}$$

This furnishes the quantization of the Higgs sector. Notice that the regularized Hamiltonians have the same structure, namely an operator which carries out a discrete operation on a spin network, divided by an appropriate power of the Planck length which compensates the power of the Planck length coming from the action of the volume operator.

## 4.2 Momentum and field covariant expansions

Let us now continue with the calculation of the contribution to (4.15) due to the momentum integral by writing the expansion

$$\hat{p}^I(B_v) : = \mathrm{tr}\left[\tau_I h_v \left(\int_{B_v} h_{\rho(r)} \hat{p}(r) h_{\rho(r)}^{-1} d^3x(r)\right) h_v^{-1}\right] \tag{4.25}$$

And by using the expressions before

$$\hat{p}^I(B_v) : = \frac{\vec{s}_I \cdot (\vec{s}_J \times \vec{s}_K)}{3!} \left[p^I(v) + T^a D_a p^I(v) + T^{ab} D_a D_b p^I(v) + \ldots\right] \tag{4.26}$$

where the $T's$ can be calculated

$$T^a = \int_{B_v} d^3x x^a = \frac{1}{4}(s_J^a + 2s_I^a + s_K^a) \tag{4.27}$$

$$T^{ab} = \int_{B_v} d^3x x^a x^b \tag{4.28}$$

$$T^a = \frac{1}{4}(s^a_J + 2s^a_I + s^a_K) \tag{4.29}$$

$$V_B = \frac{\vec{s}_I \cdot (\vec{s}_J \times \vec{s}_K)}{3!} \tag{4.30}$$

$$= \frac{J}{3!} \tag{4.31}$$

And for the derivative part

$$\underline{h}_{v+s_R}\Phi(s_R)\underline{h}^{-1}_{v+s_R} - \Phi(v) = s^a_R \mathcal{D}_a \phi(v) + \frac{1}{2!} s^a_R s^b_R \mathcal{D}_a \mathcal{D}_b \phi(v) + \ldots \tag{4.32}$$

according to the previous analysis we are using the notation $x_I = \vec{s}_I \cdot \vec{\mathcal{D}} = s^a_I \mathcal{D}_a$.

## 4.3 The Higgs effective Hamiltonian

Let us consider the total Higgs Hamiltonian without potential term

$$\hat{H}_{Higgs} = \hat{H}_{kin} + \hat{H}_{der} + \hat{H}_{\text{pot}} \tag{4.33}$$

The kinematical part is

$$\begin{aligned} \hat{H}_{kin}[N] &= \frac{(-2)^2}{2\kappa^7(3!)^2(i\hbar)^6} \sum_{v \in V(\gamma)} N(v) \left(\frac{8}{E(v)}\right)^2 \hat{p}^I(B_v)\hat{p}^I(B_v) \times \\ &\times \sum_{v(\Delta)=v(\Delta')=v} \epsilon^{IJK} \hat{Q}_{I\Delta}\hat{Q}_{J\Delta}\hat{Q}_{K\Delta} \times \epsilon^{LMN}\hat{Q}_{L\Delta'}\hat{Q}_{M\Delta'}\hat{Q}_{N\Delta'} \end{aligned} \tag{4.34}$$

and the derivative part is

$$\hat{H}_{der}[N] = \frac{1}{2\kappa^5\hbar^4}(\frac{2}{3})^4(\frac{2}{3d})^2 \sum_{v\in V(\gamma)} N(v)\epsilon^{ijk}\epsilon^{ilm} \times$$

$$\times \sum_{v(\Delta)=v(\Delta')=v} \left(\frac{8}{E(v)}\right)^2 \epsilon^{NPL}\,\mathrm{tr}\left(\tau_{\underline{I}}[\underline{h}_{v+s_N}\Phi_\Delta(s_N)\underline{h}^{-1}_{v+s_N} - \Phi(v)]\right)\,\hat{Q}_{jP\Delta}\,\hat{Q}_{kL\Delta} \times$$

$$\times \;\; \epsilon^{RST}\,\mathrm{tr}\left(\tau_{\underline{I}}[\underline{h}_{v+s_R}\Phi_{\Delta'}(s_R)\underline{h}^{-1}_{v+s_R} - \Phi(v)]\right)\,\hat{Q}_{lS\Delta'}\,\hat{Q}_{mT\Delta'} \quad (4.35)$$

Where

$$\hat{Q}_{I\Delta} = \mathrm{tr}\left(h_{s_I(\Delta)}[h^{-1}_{s_I(\Delta)}, \sqrt{\hat{V}_v}]\right) \quad (4.36)$$

$$\hat{Q}_{iP\Delta} = \mathrm{tr}\left(\tau_i h_{s_P(\Delta)}[h^{-1}_{s_P(\Delta)}, \hat{V}_v^{3/4}]\right) \quad (4.37)$$

### 4.3.1 Kinetic part

First we need to expand the relevant quantities taking part in (4.34)

$$\hat{p}^I(B_v) : \;=\; \frac{1}{3!}\epsilon^{pqr}s^p_I s^q_J s^r_K\left[p^I(v) + \frac{1}{4}(s^a_J + 2s^a_I + s^a_K)D_a p^I(v) + \ldots\right] \quad (4.38)$$

$$M^{\underline{I}}_{3IJK(\Delta)} \;:=\; \frac{1}{3!}\epsilon^{pqr}s^p_I s^q_J s^r_K p^I(v) \quad (4.39)$$

$$M^{\underline{I}}_{4IJK(\Delta)} \;:=\; \frac{1}{4!}\epsilon^{pqr}s^p_I s^q_J s^r_K(s^a_J + 2s^a_I + s^a_K)D_a p^I(v) \quad (4.40)$$

Next let us consider the gravitational contributions, which we expand as

$$\hat{Q}_{L\Delta} \;=\; s^a_L s^b_L Q_{ab} + s^a_L s^b_L s^c_L Q_{abc} + \mathcal{O}(s^4 Q), \quad (4.41)$$

with

$$Q_{ab} = \frac{1}{4}[A_{ia}, [A_{ib}, \sqrt{V}\,]], \quad Q_{abc} = \frac{1}{24}\epsilon^{ijk}[A_{ia}, [A_{jb}, [A_{kc}, \sqrt{V}\,]]]. \tag{4.42}$$

The scaling properties of the gravitational operators under the semiclassical expectation value are

$$\langle W\,\underline{\vec{E}}\,\underline{\vec{B}}|\ldots Q_{\,a_1\ldots a_n}\ldots|W\,\underline{\vec{E}}\,\underline{\vec{B}}\rangle \rightarrow \frac{{\ell_P}^{3/2}}{\mathcal{L}^n}\left(\frac{\ell_P}{\mathcal{L}}\right)^{n\Upsilon}. \tag{4.43}$$

where we have used the same scaling for the connection see Eq.(3.83)

For the product $\hat{Q}_{\,I\Delta}\hat{Q}_{\,J\Delta}\hat{Q}_{\,K\Delta}$ we need the element

$$\hat{Q}_{\,I\Delta}\hat{Q}_{\,J\Delta}\hat{Q}_{\,K\Delta} \quad = \quad U^{IJK}_{2\Delta} + U^{IJK}_{3\Delta} + \mathcal{O}\left(s^4\right) \tag{4.44}$$

where

$$U^{IJK}_{2\Delta} \quad = \quad s^a_I s^b_I s^c_J s^d_J s^e_K s^f_K Q_{ab}Q_{cd}Q_{ef} + s^a_I s^b_I s^c_J s^d_I s^e_J s^f_K s^g_K Q_{abc}Q_{de}Q_{fg} + \mathcal{O}\left(s^8\right) \tag{4.45}$$

$$U^{IJK}_{3\Delta} \quad = \quad s^a_I s^b_I s^c_I s^d_J s^e_J s^f_J s^h_K s^r_K s^t_K Q_{abc}Q_{def}Q_{hrt} + \mathcal{O}\left(s^{10}\right) \tag{4.46}$$

at this point all we need is to consider the tensorial structure and orders in the segments more than the exhaustive analysis of the exact numerical coefficients since they at last can be absorbed in the unknown coefficients coded at the end. Just to see the structure of the terms they are roughly like

Therefore

$$\hat{Q}_{\,I\Delta}\hat{Q}_{\,J\Delta}\hat{Q}_{\,K\Delta}\hat{Q}_{\,L\Delta'}\hat{Q}_{\,M\Delta'}\hat{Q}_{\,N\Delta'} = U^{IJK}_{2\Delta}U^{LMN}_{2\Delta'} + U^{IJK}_{3\Delta}U^{LMN}_{3\Delta'} + + U^{IJK}_{2\Delta}U^{LMN}_{3\Delta'} + U^{IJK}_{3\Delta}U^{LMN}_{2\Delta'} + \mathcal{O}(s^7) \tag{4.47}$$

We are considering

$$H = \frac{1}{\kappa}\int d^3x N(\vec{x}) p^{\underline{I}}(\vec{x}) \left(D^{a_1} \dots D^{a_1} p^{\underline{I}}(\vec{x})\right) \; R_{a_1 \dots a_m}(\vec{x}) \tag{4.48}$$

$$\begin{aligned}
T &= T_0 + T_1 + T_2 + \mathcal{O}(\to \ell_P^3) \qquad (4.49)\\
T_0 &= \frac{1}{2\kappa^6 3^2 (i\hbar)^6}\,\epsilon^{IJK}\,\epsilon^{LMN}\left[U^{IJK}_{2\Delta}U^{LMN}_{2\Delta'}\;M^{\underline{I}}_{3IJ(\Delta)}M^{\underline{I}}_{3LM(\Delta')}\right],\\
T_1 &= \frac{1}{2\kappa^6 3^2 (i\hbar)^6}\,\epsilon^{JKL}\,\epsilon^{MNP}\;\left[U^{IJK}_{2\Delta}U^{LMN}_{2\Delta'}\left(M^{\underline{I}}_{3IJ(\Delta)}M^{\underline{I}}_{4LM(\Delta')} + M^{\underline{I}}_{4IJ(\Delta)}M^{\underline{I}}_{3LM(\Delta')}\right)\right],\\
T_2 &= \frac{1}{2\kappa^6 3^2 (i\hbar)^6}\,\epsilon^{JKL}\,\epsilon^{MNP}\left[\;M^{\underline{I}}_{3IJ(\Delta)}\;\left(U^{IJK}_{2\Delta}U^{LMN}_{3\Delta'} + U^{IJK}_{3\Delta}U^{LMN}_{2\Delta'}\right)\;\right].
\end{aligned}$$

The calculation is performed in the appendix.B , we are left with the kinetic contribution to the Hamiltonian

$$H_{kin} \;=\; \int \frac{d^3x}{2k}\left(p^{\underline{I}}p^{\underline{I}} + \kappa_1 \ell_p^2 \left(\mathcal{D}_a p^{\underline{I}}\,\mathcal{D}_a p^{\underline{I}}\right) + \kappa_2 \ell_p^3 \epsilon^{abc}\mathcal{D}_a p^{\underline{I}}\,\mathcal{D}_b \mathcal{D}_c p^{\underline{I}}) + \dots\right)$$

or

$$H_{kin} \;=\; \int \frac{d^3x}{2k}\mathrm{Tr}\left[p\cdot p + \kappa_1 \ell_p^2\left(\vec{\mathcal{D}}p\cdot\vec{\mathcal{D}}p\right) + \kappa_2 \ell_p^3 \vec{\mathcal{D}}p\cdot(\vec{\mathcal{D}}\times\vec{\mathcal{D}}p) + \dots\right]$$

### 4.3.2 Derivative part

From the derivative part of the field we have the expansion

$$\underline{h}_{v+s_J}\Phi(s_J)\underline{h}^{-1}_{v+s_J} - \Phi(v) = s^a_R \mathcal{D}_a \phi(v) + \frac{1}{2!} s^a_J s^b_J \mathcal{D}_a \mathcal{D}_b \phi(v) + \dots \tag{4.50}$$

$$L^{\underline{I}}_{1J(\Delta)} \;:=\; s^a_J \mathcal{D}_a \phi^{\underline{I}}(v) \tag{4.51}$$

$$L^{\underline{I}}_{2J(\Delta)} \;:=\; \frac{1}{2!} s^a_J s^b_J \mathcal{D}_a \mathcal{D}_b \phi^{\underline{I}}(v) \tag{4.52}$$

$$L^{\underline{I}}_{3J(\Delta)} \;:=\; \frac{1}{3!} s^a_J s^b_J s^c_J \mathcal{D}_a \mathcal{D}_b \mathcal{D}_c \phi^{\underline{I}}(v) \tag{4.53}$$

And the gravitational contributions of the derivative part of the Hamiltonian, similarly as in the previous case the contributions are multiples of quantities like,

$$\hat{Q}_{i\,L\Delta} \;=\; s_L^a Q_{ia} + s_L^a s_L^b Q_{iab} + s_L^a s_L^b s_L^c Q_{iabc} + \mathcal{O}(s^4 Q), \tag{4.54}$$

with

$$Q_{ia} = \frac{1}{2}[A_{ia}, V^{3/4}], \quad Q_{iab} = \frac{1}{8}\epsilon_{ijk}[A_{ja}, [A_{kb}, V^{3/4}]], \quad Q_{iabc} = -\frac{1}{48}[A_{ja}, [A_{jb}, [A_{ic}, V^{3/4}]]]. \tag{4.55}$$

For the product $\hat{Q}_{i\,P\Delta}\hat{Q}_{i\,L\Delta}$ we need the element

$$\hat{Q}_{j\,P\Delta}\hat{Q}_{k\,L\Delta} \;=\; U_{2\Delta}^{PL\,jk} + U_{3\Delta}^{PL\,jk} + \mathcal{O}\left(s^4\right) \tag{4.56}$$

where

$$U_{2\Delta}^{PL\,jk} \;=\; s_P^a s_L^b Q_{ja} Q_{kb} + s_P^a s_L^b s_L^c Q_{ja} Q_{kbc} + \mathcal{O}\left(s^8\right) \tag{4.57}$$

$$U_{3\Delta}^{PL\,jk} \;=\; s_P^a s_P^b s_L^c s_L^c Q_{jab} Q_{kcd} + \mathcal{O}\left(s^{10}\right) \tag{4.58}$$

now all we need is to consider the tensorial structure and orders in the segments more than the exhaustive analysis of the exact numerical coefficients since they at last can be absorbed in the unknown coefficients coded at the end. Just to see the structure of the terms they are roughly like

Therefore

$$\hat{Q}_{\,iP\Delta}\hat{Q}_{\,jL\Delta}\hat{Q}_{\,lS\Delta'}\hat{Q}_{\,mT\Delta'} = U_{2\Delta}^{PLij} U_{2\Delta'}^{STlm} + U_{3\Delta}^{PL} U_{3\Delta'}^{STlm} + +U_{2\Delta}^{PLij} U_{3\Delta'}^{STlm} + U_{3\Delta}^{PLij} U_{2\Delta'}^{STlm} + \mathcal{O}(s^7) \tag{4.59}$$

And

$$H = \frac{1}{\kappa}\int d^3x N(\vec{x})\mathcal{D}_{r_1}\phi^{\underline{I}}(\vec{x})\ldots\mathcal{D}_{r_n}\phi^{\underline{I}}(\vec{x})\,(D^{a_1}\ldots D^{a_m})\,\mathcal{D}_r\phi^{\underline{I}}(\vec{x})\,R^{rr_1\ldots r_n}_{a_1\ldots a_m}(\vec{x}) \quad (4.60)$$

$$\begin{aligned}
K &= K_0 + K_{11} + K_{12} + \mathcal{O}(\rightarrow \ell_P^3) \qquad (4.61)\\
K_0 &= \frac{1}{2\kappa^4\hbar^4}(\frac{2}{3})^4(\frac{2}{3d})^2\,\epsilon^{NPL}\,\epsilon^{RST}\left[U^{PL}_{2\Delta}U^{ST}_{2\Delta'}\,L^{\underline{I}}_{1N(\Delta)}L^{\underline{I}}_{1R(\Delta')}\right],\\
K_{11} &= \frac{1}{2\kappa^4\hbar^4}(\frac{2}{3})^4(\frac{2}{3d})^2\,\epsilon^{NPL}\,\epsilon^{RST}\,\left[U^{PL}_{2\Delta}U^{ST}_{2\Delta'}\left(L^{\underline{I}}_{1N(\Delta)}L^{\underline{I}}_{2R(\Delta')}+L^{\underline{I}}_{2N(\Delta)}L^{\underline{I}}_{1R(\Delta')}\right)\right],\\
K_{12} &= \frac{1}{2\kappa^4\hbar^4}(\frac{2}{3})^4(\frac{2}{3d})^2\,\epsilon^{NPL}\,\epsilon^{RST}\left[\left(U^{PL}_{2\Delta}U^{ST}_{3\Delta'}+U^{IJK}_{3\Delta}U^{LMN}_{2\Delta'}\right)L^{\underline{I}}_{1IJ(\Delta)}\,L^{\underline{I}}_{1IJ(\Delta')}\right].
\end{aligned}$$

Now let us calculate the contributions to $K$ in the appendix.C. The derivative contribution to the total Higgs Hamiltonian is

$$H_{der} = \int \frac{d^3x}{2k}\left(\mathcal{D}_a\phi^{\underline{I}}\,\mathcal{D}_a\phi^{\underline{I}} + \kappa_3\ell_p\epsilon^{abc}\mathcal{D}_a\phi^{\underline{I}}\mathcal{D}_b\mathcal{D}_c\phi^{\underline{I}} + \kappa_4\ell_p^2\mathcal{D}_a\mathcal{D}_a\phi^{\underline{I}}\,\mathcal{D}_b\mathcal{D}_b\phi^{\underline{I}} + \ldots\right)$$

or

$$H_{der} = \int \frac{d^3x}{2k}\mathrm{Tr}\left(\vec{\mathcal{D}}\phi\cdot\,\vec{\mathcal{D}}\phi + \kappa_3\ell_p\vec{\mathcal{D}}\phi\cdot(\vec{\mathcal{D}}\times\vec{\mathcal{D}}\phi) + \kappa_4\ell_p^2\vec{\mathcal{D}}^2\phi\cdot\vec{\mathcal{D}}^2\phi + \ldots\right)$$

### 4.3.3 Higgs effective Hamiltonian

From (4.50) and (4.62) the total Higgs Hamiltonian is

$$\begin{aligned}
H_{kin} = \int \frac{d^3x}{2k}\mathrm{Tr}\Big[&p\cdot p + \vec{\mathcal{D}}\phi\cdot\,\vec{\mathcal{D}}\phi + \kappa_1\ell_p^2\left(\vec{\mathcal{D}}p\cdot\vec{\mathcal{D}}p\right) + \kappa_2\ell_p^3\vec{\mathcal{D}}p\cdot(\vec{\mathcal{D}}\times\vec{\mathcal{D}}p)\\
&+\kappa_3\ell_p\vec{\mathcal{D}}\phi\cdot(\vec{\mathcal{D}}\times\vec{\mathcal{D}}\phi) + \kappa_4\ell_p^2\vec{\mathcal{D}}^2\phi\cdot\vec{\mathcal{D}}^2\phi + \ldots\Big]
\end{aligned}$$

# Chapter 5

# Conclusions and Perspectives

# Appendix A

# Non-abelian electric flux integrals

The work developed in the Yang Mills sector has been done with the replacement $B \rightarrow E$, and keeping until quadratic order in the electric field, which will be now justified. In here we calculate directly the electric contribution using an appropriate parameterization in terms of the two vectors $s_I$ and $s_J$ that span the face $F_{IJ}$.

The element we want to calculate is the expression (3.4)

$$\int_{F_{IJ}} d^3x\, \mathcal{F} \;\; = \;\; \int_{F_{IJ}} d^x\, \mathbf{h}_{e+x(p)}\; \mathbf{E}(p)\; \mathbf{h}^{-1}_{e+x(p)} \tag{A.1}$$

The region of integration is a triangle with face $F_{IJ}$ and sides $s_I$ and $s_J$ Then we will use the following parameterization for the vector $\vec{x}(t,u)$

$$\vec{x}(t,u) = t\vec{s}_I + u(\vec{s}_J - \vec{s}_I) \tag{A.2}$$

the change of variables produces the jacobian $J^c$

$$\int dS^c \rightarrow \int_0^1 dt \int_0^t du J^c \tag{A.3}$$

$$J^c = |\frac{\partial(x,y)}{\partial(t,u)}|^c = \epsilon_{abc} s_I^a s_J^b \tag{A.4}$$

We will use the expression given by

$$h(\vec{v},\vec{v}+\vec{x})\vec{E}(\vec{x}+\vec{v})h(\vec{v},\vec{v}+\vec{x}) \quad = \vec{E}(\vec{v}) + D\vec{E}(\vec{v}) + \tfrac{1}{2!}D^2\vec{E}(\vec{v}) + ... \tag{A.5}$$

Where $D = x^a D_a$

Then the determination of the electric flux amounts to calculate the expression

$$\begin{aligned}\phi^E(F_{IJ}) \quad &= \quad \int(\vec{E}_v + x^a(t,u)D_a\vec{E}_v + x^a(t,u)x^b(t,u)D_aD_b\ \vec{E}_v + ... \\ &\quad +x^nD_v^n\vec{E}_v)d\vec{S}\end{aligned} \tag{A.6}$$

with

$$B_n = \int x^{a_1}x^{a_2}...x^{a_n}dS \tag{A.7}$$

we get the elements

$$\begin{aligned}&B_0 = \frac{1}{2!}\epsilon_{abc}s_I^a s_J^b \\ &B_1 = \frac{1}{3!}(s_I^d + s_J^d)\epsilon_{abc}s_I^a s_J^b \\ &B_2 = \frac{1}{4!}\{s_I^d s_I^e + \frac{1}{2}(s_I^d s_J^e + s_J^d s_I^e) + s_J^d s_J^e\}\epsilon_{abc}s_I^a s_J^b \\ &B_3 = \frac{1}{5!}(s_I^d s_I^e s_I^f + \frac{1}{3}(s_I^d s_I^e s_J^f + s_I^e s_J^d s_I^f + s_J^d s_I^e s_I^f + s_I^d s_J^e s_J^f + \\ &s_J^d s_I^e s_J^f + s_J^d s_J^e s_I^f) + s_J^d s_J^e s_J^f)\epsilon_{abc}s_I^a s_J^b\end{aligned} \tag{A.8}$$

And the corresponding fluxes are

$$\Phi_{\underline{I}}^{(2)}(F_{JK}) = \frac{1}{2!}\epsilon_{abc}s_J^a s_K^b E_{\underline{I}}^c \tag{A.9}$$

$$\Phi_{\underline{I}}^{(3)}(F_{JK}) = \frac{1}{3!}(s_J^d + s_K^d)\epsilon_{abc}s_J^a s_K^b D_d E_{\underline{I}}^c \tag{A.10}$$

$$\Phi_{\underline{I}}^{(4)}(F_{JK}) = \frac{1}{4!}\left(s_J^d s_J^e + \frac{1}{2}(s_J^d s_K^e + s_K^d s_J^e) + s_K^d s_K^e\right)\epsilon_{abc}s_J^a s_K^b D_d D_e E_{\underline{I}}^c \tag{A.11}$$

# Appendix B

# Higgs: kinetic contribution

Now let us calculate the contributions to the momentum sector of the Higgs Hamiltonian

$$\begin{aligned} R_0 &= \frac{1}{2\kappa^6 3^2 (i\hbar)^6} \sum_{v\in \mathrm{Box}} \left(\frac{8}{E(v)}\right)^2 \sum_{v(\Delta)=v(\Delta')=v} \frac{\mu_{kin}^2}{(3!)^2}\, \epsilon^{IJK}\, \epsilon^{LMN} \epsilon^{pqr} s_I^p s_J^q s_K^r \epsilon^{pqr} s_I^p s_J^q s_K^r \times \\ &\times\ s_I^a\, s_I^b\, s_J^c\, s_J^d\, s_K^e\, s_K^f\, s_I'^r\ s_I'^p\ s_J'^t\, s_J'^u\ s_K'^v\ s_K'^w\ \langle W\, \underline{\vec{E}}\, \underline{\vec{B}}|\, Q_{ab}\, Q_{cd}\, Q_{ef}\ Q'_{rp}\, Q'_{tu}\, Q'_{vw}\, |W\, \underline{\vec{E}}\, \underline{\vec{B}}\rangle. \end{aligned} \tag{B.1}$$

$$\begin{aligned} R_0 &= \frac{-1}{2\ell_p^{12} 3^2} \sum_{v\in \mathrm{Box}} \left(\frac{8}{E(v)}\right)^2 \sum_{v(\Delta)=v(\Delta')=v} \mu_{kin}^2\ \det(s)\det(s) \times \\ &\times\ s_I^a\, s_I^b\, s_J^c\, s_J^d\, s_K^e\, s_K^f\, s_I'^r\ s_I'^p\ s_J'^t\, s_J'^u\ s_K'^v\ s_K'^w\ \langle W\, \underline{\vec{E}}\, \underline{\vec{B}}|\, Q_{ab}\, Q_{cd}\, Q_{ef}\ Q'_{rp}\, Q'_{tu}\, Q'_{vw}\, |W\, \underline{\vec{E}}\, \underline{\vec{B}}\rangle. \end{aligned} \tag{B.2}$$

We see that

$$\begin{aligned} R_0 &= \frac{-1}{2\ell_p^{12} 3^2} \sum_{v\in \mathrm{Box}} \left(\frac{8}{E(v)}\right)^2 \sum_{v(\Delta)=v(\Delta')=v} \mu_{kin}^2\ \det(s)\det(s) \times \\ &\times\langle W\, \underline{\vec{E}}\, \underline{\vec{B}}|\, Q_{II}\, Q_{JJ}\, Q_{KK}\ Q'_{LL}\, Q'_{MM}\, Q'_{NN}\, |W\, \underline{\vec{E}}\, \underline{\vec{B}}\rangle. \end{aligned} \tag{B.3}$$

where we have defined

$$Q_{IJ} = \frac{s_I^a s_J^b}{4}[A_{ia},[A_{ib},\sqrt{V}\,]\,] \tag{B.4}$$

The above equation implies

$$R_0{}^{r_1 r_2} \;=\; \frac{1}{2\,Q^2}\frac{\mu^2}{\ell_P^7}\;\ell_P^6\,\frac{\ell_P^3}{\mathcal{L}^2}\left(\frac{\ell_P}{\mathcal{L}}\right)^{2\Upsilon}\;\delta^{r_1 r_2} = \frac{1}{2\,Q^2}\delta^{r_1 r_2}, \tag{B.5}$$

the flat limit is recovered by choosing

$$\mu_{kin}^2 = \left(\frac{\mathcal{L}}{\ell_P}\right)^{12\Upsilon+12}. \tag{B.6}$$

And with the relations

$$\epsilon^{KJL}\,s_K^a\,s_J^b\,s_L^c = det(s)\,\epsilon^{abc}, \quad det(s) = det(s_K^a), \qquad \epsilon^{abp}\epsilon_{abq} = 2\delta_q^p. \tag{B.7}$$

The correction arising from $T_1$ is

$$\begin{aligned} R_1^{a_1} \;=\; & \frac{-1}{2\ell_p^{12}3^2}\sum_{v\in\,\mathrm{Box}}\left(\frac{8}{E(v)}\right)^2\sum_{v(\Delta)=v(\Delta')=v}\frac{\mu_{kin}^2}{3!\,4!}\,\epsilon^{IJK}\,\epsilon^{LMN}\,[\epsilon^{pqr}\,s_I^p s_J^q s_K^r\times \\ & \epsilon^{p_1q_1r_1}\,s_I^{p_1}\,s_J^{q_1}\,s_K^{r_1}({s'}_J^{a_1}+2{s'}_I^{a_1}+{s'}_K^{a_1})+(s_J^{a_1}+2s_I^{a_1}+s_K^{a_1})\epsilon^{p_1q_1r_1}\,{s'}_I^{p_1}\,{s'}_J^{q_1}\,{s'}_K^{r_1}]\times \\ & \times\; s_I^a\,s_I^b\,s_J^c\,s_J^d\,s_K^e\,s_K^f\,{s'}_I^r\,{s'}_I^p\,{s'}_J^t\,{s'}_J^u\,{s'}_K^v\,{s'}_K^w\;\langle W\,\underline{\vec{E}}\,\underline{\vec{B}}|\,Q_{ab}\,Q_{cd}\,Q_{ef}\;Q'_{rp}\,Q'_{tu}\,Q'_{vw}\,|W\,\underline{\vec{E}}\,\underline{\vec{B}}\rangle. \end{aligned} \tag{B.8}$$

there is no symmetric tensor with one index then

$$R_1^{a_1} \;=\; 0 \tag{B.9}$$

$$
\begin{aligned}
R_2^{a_1 a_2} \;=\;& \frac{-1}{2\ell_p^{12} 3^2} \sum_{v \in \text{Box}} \left(\frac{8}{E(v)}\right)^2 \sum_{v(\Delta)=v(\Delta')=v} \frac{\mu_{kin}^2}{3!\,4!}\, \epsilon^{IJK}\, \epsilon^{LMN} \big[\epsilon^{pqr} s_I^p s_J^q s_K^r ({s'}_J^{a_1} + 2{s'}_I^{a_1} + {s'}_K^{a_1})\times \\
& \epsilon^{p_1 q_1 r_1} s_I^{p_1} s_J^{q_1} s_K^{r_1} ({s'}_J^{a_1} + 2{s'}_I^{a_1} + {s'}_K^{a_1}) + \epsilon^{pqr} s_I^p s_J^q s_K^r (s_J^{a_1} + 2s_I^{a_1} + s_K^{a_1}) \\
& \epsilon^{p_1 q_1 r_1} {s'}_I^{p_1} {s'}_J^{q_1} {s'}_K^{r_1} ({s'}_J^{a_1} + 2{s'}_I^{a_1} + {s'}_K^{a_1})\big] \times\;\; s_I^a\, s_I^b\, s_J^c\, s_J^d\, s_K^e\, s_K^f\, {s'}_I^r\, {s'}_I^p\, {s'}_J^t\, {s'}_J^u\, {s'}_K^v\, {s'}_K^w \times \\
& \times\; \langle W\, \underline{\vec{E}}\, \underline{\vec{B}}|\, Q_{ab}\, Q_{cd}\, Q_{ef}\;\, Q'_{rp}\, Q'_{tu}\, Q'_{vw}\, |W\, \underline{\vec{E}}\, \underline{\vec{B}}\rangle.
\end{aligned}
\tag{B.10}
$$

And

$$
R_2^{a_1 a_2} = \kappa_1 \frac{\mu_{kin}^2}{\ell_P^{12}} \ell_P^{16} \frac{\ell_P^9}{\mathcal{L}^{12}} \left(\frac{\ell_P}{\mathcal{L}}\right)^{12\Upsilon} \delta^{a_1 a_2} = \kappa_1 \ell_p\, \delta^{a_1 a_2} \tag{B.11}
$$

$$
R_3^{a_1 a_2 a_3} = \kappa_2 \ell_p^3 \epsilon^{a_1 a_2 a_3} \tag{B.12}
$$

$$
R_4^{a_1 a_2 a_3 a_4} = \kappa_2 \ell_p^2 (\delta^{a_1\, a_2}\;\, \delta^{a_3\, a_4} + \delta^{a_1\, a_3}\;\, \delta^{a_2\, a_4} + \delta^{a_1\, a_4}\;\, \delta^{a_2\, a_3}) \tag{B.13}
$$

# Appendix C

# Higgs: derivative contribution

We proceed to compute the derivative contributions to the Higgs Hamiltonian

$$R_0^{r_1\,r_2} = \frac{1}{2\kappa^4\hbar^4}(\frac{2}{3})^4(\frac{2}{3d})^2 \sum_{v\in\,\mathrm{Box}} \left(\frac{8}{E(v)}\right)^2 \sum_{v(\Delta)=v(\Delta')=v} \frac{\mu_{der}^2}{(3!)^2}\,\epsilon^{IJK}\,\epsilon^{LMN}\epsilon^{pqr}s_I^p s_J^q s_K^r \epsilon^{pqr} s_I^p s_J^q s_K^r \times$$
$$\times\; s_I^a\, s_I^b\, s_J^c\, s_J^d\, s_K^e\, s_K^f\, s_I^{\prime r}\, s_I^{\prime p}\, s_J^{\prime t}\, s_J^{\prime u}\, s_K^{\prime v}\, s_K^{\prime w}\;\langle W\,\underline{\vec{E}}\,\underline{\vec{B}}|\,Q_{ab}\,Q_{cd}\,Q_{ef}\;Q'_{rp}\,Q'_{tu}\,Q'_{vw}\,|W\,\underline{\vec{E}}\,\underline{\vec{B}}\rangle. \tag{C.1}$$

$$R_0^{r_1\,r_2} = \frac{-1}{2\ell_p^{12}3^2} \sum_{v\in\,\mathrm{Box}} \left(\frac{8}{E(v)}\right)^2 \sum_{v(\Delta)=v(\Delta')=v} \mu_{kin}^2\;\det(s)\det(s)\times$$
$$\times\; s_I^a\, s_I^b\, s_J^c\, s_J^d\, s_K^e\, s_K^f\, s_I^{\prime r}\, s_I^{\prime p}\, s_J^{\prime t}\, s_J^{\prime u}\, s_K^{\prime v}\, s_K^{\prime w}\;\langle W\,\underline{\vec{E}}\,\underline{\vec{B}}|\,Q_{ab}\,Q_{cd}\,Q_{ef}\;Q'_{rp}\,Q'_{tu}\,Q'_{vw}\,|W\,\underline{\vec{E}}\,\underline{\vec{B}}\rangle. \tag{C.2}$$

We see that

$$R_0^{r_1\,r_2} = \frac{-1}{2\ell_p^{12}3^2} \sum_{v\in\,\mathrm{Box}} \left(\frac{8}{E(v)}\right)^2 \sum_{v(\Delta)=v(\Delta')=v} \mu_{kin}^2\;\det(s)\det(s)\times$$
$$\times\langle W\,\underline{\vec{E}}\,\underline{\vec{B}}|\,Q_{II}\,Q_{JJ}\,Q_{KK}\;Q'_{LL}\,Q'_{MM}\,Q'_{NN}\,|W\,\underline{\vec{E}}\,\underline{\vec{B}}\rangle. \tag{C.3}$$

where we have defined

$$Q_{IJ} = \frac{s^a_I s^b_J}{4}[A_{ia},[A_{ib},\sqrt{V}\,]] \tag{C.4}$$

The above equation implies

$$R_0{}^{r_1 r_2} \;=\; \frac{1}{2\,Q^2}\frac{\mu^2}{\ell_P^7}\,\ell_P^6\,\frac{\ell_P^3}{\mathcal{L}^2}\left(\frac{\ell_P}{\mathcal{L}}\right)^{2\Upsilon}\;\delta^{r_1 r_2} = \frac{1}{2\,Q^2}\delta^{r_1 r_2}, \tag{C.5}$$

the flat limit is recovered by the choose of

$$\mu^2_{der} = \left(\frac{\mathcal{L}}{\ell_P}\right)^{12\Upsilon+12}. \tag{C.6}$$

And with the relations

$$\epsilon^{KJL}\, s^a_K\, s^b_J\, s^c_L = det(s)\,\epsilon^{abc}, \quad det(s) = det(s^a_K), \qquad \epsilon^{abp}\epsilon_{abq} = 2\delta^p_q. \tag{C.7}$$

The correction arising from $T_1$ is

$$\begin{aligned} R_1{}^{r_1\,r_2}_{\;a_1} \;=\; & \frac{-1}{2\ell_p^{12}3^2}\sum_{v\in \mathrm{Box}}\left(\frac{8}{E(v)}\right)^2 \sum_{v(\Delta)=v(\Delta')=v}\frac{\mu^2_{kin}}{3!\,4!}\,\epsilon^{IJK}\,\epsilon^{LMN}\,[\epsilon^{pqr}\,s^p_I s^q_J s^r_K \times \\ & \epsilon^{p_1q_1r_1}\,s^{p_1}_I\,s^{q_1}_J\,s^{r_1}_K(s'^{a_1}_J + 2s'^{a_1}_I + s'^{a_1}_K) + (s^{a_1}_J + 2s^{a_1}_I + s^{a_1}_K)\epsilon^{p_1q_1r_1}\,s'^{p_1}_I\,s'^{q_1}_J\,s'^{r_1}_K\big] \times \\ & \times\; s^a_I\,s^b_I\,s^c_J\,s^d_J\,s^e_K\,s^f_K\,s'^r_I\,s'^p_I\,s'^t_J\,s'^u_J\,s'^v_K\,s'^w_K\;\langle W\,\underline{\vec{E}}\,\underline{\vec{B}}|\,Q_{ab}\,Q_{cd}\,Q_{ef}\;Q'_{rp}\,Q'_{tu}\,Q'_{vw}\,|W\,\underline{\vec{E}}\,\underline{\vec{B}}\rangle. \end{aligned} \tag{C.8}$$

with

$$R_1{}^{r\,r_1}_{\;a_1} \;=\; \kappa_3\ell_p\epsilon^{a_1 r r_1} \tag{C.9}$$

$$
\begin{aligned}
R_{2\;a_1\,a_2}^{r_1\,r_2} \;\; = \;\; & \frac{-1}{2\ell_p^{12}3^2}\sum_{v\in\,\mathrm{Box}}\left(\frac{8}{E(v)}\right)^2\sum_{v(\Delta)=v(\Delta')=v}\frac{\mu_{kin}^2}{3!\,4!}\,\epsilon^{IJK}\,\epsilon^{LMN}\,\big[\epsilon^{pqr}s_I^p s_J^q s_K^r(s'^{a_1}_J+2s'^{a_1}_I+s'^{a_1}_K)\times \\
& \epsilon^{p_1q_1r_1}s_I^{p_1}s_J^{q_1}s_K^{r_1}(s'^{a_1}_J+2s'^{a_1}_I+s'^{a_1}_K)+\epsilon^{pqr}s_I^p s_J^q s_K^r(s_J^{a_1}+2s_I^{a_1}+s_K^{a_1}) \\
& \epsilon^{p_1q_1r_1}s'^{p_1}_I s'^{q_1}_J s'^{r_1}_K(s'^{a_1}_J+2s'^{a_1}_I+s'^{a_1}_K)\big]\times\;\; s_I^a\,s_I^b\,s_J^c\,s_J^d\,s_K^e\,s_K^f\,s'^r_I\,s'^p_I\,s'^t_J\,s'^u_J\,s'^v_K\,s'^w_K\times \\
& \times\,\langle W\,\underline{\vec{E}}\,\underline{\vec{B}}|\,Q_{ab}\,Q_{cd}\,Q_{ef}\;Q'_{rp}\,Q'_{tu}\,Q'_{vw}\,|W\,\underline{\vec{E}}\,\underline{\vec{B}}\rangle.
\end{aligned}
\tag{C.10}
$$

And

$$
R_{2\;a_1\,a_2}^{r_1\,r_2}=\,\kappa_8\,\frac{\mu_{kin}^2}{\ell_P^{12}}\,\ell_P^1 6\,\frac{\ell_P^9}{\mathcal{L}^{12}}\,\left(\frac{\ell_P}{\mathcal{L}}\right)^{12\Upsilon}\delta^{a_1a_2}=\kappa_4\ell_p^2(\delta_{a_1}^{r_1}\delta_{a_2}^{r_2}+\delta_{r_1}^{r_2}\delta_{a_1}^{a_2}) \tag{C.11}
$$

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