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Lorentz Invariance Violation in QED: Theoretical and phenomenological searches

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Abstract

In this thesis we study some aspects of Lorentz invariance violation (LIV) as well as violations of CPT symmetry. Particular attention will be given to the implications of these violations in QED. It must be said form the outset that these have not been experimentally seen so far, however its study is extremely relevant since in many approaches to Quantum Gravity these symmetries seem to be no longer valid at least in some very restricted regimes. Therefore exploring its consequences at least phenomenologically could tell us important features of the would-be theory of quantum gravity.

We start presenting some formal issues and giving some context on the theoretical and experimental facets of LIV. Then we will specifically study the case of LIV in QED. To this end we will start from a model in which fermions are coupled to a constant background axial vector b^μ which in this thesis will be the responsible for LIV, and henceforth call LIVQED. Some motivation for such a scenario will be given in the text. Then we will study the one-loop radiatively induced effects of the above in the photon sector. To this end we compute the vacuum polarization tensor and use the 't Hooft-Veltman-Breitenlohner-Maison Dimensional Regularization scheme. The LIV vector introduces new subtleties at the quantum level of the theory which need to be critically examined for consistency. At this stage the LIV effects will produce both CPT-even and CPT-odd contributions, due to the axial coupling. The latter have already been studied in the literature, while the former had not been studied until the author in collaboration with other people addressed the issue in [4].

The phenomenological implications of this model are presented and by contrasting it to experiments we obtain bounds on the extent to which Lorentz symmetry could be violated in this scenario. Equally important, we analyze the quantum consistency of the CPT-even part of LIVQED opening the possibility for a new way in which LIV may occur which had not been found in the study of LIVQED in the CPT-odd sector. In this respect, this work is an important contribution in the study of the whole LIVQED programme since for the model to be fully consistent, both parts need to be studied.

Notations and Conventions

Throughout this thesis we will work in "natural" units in which $c = \hbar = 1$, where c is the speed of light and $\hbar$ is (the reduced) Planck constant.

Tensors will be written following the notation of Peskin and Schroeder [8]. g denotes the spacetime metric and vectors are denoted by italic case. Three-vectors are denoted by an arrow or boldfaces:

$$\begin{aligned} g_{\mu\nu} = g^{\mu\nu} \quad &= \quad \mathrm{diag}(1, -1, -1, -1)\,, \\ A^\mu = (A^0, \vec{A}) = (A^0, \mathbf{A}), \quad & \quad A_\mu = g_{\mu\nu} A^\nu = (A_0, -\mathbf{A}); \\ A \cdot B = g_{\mu\nu} A^\mu B^\nu \quad &= \quad A^0 B^0 - \mathbf{A} \cdot \mathbf{B}\,. \end{aligned}$$

In the above it is implicit that spacetime is four-dimensional. Unless otherwise stated, greek indices run from 0 to 3 while latin indices run from 1 to 3. However, we will consider d-dimensional spacetime as well, with a Minkowsky metric for dimensional regularization. There $\mu, \nu, \cdots = 0, 1, \ldots, d$. The temporal component of a d-dimensional vector will be p^0 and the $d-1$-dimensional part as $\tilde{p}^i$, with $i = 1, 2, \ldots d$. Thus in d-dimensional space, $p^2 = p \cdot p = p_0^2 - \tilde{p}^2$.

For dimensional regularization in the presence of fermions we use the notation of [25, 26]. For vector manipulations we use: Thus,

$$\begin{aligned} \bar{g} \;= \mathrm{diag}\,(+, -, -, -)\,, \quad & \quad \hat{g} \;= \mathrm{diag}\,(-, -, -, \cdots)\,; \\ p \cdot q \quad &= \quad \bar{p} \cdot \bar{q} - \hat{p} \cdot \hat{q}\,. \end{aligned}$$

Where the physical and "evanescent" components of vectors, $\bar{p}$, $\hat{p}$, respectively are:

$$\begin{aligned} \bar{g}^{\mu\nu} \quad &= \quad \begin{cases} g^{\mu\nu}, & \text{If } \mu \text{ and } \nu \text{ are less than 4,} \\ 0, & \text{otherwise;} \end{cases} \\ \bar{p}^\mu \quad &= \quad \bar{g}^{\mu\nu} p_\nu\,. \end{aligned}$$

$$\hat{g}_{\mu\nu} = \begin{cases} g_{\mu\nu}, & \text{If } \mu \text{ and } \nu \text{ are 4 or larger,} \\ 0, & \text{otherwise;} \end{cases}$$
$$\hat{p}^{\mu} = \hat{g}^{\mu\nu} p_{\nu}.$$

For the algebra of Dirac matrices we use:

$$\gamma^{\mu} = \bar{\gamma}^{\mu}\,, \qquad \mu = 0, 1, 2, 3;$$
$$\gamma^{\mu} = \hat{\gamma}^{\mu}\,, \qquad \mu = 4, \ldots, 2\omega - 1\,.$$
$$\{\bar{\gamma}^{\mu}, \bar{\gamma}^{\nu}\} = 2\bar{g}^{\mu\nu}\,\mathbf{1}\,; \quad \{\hat{\gamma}^{\mu}, \hat{\gamma}^{\nu}\} = 2\hat{g}^{\mu\nu}\,\mathbf{1}\,; \quad \{\bar{\gamma}^{\mu}, \hat{\gamma}^{\nu}\} = 0\,.$$
$$\gamma_5 \equiv i\bar{\gamma}^0\bar{\gamma}^1\bar{\gamma}^2\bar{\gamma}^3\,; \qquad \gamma_5^2 = \mathbf{1}\,; \qquad \{\bar{\gamma}^{\mu}, \gamma_5\} = 0 = [\,\hat{\gamma}^{\mu}, \gamma_5\,]$$

where $\mathbf{1}$ denotes the identity $2^{\omega} \times 2^{\omega}$ square matrix, whereas $\hat{\mathbf{1}}$ denotes the identity matrix in the $2\omega - 4$ dimensional Euclidean space.

0.1 Acronyms used

SR: Special Realtivity	GR: General Relativity ,
LT: Lorentz Transformation	LIV: Lorentz Invariance Violation,
QM: Quantum Mechanics	QED: Quantum electrodynamics,
QFT: Quantum Field Theory	GT: Group Theory,
SM: Standard Model	SME: Standard Model Extension,
LQG: Loop Quantum Gravity	NC: Non-commutative ,
MCS: Maxwell-Chern-Simons	CS: Chern-Simons,
CFJ: Carroll-Field-Jackiw	ABJ: Adler-Bell-Jackiw ,
tHV: 't Hooft-Veltman	DR: (tHV) Dimensional regularization,
tHVBM: 't Hooft-Veltman-Breitenlohner-Maison	$\overline{DR}$: (tHVBM) Dimensional regularization,
MDR: Modified Dispersion Relations	LIVDRS: LIV dimensional regularization scheme,
UHECRs: Ultra High Energy Cosmis Rays	CMB: Cosmic Microwave Background,
FIBR: Far Infrared Background Radiation	AGASA: Akeno Giant Air Shower Array,
HiRes: High Resolution Fly's Eye	GZK: Greisen-Zatsepin-Kuzmin,

Contents

Introduction

Although theoretical particle physics is far from simple, one could say that its goal is rather simple: the study of *matter* and its *interactions*, which are the basic quantities of Newton's 2^{nd} Law $a = F/m$. In this respect, an enormous step forward has been given in the understanding of these concepts, from the Greek's *atoms*, to the *quarks* unveiled in nowadays particle accelerators and from forces such as tensions, friction or contact forces to the forces of gravitational interaction, electromagnetism and ultimately the forces that keeps matter *together* at a subatomic level, or make it decay.

It has been a mayor success of theoretical physics (both from a philosophical and a phenomenological point of view) to understand all phenomena in Nature as a consequence of the, possibly combined, action of only four, and hence fundamental, forces: *gravity, electromagnetism, weak and strong forces*. Naturally this statement is only meaningful bearing in mind the reductionist point of view of physics.

In this sense, with the Law of Universal Gravitation, Newton was a pioneer in what we now call "unification", the moment he understood that the behaviour of bodies falling near the Earth's surface and those of the Planets about the Sun (or that of the Moon about the Earth) could all be explained with only one law. Since then and with rather astonishing success, the other fundamental forces of Nature have been continuously tried to be unified in a similar sense. First, and also within the context of classical physics, it was the turn of electric and magnetic phenomena to be unified into electromagnetism. Then, only with the advent of quantum mechanics could we really unravel the inner workings of the weak and strong force and also gain more insight on the nature of the electromagnetic force, allowing yet another unification by the late 1960's due to Glashow, Weinberg and Salam, namely, the merging of the weak and the electromagnetic force into the so called *electroweak* force.

And it has been also very relevant throughout the development of physics the change in paradigm to describe *forces* and *matter*, where another sort of unification takes place in the way in which the laws of physics are formulated. The former now understood as due to *fields* in space and the latter as *particles*, which are the quanta of fields. Surprisingly enough, both fields and particles can be described by Relativistic Quantum Field Theory or simply QFT, allowing for a natural union of Quantum Mechanics (QM) and Einstein's theory of Special Relativity (SR). The reason for these two to be basic ingredients in the description of Nature is straightforward. The first guarantees the coordinate independence of the phenomena described, while the second insures that particles obeys the proper mechanical laws when these are sub-atomic ones. Of course one could wonder if this unified description were necessary at all, and the answer is yes. For sub-atomic particles cannot be properly described without QM, and typical energies involved in sub-atomic interactions are comparable and sometimes even larger to the rest mass energy of such particles, mc^2, therefore their velocities are near to that of light, c, making relativistic effects inescapable.

As commented above, QFT is the mathematical framework in which sub-atomic particles and its interactions are described. The basic foundations of QFT being the axioms of SR and QM. Although sometimes quite disregarded, Group Theory (GT) should also be considered as another pillar of QFT, since it naturally implements the physical principles of SR and QM. In fact, the laws of physics, not only those of QFT, are built on a very strong and profound principle, that of Symmetry. Well known postulates as those of conservation of energy, linear momentum, electric charge, etc. . . can be understood as consequence of some symmetry. Let us also recall that the Special and General theories of Relativity are founded on underlying <u>spacetime</u> symmetries, while Quantum Mechanics is also founded on <u>internal</u> symmetries as well, which are symmetries associated with *intrinsic* degrees of freedom of particles.

Here is where the relevance of GT lies, because it can naturally implement the symmetry principles that a physical system may exhibit into the mathematics of its description.

Despite the *electroweak* and the *strong* forces are both successfully described by extremely accurate QFTs, *gravity* has remained quite elusive to such a programme. The need for achieving a proper description of the gravitational interaction in terms of a QFT, and finally unify it with the other fundamental forces in a so-called Quantum Gravity (QG) theory, is not a merely aesthetic one, nor is it purely academic. It would be imperative when the subject of study were, for example, *Black Holes*, or the Universe at its very early stages, or even an eventual quantum structure of spacetime.

Therefore, the quest for a QG theory is one of the most important in theoretical physics. However, years of untiring work of many researchers have resulted in important progress but with no definite success. This advise us to keep a less ambitious, yet more fruitful bearing, aiming at more specific issues. In this sense, in several approaches towards a QG theory, there is one particular point in common to many of them, that deserves attention. This is the change in status of some well established symmetry principles, which may no longer have the cherished place they have so far enjoyed.

Thus, this thesis will be devoted to the study of eventual violations, under specific circumstances, of Lorentz symmetry or Lorentz invariance, which states that the laws of physics should be unaltered by the state of motion of the observer that describes the theory.

Particularly I will focus on the possibility of having a specific kind Lorentz invariance violation (LIV for short) in QED, the consistency of such quantum theory by the anlaysis of the one-loop radiative effects implied, its phenomenological implications and on plausible arguments to explain such scenarios. Also I will study the emergence of LIV in certain approaches to quantum gravity and Planck scale physics.

Naturally this thesis will not give definite results regarding the general fate of Lorentz symmetry, nor will it be a complete treatise on the matter. Nonetheless, it is expected to serve the reader as a guide to some advancements made on the field, and also to shed some lights on the subtleties at the quantum level of having LIV in QED. Last but not least, throughout the whole thesis, and particularly in connection with the above, we will always be aware of the phenomenological implications of these assumptions, to contrast predictions with experiments.

It is also important to anticipate that the idea of LIV has also been considered as a way of solving long standing puzzles in physics of diverse areas. The most striking case in which this is so is in Cosmic Ray Physics and we will pay attention to this issue further in the text. Obviously, LIV may not be the only possible explanation to these conundrums, but in many of them, LIV seems a very plausible one. Presumably this would indicate that LIV may not only be just a detail in the far corner of hardly explorable physics, with little implications due to our experimental capacity (or lack thereof), rather, if no other satisfactory explanations can be put forward in order to explain such puzzles, LIV may be a necessary feature of the ultimate theory or any other theory that supersedes both the Standard Model (SM) of particle physics[1] and the

[1]The SM is one just one example of a QFT, which, with remarkable accuracy describes all known

theory of General Relativity.

It is worthwhile to mention that in the history of particle physics it has proven of paramount importance the study of those symmetries that are not exactly preserved when they were considered to be so in the very beginningtherefore, our inquiries are not doomed to fail *a priori*.

Plan of the Thesis

This thesis will be organized as follows. Chapter [1] contains some formal issues which must be taken under consideration and some background material that will be used throughout the thesis. Most of it will concern specific aspects QFT that will be relevant for the calculations presented in chapter 3 where we study LIV in the context of QED. Special attention will be given to the technique of Dimensional Regularization of Feynman integrals for the cases involving fermions. This topic, though known for long time, has some conceptual and calculational subtleties. For many important results of this thesis Dimensional Regularization was crucial, which is why we present it with some detail.

Chapter [2] contains a brief exposition of some frameworks that incorporate LIV. Paying particular attention to the Standard Model Extension (SME). A general framework elaborated in [1, 2, 3], in which the SM Lagrangian is supplemented with all the LIV interaction terms which are Lorentz scalars. A brief account on how it is built and its motivations will be given. Also in this chapter I will review different contexts where the idea of LIV has emerged, exposing the main motivation and the puzzles it aims to solve. Special attention will be given to those models which can be experimentally tested. Although LIV appears in highly diverse contexts, in *most* of them its effects entail modifications to usual particle kinematics. Thus, despite the diversity, Lorentz symmetry violations **can** be generically parameterized in a systematic manner. The chapter ends with a brief account on some experimental bounds that have been imposed on the parameters of Lorentz symmetry violation. Some of these were reported in [4].

In chapter [3] a detailed study of LIVQED will be exposed, focusing on the CPT-even part of LIV induced by coupling an external axial-vector with fermions, trying to motivate such a LIV. The CPT-odd part had already been studied in the literature, but the CPT-even part had only been kinematically studied and characterized. Thus, special emphasis will be given to the quantum consistency of this LIVQED and to its phenomenological implications. This is by no means a trivial task and for a completely

sub-atomic phenomena in a wide range of orders of magnitude.

satisfactory and consistent description of this LIVQED model both the CPT-odd and the CPT-even parts must be analyzed. In this sense, the result obtained constitute a novel and independent contribution to the field. Some of the results presented in this chapter concerning the quantum consistency of the CPT-even sector of LIV in QED were reported for the first time in [5].

Finally in chapter [4] we will present the general conclusion of this thesis. Obviously no definite conclusion will be arrived at about whether Lorentz symmetry is broken or will remain an exact symmetry of Nature. In fact, so far, Lorentz symmetry has withstood all experimental tests and bottom line physical facts are supported by experiments only. Rather, I will only present how far could one go considering LIV with specific models as a framework, the recent developments on the field and also give an outlook for relevant future researches.

Chapter 1

Formal Aspects

In this chapter I will review some background material that will be relevant for our study in chapter 3. Most of it can be found on field theory textbooks, for example [8, 6, 9, 10], and therefore will not be presented in full detail, unless necessary. I will only focus on specific points and arguments that are worth recalling and emphasizing, for the development of this thesis, either form a theoretical or computational point of view.

1.1 The meaning of LIV – Breaking paradigms

In Physics, the paradigms that could be considered as "physically meaningful" (intuitive and plausible or absolutely disrupting, as is customary in the quantum realm), are those which enjoy a good relation between predictions and accurate experimentation. However, as far as the relation above is not hampered, sensible departures from the common lore can indeed be found.

Lorentz symmetry or more precisely, invariance under Lorentz transformations (LTs), is a consequence of one such paradigm, that of SR, and so far experiments continue to confirm its validity up to present day capabilities. However, when speaking of "Lorentz symmetry" many issues are encompassed altogether and sometimes erroneously thought of as equivalent, (such as general covariance, energy-momentum conservation, invariance under *active* or *passive* LTs). Typical confusions of this kind are found as objections in seminars where the issue of Lorentz symmetry violation is discussed, where it is not rare to hear questions such as *"How can you consider usual relativistic kinematics for studying reaction/decay thresholds if you are violating Lorentz symmetry?", "If you are introducing LIV by means of a vector b^μ why don't you consider its transformation under Lorentz?", "If you consider LIV, haven't you got problems interpreting your Lagrangian as a Lorentz scalar?", "If Lorentz symmetry is violated, how can you consider fundamental particles* *as definite representations of the Lorentz group... what will be the meanings of mass and spin in this Lorentz violating scenario?"* etc... These doubts however, originate either from disregarding the more specific sense, discussed below, in which LIV is understood, or from a failure of the speaker to clear them out. Therefore, it is of utmost importance to clarify what is precisely meant by Lorentz symmetry or (Lorentz symmetry breaking).

1.1.1 Lorentz transformations

We start by making a distinction between different kinds of LTs referred to in the literature. We will explain them below and they are: (*i*) *Observer* (or *Passive*) LTs, (*ii*) *Particle* LTs and (*iii*) *Active* LTs. Unfortunately these names vary from one author to the other so we focus more on their meaning rather than on their names. Thus, some natural questions arise. Are all the above equivalent? If not, is any of them more fundamental than the others? and if so why? Answering these questions and grasping the subtle differences between the kinds of transformations is very important considering that in this thesis we will be concerned in scenarios where it is said that invariance under LTs is broken.

It will turn out that when we speak of Lorentz symmetry breaking we will mean a theory or model which is observer/coordinate independent but which is not invariant under particle LTs due to the inclusion of background fields. To see that this is by no means contradictory, let us make a brief stop at some basic concepts and examples.

(*i*) **Observer or Passive Lorentz transformations**[1]**:**

Transformations relating the coordinates, on which physical quantities depend, from one reference frame to other reference frame possibly rotated or boosted relative to each other. Invariance under this kind of LT is important since it guarantees that the physics being described is independent on the choice of reference frame chosen. Otherwise, for example, *The Particle Data Group* (`http://pdg.lbl.gov/`) should publish location-dependent lists of particles' masses, spins *etc.*

This really is a principle or paradigm, (<u>coordinate</u> or <u>observer independence</u>), we do want to respect. Therefore, throughout this thesis, we will preserve invariance under observer Lorentz transformations.

Usually this is achieved by setting the physical theory in a spacetime-manifold $\mathcal{M}$ (possibly endowed with a metric $\mathfrak{g}$. Physical observables being represented by

[1]These will be the usual LTs of Special Relativity *i.e.* those which relate the observations of two inertial observers with different velocities and spatial orientations. Obviously this amounts to the same thing as, for a given inertial observer, rotating or boosting the coordinate frame by the same amount.

tensor or spinor fields $\Phi(x)$ and by writing the laws of physics in terms of covariant equations as derived from the usual action principle. The action typically takes the form:

$$S[\mathfrak{g}, \Phi(x)] = \int_{\mathcal{M}} \mathcal{L}(\mathfrak{g}, \Phi(x), \nabla\Phi(x)). \tag{1.1}$$

which must be a Lorentz scalar. Under coordinate transformations, $x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu$ the fields transform accordingly, $\Phi(x) \to \Phi'(x) = M(\Lambda)\Phi(\Lambda^{-1}x)$, and so do their gradients. The matrices $M(\Lambda)$ furnish a representation of the Lorentz group, *etc.* The requirement that S be scalar implies that the equations of motion, will be invariant under observer LTs. Thus observations between relatively boosted or rotated inertial observers are physically equivalent.

(*ii*) **Particle Lorentz transformations:**

Transformations, in a given oriented inertial frame, relating the properties of two distinct particles or fields with different momentum or spin orientation. In the case of free particles these LTs are simply the inverse of the LTs above. However, for particles interacting with background fields this relation no longer holds and the difference between them can be grasped by considering a particle of mass m and charge q moving in a circular trajectory perpendicular to a magnetic field $\vec{B}$. This situation is described by

$$m\frac{dv^\mu}{d\tau} = qF^{\mu\nu}v_\nu, \tag{1.2}$$

where v^μ is the particle's four-velocity, τ the proper time and $F_{\mu\nu} = \partial_{[\mu}A_{\nu]}$; ($A^\mu = (\varphi, \vec{A})$; $\vec{B} = \nabla \times \vec{A}$). The equation (1.2) is a covariant one, therefore valid in all reference frames or coordinate systems. However, suppose we perform a particle LT (boost to be more precise) along the charge's trajectory. This will increase its momentum and make it go round in a larger circle, without any effect over the (external) magnetic field. On the contrary, an observer boost along the particle's trajectory, will have a completely different result, because the magnetic field although being fixed, will unfold its electric counterpart and the motion as seen by this boosted observer will no longer be circular due to the $\vec{E} \times \vec{B}$ drift caused by the emergent electric field.

(*iii*) **Active Lorentz transformations:**

Similar to the particle LTs, these are transformations, in a given oriented inertial frame, in which all particles or fields (including background ones) are rotated and/or boosted. Still we could make the distinction between physical and background fields but the reader will certainly make sense out of this expressions. There are however, situations where background fields are also dynamical, as is the case in some new theories of gravity with a background field[2], but that is out of the scope of this thesis.

1.1.2 Breaking (which?) Lorentz symmetry

In the context of this thesis we will focus on specific scenarios where Lorentz symmetry/invariance is violated, precisely by the presence of a background vector field, say b^μ. In the light of the previous discussion we must stress that it will be only particle Lorentz symmetry which is violated, while observer Lorentz symmetry is preserved. For example in this thesis will be generated the coupling of fermions to a constant background axial vector b^μ. With the above considerations, these four quantities transform as the components of a four-vector under observer LTs while they transform as 4 scalars under particle LTs. Possible physical interpretation for the LIV vector b^μ and its origin will be given further on the thesis. Also it is important to note that preserving observer Lorentz symmetry allows for a meaningful discussion of the (observer) LT properties of the LIV vectors (or tensors) included. For example, at some point we will argue that a spacelike LIV vector b^μ coupled axially to fermions by an extra term in the Dirac Lagrangian, ($\mathcal{L}_{\mathrm{Dirac}} = \bar{\psi} b^\mu \gamma_\mu \gamma_5 \psi$), is experimentally ruled out. This spacelike nature of b^μ is respect to observer LTs, therefore if it is spacelike in one inertial reference frame it will be so in all inertial frames.

Henceforth, unless it is extremely necessary, we will no longer write the terms *Particle*, *Observer*, or *Active* Lorentz transformations, understanding that whenever we talk about LIV we mean violation under particle LTs while observer independence is fully respected.

1.2 Symmetries in physics

In the introduction we already gave an idea of the importance of symmetries in Physics. Indeed one should say that symmetry is present in Nature[3]. By noticing and by carefully studying the symmetries in Nature, (typically accompanied by a great simplification), scientist have learnt tremendous facts about it.

[2]Some theories have been precisely studied in the context of Lorentz symmetry violation where the existence of a preferred reference frame is considered. The point is that if one aims for a theory that also includes gravity, then background fields are dynamical objects. The opposite would be equivalent to consider the metric as a non-dynamical, which is not the case form the general relativistic point of view. See for example [11, 3].

[3]To give just one amongst many examples, the non-technical reader is invited amuse him/herself in the Internet searching for the connection between Fibonacci numbers and phyllotaxy, the study of patterns in leaves around a stem, scales on a pine cone or on a pineapple, florets in the head of a daisy, and seeds in a sunflower.

In this section we will delve into some more technical issues related to the role of symmetries in classical and quantum physics, that will be pertinent for this thesis. We will leave for the next section a discussion concerning the emergence of anomalies where yet again symmetries are fundamental.

Particularly we will focus on the importance of Lorentz symmetry and Gauge symmetry in Quantum Electrodynamics (QED), which is the quantum theory that describes the interaction of light and electrons in a relativistic manner.

1.2.1 Noether's Theorem

In classical field theory, the relationship between symmetries and conservation laws was elevated to the status of theorem by Emmy Noether in 1918 [12], stating that to every continuous symmetry of the action there corresponds a conserved current.

Classically the failure of a symmetry to hold results in a non-conservation of the charge associated to the given current. Now given a symmetry of a system at the classical level, one could wonder what happens upon quantization. This will prove of utmost importance and we leave this discussion for sections (1.3).

1.2.2 Spacetime translational Invariance - Threshold energies

In different LIV scenarios it is customary that a phenomenological imprint of LIV is the modification of particle dispersion relations which can be employed to compute, for example, the threshold energy for a given reaction to take place. Thus the detailed (experimental) observation of such a reaction or its absence can be used to probe the LIV model proposed. However, for the computation of the threshold energies it is necessary to use energy-momentum conservation, which, in terms of Noether's theorem is due to translational invariance of spacetime. Therefore for such an analysis to be valid, LIV effects are considered to pertain only the particle's kinematics, while the underlying spacetime is supposed to be translation invariant. This choice is rather *ad hoc*, however, its main motivation is merely phenomenological as occurs, for example, in some astrophysical scenarios. In sections (2.2) and (2.3) we will present some of these.

1.2.3 CPT Invariance

Although Lorentz symmetry is a continuous spacetime symmetry it is intimately connected with other spacetime <u>and</u> internal discrete symmetries. These are Parity P, time reversal T, and charge conjugation C. The first sends $(t, \mathbf{x}) \to (t, -\mathbf{x})$ while the second does $(t, \mathbf{x}) \to (-t, \mathbf{x})$. Charge conjugation when applied to a particle it transforms into its anti-particle. The experimental status of these symmetry operations are well measured but not fully well understood, as to the CP violation in rare processes involving neutral Kaons.

Altogether the simultaneous transformation CPT has been experimentally confirmed in all observations so far. Actually it has been demonstrated that every QFT with a hermitian Hamiltonian, is CPT invariant as far as Lorentz symmetry is preserved! In [13] this and also the spin-statistics theorems are proved in full detail.

As from the theorem, Lorentz violation does not necessarily implies CPT violation, while the converse is indeed true for local theories [14]. Now, there are certain experiments (usually related to chiral effects) more sensitive to CPT violation rather that LIV, therefore it is extremely important the study of CPT symmetry as well as Lorentz symmetry (or possible violations of any of these).

1.2.4 Gauge Symmetry and Ward Identities in QED

In physics a *gauge* is a degree of freedom in a theory with no observable consequences. A *gauge transformation* is a transformation of that degree of freedom and the theory is said *gauge invariant* or to posses gauge symmetry if nothing changes after a gauge transformation.

Classical electromagnetism is governed by the Maxwell equations which are known to imply the conservation of electric charge and to be invariant under gauge transformations of the electromagnetic potentials $\varphi, \mathbf{A}$. In the relativistic and quantum mechanical theory an electron is described by a quantized field ψ governed by the Dirac Lagrangian $\mathcal{L}_{\text{Dirac}} = \overline{\psi}(i\partial - m)\psi$. This theory has spacetime symmetries as well as internal symmetries. Among the latter we find the invariance of the theory under the gauge transformation of the field $\psi(x) \to e^{i\lambda}\psi(x)$, whose Noether current, $j^\mu(x) = \overline{\psi}(x)\gamma^\mu\psi(x)$ is in fact conserved provided ψ satisfies the Dirac equation. Once the fermion field is coupled to the electromagnetic field, the current j^μ becomes precisely the electric current and the charge associated with it becomes the electric charge. Thus in field theory, electric charge conservation can be understood as a consequence of the invariance of the Dirac Lagrangian under the gauge transformation of the Dirac field above. Yet the above transformation is a global $U(1)$ one *i.e.* the phase of Dirac field at all spacetime points is changed simultaneously by the same amount, which is not the most general situation one could encounter. As C. N. Yang and R. Mills said in motivation for their paper that would give birth to gauge theories, *"It seems that this is not consistent with the localized field concept that underlies the usual physical theories..."* [15]

Nowadays, all theories of fundamental interactions are gauge theories, similar to the above. In 1918 H. Weyl stated the so-called "gauge principle", saying that physics should be unaltered by the local choice of coordinates. The freedom to choose different coordinates at different spacetime points demands the existence of a long-range gauge field, (sometimes called *connection*) coupled to the corresponding charges. Thus, application of this principle to local choice of spacetime coordinates leads to Einstein's General Relativity whereas application to *internal* coordinates leads to electromagnetism, and also the weak and strong interactions. The long-range property in turn, implies the masslessness of the gauge field. At the same time, the gauge field (the *connection*) determines the form of the interaction term between matter fields[4].

Since this thesis concerns LIV in QED, let us review the theory of QED and the role guage symmetry plays in it.

The QED Lagrangian

Electrons are described relativistically and quantum mechanically by the Dirac Field Ψ and the Dirac Lagrangian:

$$\mathcal{L}_{Dirac} = \overline{\psi}(i\partial\!\!\!/ - m_e)\psi. \tag{1.3}$$

This Lagrangian is invariant under global $U(1)$ transformations of the Dirac field $\psi \to \psi' = e^{ie\lambda}\psi$. According to Noether's theorem, global $U(1)$ symmetry implies the existence of a classically conserved current

$$j^\mu = e\overline{\psi}\gamma^\mu\psi, \qquad \partial_\mu j^\mu = 0. \tag{1.4}$$

Nevertheless, this does not describe the interaction of light and electrons. To this aim one usually "gauges" the global $U(1)$ symmetry[5] . That is, by making the transformation depend on a local fuction $\lambda(x)$, the action of the derivative in $\mathcal{L}_{Dirac}$ produces an extra term $-e\overline{\psi}i\partial\!\!\!/\lambda(x)\psi$ and the Lagrangian would not be invariant under $\psi \to \psi' = e^{ie\lambda(x)}$. This can be amended by the introduction of gauge field A_μ and replacing the ordinary derivative by $\partial_\mu \to \partial'_\mu = D_\mu = \partial_\mu - ieA_\mu$, yielding:

$$\mathcal{L}'_{Dirac} = \overline{\psi}(iD\!\!\!\!/ - m_e)\psi, \tag{1.5}$$

and this Lagrangian is invariant under the local gauge transformations of both the Dirac and the gauge fields. In this procedure, electron-photon interactions are introduced almost without even noticing, since the Lagrangian (1.5) contains the interaction term:

$$\mathcal{L}_I = e\overline{\psi}A\!\!\!/\psi. \tag{1.6}$$

[4] This is why C. N. Yang claimed *"Symmetry dictates interaction"*.

[5] Global since the function λ is the same for all spacetime points

Diagrammatically this term generates the electron-photon vertex of the theory. Also, once we introduced the gauge field A_μ we must consider it's own dynamics. This in fact is obtained through the Maxwell Lagrangian:

$$\mathcal{L}_{Maxwell} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \tag{1.7}$$

with $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength tensor, and the Euler-Lagrange equations for the A_μ field derived form (1.7) are:

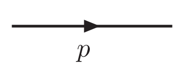
$$\partial_\mu F^{\mu\nu} = j^\nu, \tag{1.8}$$

with j^ν defined as in (1.4). With the gauge field A_μ assembling the electromagnetic potentials $A^\mu = (\varphi, \vec{A})$ with $\varphi = -\nabla\vec{E}$ and $\vec{B} = \nabla \times \vec{B}$, the equations of motion (1.8) yield the usual Maxwell equations. Thus the gauge field A_μ is termed Maxwell gauge field and represents the electromagnetic field. As the photon is the mediator of the electromagnetic interaction, A_μ is also called the photon field.

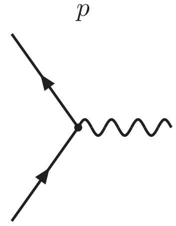
As commented after eqn. (1.18), for a completely satisfactory QFT of electrodynamics, to the free Maxwell Lagrangian one must add a gauge fixing term $\mathcal{L}_{gf} = -\frac{(\partial_\mu A^\mu)^2}{2\xi}$. All in all the QED Lagrangian is:

$$\mathcal{L}_{QED} = \overline{\psi}(i\not{D} - m_e)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{(\partial_\mu A^\mu)^2}{2\xi}. \tag{1.9}$$

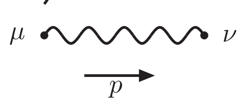
The Feynman rules for QED are [8]:



$$= S_F(p) = \frac{i}{\not{p} - m}, \tag{1.10}$$



$$= -ie\gamma_\mu, \tag{1.11}$$



$$= \Delta_F^0(p)_{\mu\nu} = \frac{-ig_{\mu\nu}}{p^2 + i\epsilon}. \tag{1.12}$$

The first and third diagrams correspond to the fermion (of charge eq_f) and photon Feynman propagators respectively and the second diagram corresponds to the coupling of the photon field to fermions.

The procedure done above is sometimes called "gauging" a symmetry. Its application to other symmetries such as $SU(2)$ and $SU(3)$ is much more involved and leads to the

QFTs for the *weak* and *strong* interactions as well.

In the preceding section two point are worth emphasizing[6].

- The fact that gauge symmetry determines the form of the interaction means that it imposes conditions on the dynamics of the theory, which in turns reflects on its diagrammatics,
- The Noether currents associated with a given gauge symmetry are classically conserved, but there is no *a priori* guarantee that quantum mechanically this will be so.

The first point will lead us below to the Ward identity and the second point will lead us in section (1.3) to consider the role of anomalies in the theory.

Ward Identities

Let $\mathcal{M}(k)$ be the amplitude of an arbitrary QED process allowed by the Feynman rules above, involving an outgoing external photon with momentum k and polarization vector $\epsilon^*_\mu(k)$ and with external electrons *on-shell*. If $\mathcal{M}^\mu$ is the part of the amplitude with the $\epsilon^*_\mu(k)$ dependence factored out, then

$$\mathcal{M} = \mathcal{M}^\mu(k)\epsilon^*_\mu(k). \tag{1.13}$$

Now if the classically conserved vector current introduced in (1.4) is in fact conserved at the quantum level, then it holds that

$$k_\mu \mathcal{M}^\mu(k) = 0. \tag{1.14}$$

This can be roughly seen by recalling that for an arbitrary QED process, $\mathcal{M}^\mu(k)$ is given by the matrix element of the operator that creates external photons, between all possible initial and final states excluding the photon in question. This operator is precisely the Heisenberg field of the Dirac vector current, which creates external photons by the interaction term $ej^\mu A_\mu$ coming form (1.6). Thus, computing $k_\mu \mathcal{M}^\mu(k)$ amounts to $\langle \cdot | k_\mu j^\mu | \cdot \rangle$. With k_μ beign ∂_μ in momentum representation one ends up with:

$$k_\mu \mathcal{M}^\mu \sim \langle \cdot | \partial_\mu j^\mu | \cdot \rangle, \tag{1.15}$$

which vanishes under the assumption of a conserved current ($\partial_\mu j^\mu = 0$) at the quantum level[7] .

[6] A third point which will be commented further on is the importance of gauge symmetry and Ward identities in proving the unitarity of a theory.

[7] Of course, this is only a hand waving argument, and obviously a totally rigorous derivation exists, which can be found on common QFT textbooks. Also a more general identity can be proved for any QED

This is known as the *Ward identity* [16]. (Note that it is of utmost importance that gauge symmetry must hold at the quantum level, hence the importance of anomalies to be reviewed below).

Gauge symmetry and the Photon propagator

Let us recall that the Ward identity allow us to properly determine the form of the photon propagator, and following the same lead, obtain a more general photon propagator corresponding to different gauges. The action of the free electromagnetic field is

$$S = \int d^4x \left[-\frac{1}{4}(F_{\mu\nu})^2 \right] \tag{1.17}$$

and can be written after expanding the field strength tensor and integrating by parts as:

$$S = \frac{1}{2}\int d^4x A_\mu(x)\Delta^{\mu\nu}A_\nu(x), \qquad \Delta_{\mu\nu} = (\partial^2 g_{\mu\nu} - \partial_\mu\partial_\nu), \tag{1.18}$$

As is customary in QFT the propagator is read from an expression as the above as the inverse of the Δ operator, however, in this case this operator is not invertible. In fact, it fails to be so because it vanishes when acting on the very particular field configurations $A'_\mu = \frac{1}{e}\partial_\mu\Lambda(x)$, precisely those related by gauge transformation to $A_\mu(x) = 0$, *i.e.* $\Delta^{\mu\nu}(\frac{1}{e}\partial_\mu\Lambda(x)) = 0$, for any scalar function $\Lambda(x)$. The bottom line reason for this problem is the over-counting of infinitely physically equivalent field configurations, which lead Faddeev and Popov [18] to a very insightful path for the quantization of gauge fields. But let us see if we can cheat a little bit. If we consider a theory for a massive vector boson of mass m, this would be similar to the case of the electromagnetic field, except for the mass. The action for such a theory is:

$$S[A] = \frac{1}{2}\int d^4x A_\mu(x)\mathfrak{D}^{\mu\nu}A_\nu(x), \qquad \mathfrak{D}_{\mu\nu} = ((\partial^2 + m^2)g_{\mu\nu} - \partial_\mu\partial_\nu), \tag{1.19}$$

and the momentum space propagator derived from it is:

---

process involving an incoming or outgoing photon with momentun k, an arbitrary number of incoming and external electrons which may or may not be *on-shell* and additional external photons, called Ward-Takahashi identity [17]. Although for this thesis it suffices with the above, we briefly present the derivation below. In this case, the identity implies

$$k_\mu\mathcal{M}^\mu(k) \neq 0. \tag{1.16}$$

However, with the use of the LSZ reduction formula, one can show that the nonzero terms on the RHS of eq. (1.16) do not contribute to the S-matrix, thus if the amplitude $\mathcal{M}$ is an S-matrix element then we recover the *Ward identity* (1.14). Thus one could say that if the Ward identity is satisfied then gauge symmetry is being enforced, or equivalently, one can enforce gauge symmetry by demanding that the amplitude of a given correlation function in QED satisfies the Ward identity.

$$\mathfrak{D}_{\mu\nu}(k) = i\frac{-g_{\mu\nu} + k_\mu k_\nu/m^2}{k^2 - m^2}. \tag{1.20}$$

If this were to tell us something about the photon propagator, then we must set $m = 0$. This can be done in the denominator without problem, the question is what to do with the $k_\mu k_\nu/m^2$, and the answer comes form the Ward identity. It precisely tell us that any term in the photon two-point function (its propagator) that is proportional to k_μ do not contribute to any S-matrix element and can thus be safely omitted, yielding the usual expression

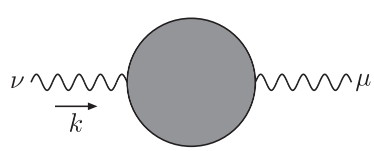
$$\mathfrak{D}_{\mu\nu}(k)\Big|_{m\to 0} = -i\frac{g_{\mu\nu}}{k^2}. \tag{1.21}$$

Furthermore, by the same token we can add such a term with an arbitrary coefficient to obtain:

$$\mathfrak{D}_{\mu\nu}(k)\Big|_{m\to 0} = \frac{-i}{k^2}\left(g_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2}\right). \tag{1.22}$$

The arbitrariness in the choice of ξ then encodes the gauge invariance of the electromagnetic field. As the reader might guess, the above procedure entails the inclusion of the gauge fixing term to the Maxwell Lagrangian done in (1.9).

Another point where we see the importance and use of the Ward identity is when it comes to regulate divergent loop integral in higher order processes. When computing the exact photon propagator one finds:



$$= \frac{-ig_{\mu\nu}}{k^2(1-\Pi(k^2))}, \tag{1.23}$$

where $\Pi(k^2)$ is regular at $k^2 = 0$ and thus the exact propagator has a pole at $k^2 = 0$, which means that the physical mass of the photon is zero and receives no contribution from higher order corrections. This result, however, makes use of the Ward identity. Now if we wanted to compute $\Pi_2^{\mu\nu}(k)$, the correction to the photon propagator up to second order in e we would find an expression for the loop integral which is UV-divergent. Therefore a regularization method must be chosen. And to do so we will choose a regularization method that preserves (gauge) symmetry, *i.e.* that satisfies the Ward identity[8].

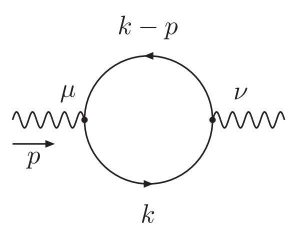
As it has already been mentioned, in the main part of this thesis we will study the effects of coupling fermions to a constant background vector in an axial manner and interpreting its effects as a violation of Lorentz symmetry in the sense discussed in section (1.1.2). This will give a contribution to the QED Lagrangian of the kind:

$$\mathcal{L}_b \,=\, \overline{\psi} b^\mu \,\gamma_\mu \gamma_5 \,\psi \,. \tag{1.24}$$

Quantum mechanically this interaction term leads to a modified fermion propagator exhibiting the dependence on the LIV vector b^μ. Naturally in the limit of vanishing b^μ, we recover the usual Feynman propagator for the fermion.

$$\begin{aligned} \longrightarrow \quad &= \quad S_F(p;b) \quad = \frac{i}{\not{p} - m - \not{b}\gamma_5}, \\ S_F(p;b)\Big|_{b=0} \quad &= \quad \frac{i}{\not{p} - m} = S_F(p). \end{aligned} \tag{1.25}$$

The QED vacuum can still create virtual (fermionic) particle-antiparticle pairs depicted in the figure below. However, with the new fermion propagators the photon self-energy acquires a LIV contribution, which to second order in e is:



$$= \quad \Pi_2^{\mu\nu}(p;b), \tag{1.26}$$

where p is the total incoming photon momentum and the propagators are to be understood as the modified ones, $S_F(p;b)$.

The vacuum polarization tensor of this modified electrodynamics, (later we shall call this model LIVQED) will be one of the central objects of our study. The modified fermion propagator will result in a complicated tensor structure for the vacuum polarization tensor yielding a rather lengthy Feynman integral to be done. As I have said, we will take gauge symmetry as a guiding principle, and therefore imposing the Ward identity on $\Pi_2^{\mu\nu}(p;b)$ will serve as a consistency check of our calculations.

[8] For example, a naive regularization by the introduction of a cut-off M in loop momemtum endows the photon with mass, $m_\gamma \propto M$. To remove the regulator we do $M \to \infty$, resulting (catastrophically) in an infinitely massive photon. Fortunately there is no unique regularization that preserves the (gauge) symmetry, for example Pauli-Villars and dimensional regularization do, however the latter is more straightforward, which is one of the reasons we will use it in this thesis.

1.2.5 Stueckelberg mechanism and gauge restoration

The important lesson of the previous subsection is that gauge symmetry is fundamental and as such it will be one of the paradigms on which we will rely. In QED we saw how gauge invariance, current conservation, the Ward identity and the masslessness of the photon are related. Sometimes, however, it is thought that a massive photon implies the violation of gauge invariance. Of course this is true if we include just a mass term for the A_μ field in the Maxwell Lagrangian, but in other theories need not bo so. In fact, the whole point of the Higgs-mechanism is to endow some of the gauge bosons of the electroweak force with mass, namely the $W^\pm$ and the Z^0 bosons, retaining gauge invariance. Nevertheless, there are other means of formulating a model for a massive vector field in a gauge invariant and also renormalizable way when the gauge group is $U(1)$ and it's due to Stueckelberg [19][9]. Extensions of the Stueckelberg mechanism to the non-Abelian case in order to find alternative descriptions of the Standard Model have been looked for, however renormalizability and unitarity are not achieved leaving the way for the Higgs-mechanism with SSB of the $SU(2)_L \times U(1)_Y$ as the only successful possibility.

In this thesis we will consider a mass term for the photon. In fact, due to the LIV effects to be studied, the photon mass m_γ will in principle have a bare mass μ_γ and also an induced one δm_γ due to radiative corrections as can be guessed from (1.26), such that $m_\gamma^2 = \mu_\gamma^2 + \delta m_\gamma^2$. So, in order to restore the broken gauge invariance by the photon's mass we will employ the "Stueckelberg mechanism". I will very briefly motivate it with a simple calculation.

Consider the Lagrangian of massive vector field, known as the Proca Lagrangian:

$$\mathcal{L} = -\frac{1}{4}(F_{\mu\nu})^2 + \frac{1}{2}m^2(A_\mu)^2. \tag{1.27}$$

where the field strength tensor is as usual $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. This Lagrangian is obviously not invariant under the gauge transformations

$$\delta A_\mu = \partial_\mu \Lambda, \tag{1.28}$$

however, if we make the replacement $A_\mu \to \tilde{A}_\mu = A_\mu + \frac{1}{m}\partial_\mu B$, then the new Lagrangian is:

$$\tilde{\mathcal{L}} = -\frac{1}{4}(F_{\mu\nu})^2 + \frac{1}{2}m^2(A_\mu)^2 + mA_\mu\partial^\mu B + \frac{1}{2}(\partial_\mu B)^2. \tag{1.29}$$

[9]It is worth mentioning that E. C. G. Stueckelberg did many influential works for modern physics. Among them is the interpretation of the positron as a negative energy electron travelling backwards in time back in 1942 [20]

And this Lagrangian is indeed invariant under the (gauge) transformations:

$$\delta A_\mu = \partial_\mu \Lambda, \qquad\qquad \delta B = -m\Lambda. \tag{1.30}$$

So let us see what has happened here. Recall that in QED gauge invariance could be understood as a consequence of having redundant degrees of freedom to describe the theory. A vector field has four degrees of freedom, however we know that the physical photon has only two transverse polarizations, or two helicities. And precisely this gives a hint on how to account for a massive photon. The mass term breaks gauge invariance and now, out of the original four, only one degree of freedom (instead of two) can be eliminated through the Lorentz condition $\partial_\mu A^\mu = 0$. Thus the three degrees of freedom of the A_μ field are interpreted as belonging to a massive (spin-one) vector field, with two transverse polarizations and also a longitudinal one. Thus, the Stueckelberg mechanism consists in introducing an additional scalar field $B(x)$ making up (together with the vector field) a total of five degrees of freedom to describe all three polarizations of a (now) massive vector field, *i.e.* the introduction of additional or redundant degrees of freedom some of which are gauge-fixed, resulting in a gauge invariant theory.

1.3 Anomalies and the role of γ_5 in Dimensional Regularization

In the previous subsection we emphasized that some classical symmetries of a theory may not hold once the theory is quantized. If there is no spontaneous breaking of such symmetry and despite the absence of explicit terms in the action that break it, we say we encounter an *anomaly*. Considering the "status" of guiding principle we gave to symmetries this issue deserves attention.

The reason why anomalies emerge in QFT, typically is due to the fact that in many cases we need to deal with divergent expressions arising form quantum corrections obtained in a perturbative fashion. To tackle these divergences, the so-called *renormalization programme* has been devised, which is done in two steps. (i) First one must *regularize* the divergent expressions after which one speaks of regulator dependent rather than divergent quantities[10]. and (ii) then one must *renormalize* the quantities in question, a systematic procedure by which the predicted observables are rendered finite at <u>all</u> energies and to all orders in perturbation theory, at the price of the introduction

[10] Regularization amounts to "isolating", from its finite part, the divergent part of an expression, for its later "removal" by renormalization. Of course, there is no unique method, but certainly, this is a sensible procedure. By the way, the choice of regularization method employed will be determined by our point of view of taking (gauge) symmetry as fundamental.

of a finite number of arbitrary parameters in the theory.

It is very important to emphasize that anomalies do not emerge by a bad choice of regularization method, rather they reflect the fact that some classical symmetries simply cannot be realized in the quantum theory, despite our attempts to regularize it in one way or another. Furthermore, in order to speak of a proper anomaly one must insure that the anomaly persist once the regulator is removed, for in some cases the would-be broken symmetry may be restored after the regulator is removed.

Then in general one must ask how are symmetries influenced by the quantization procedure and its concomitant “renormalization programme” (if needed). In this thesis we will not be concerned with anomalies *per se*, however, they will be relevant for supporting our choice of regularization method. And to see the connection between these topics, let us review the most typical symmetry with *anomalous breaking* in QFTs, namely chiral symmetry in QED.

1.3.1 Chiral anomaly

The Dirac Lagrangian $\mathcal{L}_D = \overline{\psi}(i\partial\!\!\!/ - m)\psi$ can be written as:

$$\mathcal{L}_D = \overline{\psi}_- i\partial\!\!\!/\, \psi_- + \overline{\psi}_+ i\partial\!\!\!/\, \psi_+ - m(\overline{\psi}_-\, \psi_+ + \overline{\psi}_+\, \psi_-)\,, \tag{1.31}$$

where as usual, $\psi_\pm = P_\pm\, \psi$ and the projection operators over helicity states are $P_\pm = \frac{1}{2}(1 \pm \gamma_5)$. More on the definition of γ_5 later. It is well known that in the massless case $m = 0$, apart from the usual invariance under global $U(1)_V$ transformation of the fields

$$U(1)_V: \qquad \psi_\pm \to e^{i\theta}\psi_\pm, \tag{1.32}$$

the Dirac Lagrangian possesses an “enhanced” symmetry since it is also invariant under another kind of global $U(1)_A$ transformation of the fields that acts <u>differently</u> on left- and right-handed components of massless Dirac-fermions, namely:

$$U(1)_A: \qquad \psi_\pm \to e^{i\pm\phi}\psi_\pm. \tag{1.33}$$

According to Noether’s theorem there are two conserved currents, one of them a vector current and the other an axial vector current:

$$j^\mu_V = \overline{\psi}\gamma^\mu\psi \qquad \Longrightarrow \qquad \partial_\mu\, j^\mu_V = 0\,, \tag{1.34}$$

$$j^\mu_A = \overline{\psi}\gamma^\mu\gamma_5\psi \qquad \Longrightarrow \qquad \partial_\mu\, j^\mu_A = 0\,. \tag{1.35}$$

As we anticipated above, we will encounter an anomaly and it will occur for the axial vector current. However, we must stress that the anomaly of the axial vector current is in fact a choice, rather than a "property" of QED. A "property" that we do not want to be violated by quantum corrections is charge conservation, otherwise real electrons could vanish completely or be created out of the blue! Recall that, generically the Noether charge associated with a conserved current is the spatial integral of the time component of the current, thus electric charge is precisely obtained form the vector current as $Q = \int d^3x j_V^0$. However, the axial charge $Q_A = \int d^3x j_A^0$ has no direct physical interpretation, neither must it be conserved! Also, consider we coupled a fermion with a photon (as we are indeed allowed in QED). If the vector current were not conserved, $\partial_\mu j_V^\mu \neq 0$, as we saw in (1.15) we would violate gauge symmetry (the Ward identity), resulting in a theory without the correct degrees of freedom for the photon (1.2.4)! Nevertheless, none of this happens if $\partial_\mu j_A^\mu \neq 0$. So far we have only argued that the vector current must be conserved in the quantum theory regardless of what happens to the axial vector current. In the next section we will see how the conservation of the former results in a violation of the latter, because we choose to conserve gauge symmetry rather than the axial current. Of course this choice is based on physics, yet it is still a choice.

1.3.2 Dimensional Regularization

As mentioned before, the choice of regularization method should not alter the final physical result, and certainly, more than one choice could be made. In this subsection I will point out the most salient features of the *Dimensional Regularization* (DR) method devised and perfected by 't Hooft and Veltman [21], making it our choice for this thesis. See also the work by Bollini and Giambiagi whom also made progress on the idea of continuation in the number of dimensions [22, 23].

Apart form simplifying some calculations if compared with other gauge preserving regularization methods, such a Pauli-Villars' method [24], we could mention that DR:

- Preserves gauge invariance because the Ward Identities are valid. This is so because, as will be seen when we describe the DR in detail, the Ward identities do not involve the dimensionality of spacetime,

- Preserves Poincaré invariance. Typically in loop integrals, after the introduction of Feynman parameters one does a *momentum shift*. If the regularization method did not preserve Poincaré invariance, such an operation would be invalid!

Furthermore, and in connection with the previous subsection, DR turns out to be an appropriate regularization method even in presence of γ_5. At first sight it may seem that any deviation form $d = 4$ (as suggested by the name DR, and as will be explained soon) is inappropriate for treating γ_5 since it is intrinsically a four-dimensional object. However, 't Hooft and Veltman circumvented this difficulty by giving specific definitions for γ_5 and for the algebra of γ matrices in d dimensions.

The amplitudes for higher order processes in QFT as depicted by their Feynman diagrams can have two kinds of divergences, ultraviolet (UV) and infrared (IR) divergences coming form the very large and low momentum values in loop integrals, respectively. For example, for UV-divergences we could loosely say that the integrand does not have enough "weight" to cancel the integration measure, and the integral diverges as the integrated momentum acquires arbitrary large values. However, by going to a small enough spacetime dimension the contribution from the measure can be "canceled" by the integrand and the UV-divergences may be eliminated. This considers the spacetime dimension as a regulator, which must be removed by doing $d \to 4$ from below, *i.e.* increasing d, therefore we need to treat d as a continuous variable.

Also another relevant feature of DR is that for some theories, IR divergences can also be regularized in a gauge invariant way if the parameter d is a continuous variable, by slightly increasing its value and then removing the regulator tending d to the spacetime dimension from above, *i.e.* reducing d.

't Hooft-Veltman Regularization

Let us review the idea of DR. The point of 't Hooft and Veltman is to introduce a parameter d. This parameter will be related with the spacetime dimension but the reader should not be confused with theories in higher dimensions in the case $d > 4$, for example. As 't Hooft and Veltman emphasize "... d in some sense can be visualized as the dimension of space time ... " [[6], pp.76]. In fact, **the parameter d may well be a non-integer and even a complex number**! Then take any Feynman integral and consider the same integral when the spacetime dimension is precisely d, for example,

$$I_d = \int d^d p \ f(p), \tag{1.36}$$

here $f(p)$ is any given function of the vector p belonging to a d dimensional space. Consider a function f depending on a loop momentum p^μ and possibly on a finite number of external momenta $q_a^\mu, a = 1, \ldots, N$. Then the function f will in general be a tensor function of the vectors p^μ and q_a^ν. If f is a scalar function with only one external momentum q^μ, it will be $f = f(p^2, p \cdot q, q^2)$. If f is a tensor function then it must be written out of the explicit vectors p_μ, q_ν, for example, $f^\mu(p,q) = p^\mu f_1(p^2, p\cdot q, q^2) + q^\mu f_2(p^2, p \cdot q, q^2)$, with f_1, f_2 scalar functions, *etc.*

It may be the case that the integral makes sense only for some values of d. Now suppose a representation for this integral is found, satisfying all the properties that could be expected from a d-dimensional integration operation, allowing d to be a complex number. Then one can use the found representation as the definition of I_d in the region where the integral exists and outside that region I_d is defined as the analytic continuation in d of the found representation. In turns out that meaningful definitions for I_d can be given when $d < 4$ and when $d > 4$, call them I_d^- and I_d^+ and in fact they correspond to the analytic continuation in d of I_d for arbitrarily small and large d respectively. At the end of the calculation one does $d \to 4$ and the formerly untractable divergences are "isolated" and seized control of in the form of poles as $d \to 4$. Thus UV- and IR-divergences can be regularized by means of DR, with due care when both kinds of divergences are present simultaneously.

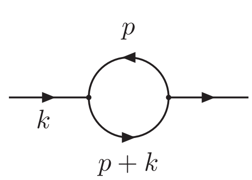
"Symmetrization" properties, also proceed in the usual manner, for example,

$$\int d^d p \; p^\mu p^\nu f(p^2) = \frac{\delta^{\mu\nu}}{d} \int d^d p \; p^2 f(p^2). \tag{1.37}$$

For a detailed discussion on the definitions and axioms of integration in d-dimensional space, with $d \in \mathbb{C}$ and for rigorous demonstrations of existence and uniqueness of d-dimensional integration, see Collins [7].

In this thesis we will only be concerned with UV-divergencies only in our one-loop calculations. This are regulated making the d parameter tend to 4 from below, thus we will only focus on the analytic continuation for $d < 4$. Whenever UV/IR-divergencies must be regulated simultaneously need we worry about the analytic continuation for both cases $d < 4$ and $d > 4$. To see the above procedure in action let us consider a very simple example.

The following computation will be done in detail in order to illustrate the process of analytic continuation. It is important to remark that the result is absolutely independent of the use of Feynman parameters. Consider for example a typical self energy diagram. The physical amplitude to compute is in 4-dimension and thus denoted by I_4.



$$\begin{aligned} &= \quad I_4 \quad = \quad \int d^4 p \; F(k,p,m). \\ F(k,p,m) \quad &= \quad \frac{1}{(p^2 + m^2 - i\epsilon)} \frac{1}{((p+k)^2 + m^2 - i\epsilon)} \end{aligned} \tag{1.38}$$

For large momenta this integral behaves as $\int \frac{dx}{x}$ and thus is logarithmically UV-divergent. Thus consider the following formal expression where d is to be consider as a parameter:

$$I_d = \int d^d p \; F(k,p,m), \quad \text{where}\, p^2 = p_0^2 - \tilde{p}^2, \tag{1.39}$$

and $\tilde{p}$ is the length of the $n-1$-dimensional spatial p. As usual one can introduce polar coordinates in the $n-1$-dimensional space and write:

$$\int d^d p = \int dp_0 \int d^{d-1}\tilde{p} = \int dp_0 \int d^{d-1}\Omega \int d\tilde{p}\tilde{p}^{d-2}, \tag{1.40}$$

where $d^{d-1}\Omega$ is an element of solid angle in $(d-1)$-dimensions and:

$$\begin{aligned}\Omega_{d-1} &= \int_0^{2\pi} d\theta_1 \int_0^{\pi} d\theta_2 \sin\theta_2 \int_0^{\pi} d\theta_3 \sin^2\theta_3 \ldots \int_0^{\pi} d\theta_{d-2} \sin^{d-3}\theta_{d-2} \\ &= \frac{2\pi^{(d-1)/2}}{\Gamma\left(\frac{d-1}{2}\right)}.\end{aligned} \tag{1.41}$$

Altogether allows to write:

$$I_d = \frac{2\pi^{(d-1)/2}}{\Gamma\left(\frac{d-1}{2}\right)} \int_{-\infty}^{\infty} dp_0 \int_0^{\infty} d\tilde{p}\tilde{p}^{d-2} F. \tag{1.42}$$

The starting expression (1.39) is defined only for $d = 1, 2, 3$ and as already commented diverges for $d = 4$. However, the expression (1.42) is defined for some non-integer and even complex values of d in the range $1 < d < 4$. Also we note the RHS of (1.42) coincides with (1.39) for $d = 2, 3$. Now, we can take the RHS of (1.42), in the region where this expression exist, to define I_d in the given region. Outside of the region $1 < d < 4$ we will define I_d as the analytic continuation in d of the RHS of (1.42). Then the physical theory is recovered as $d \to 4$. By no means will the UV-divergent behaviour as $d \to 4$ will disappear. What we will do now is to find an expression which has at most a simple pole at $d = 4$ and a corresponding (finite) residue. If this is achieved, then an appropriate counterterm can be introduced in the Lagrangian consisting of minus the pole times the residue and the divergence is thus cancelled.

Analytic continuation for arbitrary small values of d is indeed possible but we will concern on the analytic continuation to the region that includes the point $d = 4$. To this end we consider the RHS of expression (1.42) and take d to be in the convergence region $1 < d < 4$, next we introduce in the integrand the expression:

$$1 = \frac{1}{2}\left(\frac{d}{dp_0}p_0 + \frac{d}{d\tilde{p}}\tilde{p}\right), \tag{1.43}$$

and integrate by parts with respect to p_0 and $\tilde{p}$. The surface terms do indeed vanish in the convergence region and we are left with:

$$I_d = \frac{-1}{2} \frac{2\pi^{(d-1)/2}}{\Gamma\left(\frac{d-1}{2}\right)} \int_{-\infty}^{\infty} dp_0 \int_0^{\infty} d\tilde{p} \left(p_0 \tilde{p}^{d-2} \frac{dF}{dp_0} + \tilde{p} \frac{d(\tilde{p}^{d-2}F)}{d\tilde{p}} \right). \tag{1.44}$$

The self-energy diagram from which we started was only to present a typical Feynman integral. What we want to stress is the mechanism behind the analytic continuation rather than the details of a particular calculation. Taking a simpler form for the function F:

$$F(k=0,p,m) = \frac{1}{(p^2+m^2)^2}, \tag{1.45}$$

and plugging the derivatives of F in (1.44) we find:

$$I_d = \frac{2\pi^{(d-1)/2}}{\Gamma\left(\frac{d-1}{2}\right)} \int_{-\infty}^{\infty} dp_0 \int_0^{\infty} d\tilde{p} \left[\frac{2p_0^2 - 2\tilde{p}^2}{(p_0^2 - \tilde{p}^2 + m^2)} \right] \tilde{p}^{d-2} F - \left(\frac{d-2}{2} \right) I_d\,, \tag{1.46}$$

where the expression in square brackets is equal to $2 - \frac{2m^2}{(p^2+m^2)}$. Finally we arrive to:

$$I_d = -\frac{2}{(d-4)} I'_d, \qquad \text{where}\; I'_d = 2\frac{2\pi^{(d-1)/2}}{\Gamma\left(\frac{d-1}{2}\right)} \int_{-\infty}^{\infty} dp_0 \int_0^{\infty} d\tilde{p} \frac{m^2 \tilde{p}^{d-2}}{(p^2+m^2)^3}. \tag{1.47}$$

This is the relevant result. I'_d is convergent for $1 < d < 5$ and the first term of expression (1.47) coincides with expression (1.42) for $1 < d < 4$. The UV-divergence of the original integral is seen from I_d as $d \to 4$ in the form of a simple pole. In this case its residue being $-2I'_d$. If the original integral had had a more violent UV-divergence the above procedure of partial integrations can be repeated indefinitely yielding:

$$I_d = \Gamma\left(\frac{4-d}{2}\right) \tilde{I}_d\,, \tag{1.48}$$

with $\tilde{I}_d$ a UV-convergent integral for arbitrary large values of d. Thus we can extend the convergence region to even larger values of d resulting in an analytic function for I_d with simple poles at $d = 4, 6, 8, \ldots$. Nowadays the outstanding original formulation of 't Hooft and Veltman is somehow forgotten and a more straightforward dimensional regularization of Feynman integrals is done. This exploits the use of Feynman parameters, the propagators denominators are usually written in terms of the integral representation of the causal propagators (also called Schwinger parametrization) with the aid of the definition of Euler's Gamma function and so on. This can be found on the appendix (A) where more integrals pertinent for our aims are computed.

't Hooft-Veltman-Breitenlohner-Maison Regularization

Now let us see how a proper dimensional regularization can be devised in presence of γ_5. This is very important in general and particularly for this thesis were we will encounter the need to perform a Feynman integral with a γ_5 term. In this section we will show how can this be done. Furthermore, in relation with the previous section where we talked about the chiral anomaly, here we will show how the demand of gauge symmetry preservation at the quantum level leads to a violation of the axial vector current due to quantum corrections and how does the 't Hooft-Veltman-Breitenlohner-Maison [21, 25, 26] rules for extending the Dirac algebra and γ_5 to arbitrary dimensions yields the correct numerical factor for the axial vector current violation. In fact in 1969 Adler [27] and Bell together with Jackiw [28] had obtained that although the axial vector current was classically conserved it was not so at the quantum level, obtaining a definite result:

$$\partial_\mu j^\mu_A = -\frac{e^2}{16\pi^2}\epsilon^{\alpha\beta\mu\nu}F_{\alpha\beta}F_{\mu\nu}. \tag{1.49}$$

Furthermore, Adler and Bardeen [29] demonstrated that the ABJ anomaly as it came to be called, is correct to all orders in QED, *i.e.* that it receives no further corrections from higher orders.

Apart from vector algebra as the one used in the example above, for Feynman integrals one also needs γ-matrices algebra when it comes to dealing with fermions. In 4-dimension these satisfy

$$\begin{aligned}\{\gamma^\mu,\gamma^\nu\} &= 2g^{\mu\nu}\mathbf{1},\\ \{\gamma_5,\gamma^\mu\} &= 0,\\ \gamma_5^2 &= \mathbf{1}.\end{aligned} \tag{1.50}$$

where $\mathbf{1}$ is the identity matrix in 4-dimensions. However, if one extended these definitions for d-dimensions, and computed the anomaly of the axial vector current an (awkward) vanishing result would be obtained, [21]. Thus, an alternative definition for the Dirac algebra and for γ_5 is adopted, which altogether with the prescriptions for d-dimensional integration we will call the $\overline{DR}$ dimensional regularization due to 't Hooft-Veltman-Breitienlohner-Maison.

Here we list some useful identities concerning dimensional regularization that will be used in this thesis. The Levi-Civita symbol in the four dimensional Minkowski's space-time is normalized according to

$$\epsilon^{0123} = -\epsilon_{0123} \equiv 1\,, \tag{1.51}$$

in such a way that the following identity holds true in the four dimensional Minkowski's space-time: namely,

$$\begin{aligned}\epsilon^{\mu\nu\alpha\beta}\epsilon_\mu{}^{\lambda\rho\sigma} &= g^{\nu\rho}g^{\alpha\lambda}g^{\beta\sigma} + g^{\alpha\rho}g^{\beta\lambda}g^{\nu\sigma} + g^{\beta\rho}g^{\nu\lambda}g^{\alpha\sigma}\\ &- g^{\nu\lambda}g^{\alpha\rho}g^{\beta\sigma} - g^{\alpha\lambda}g^{\beta\rho}g^{\nu\sigma} - g^{\beta\lambda}g^{\nu\rho}g^{\alpha\sigma}\,.\end{aligned} \tag{1.52}$$

For the algebra of Dirac matrices in a 2ω-dimensional spacetime (where the γ-matrices are $2^\omega \times 2^\omega$) with a Minkowski's signature, we have:

$$\begin{aligned} \gamma^\mu &= \bar{\gamma}^\mu\,, & \mu &= 0,1,2,3; \\ \gamma^\mu &= \hat{\gamma}^\mu\,, & \mu &= 4,\ldots,2\omega-1\,. \end{aligned} \tag{1.53}$$

$$\{\bar{\gamma}^\mu,\bar{\gamma}^\nu\} = 2\bar{g}^{\mu\nu}\,\mathbf{1}\,; \quad \{\hat{\gamma}^\mu,\hat{\gamma}^\nu\} = 2\hat{g}^{\mu\nu}\,\mathbf{1}\,; \quad \{\bar{\gamma}^\mu,\hat{\gamma}^\nu\} = 0\,. \tag{1.54}$$

$$\gamma_5 \equiv i\bar{\gamma}^0\bar{\gamma}^1\bar{\gamma}^2\bar{\gamma}^3\,; \qquad \gamma_5^2 = \mathbf{1}\,; \qquad \{\bar{\gamma}^\mu,\gamma_5\} = 0 = [\,\hat{\gamma}^\mu,\gamma_5\,] \tag{1.55}$$

where $\mathbf{1}$ denotes the identity $2^\omega \times 2^\omega$ square matrix, whereas $\hat{\mathbf{1}}$ denotes the identity matrix in the $2\omega - 4$ dimensional Euclidean space.

For vector manipulations we use the notation of [25, 26] of projectors:

Onto the physical dimensions,

$$\bar{g}^{\mu\nu} = \begin{cases} g^{\mu\nu}, & \text{If } \mu \text{ and } \nu \text{ are less than 4,} \\ 0, & \text{otherwise;} \end{cases} \tag{1.56}$$

$$\bar{p}^\mu = \bar{g}^{\mu\nu}p_\nu. \tag{1.57}$$

And onto the unphysical ones,

$$\hat{g}_{\mu\nu} = \begin{cases} g_{\mu\nu}, & \text{If } \mu \text{ and } \nu \text{ are 4 or larger,} \\ 0, & \text{otherwise;} \end{cases} \tag{1.58}$$

$$\hat{p}^\mu = \hat{g}^{\mu\nu}p_\nu. \tag{1.59}$$

Note also that we are considering an extended $d\,(=2\omega)$ spacetime with Minokwski metric, therefore,

$$\bar{g} = \mathrm{diag}\,(+,-,-,-)\,, \qquad \hat{g} = \mathrm{diag}\,(-,-,-,\cdots)\,; \tag{1.60}$$

$$p\cdot q = \bar{p}\cdot\bar{q} - \hat{p}\cdot\hat{q}\,. \tag{1.61}$$

Taking all the above listed equations into account, it is not difficult to check the following trace formulae: i.e.,

$$
\begin{aligned}
\mathrm{tr}\left(\gamma^\mu\gamma^\nu\right) &= g^{\mu\nu}\,\mathrm{tr}\mathbf{1} \;=\; 2^\omega\, g^{\mu\nu} \\
2^{-\omega}\mathrm{tr}\left(\gamma^\kappa\gamma^\lambda\gamma^\mu\gamma^\nu\right) &= g^{\kappa\lambda}\, g^{\mu\nu} \;-\; g^{\kappa\mu}\, g^{\lambda\nu} \;+\; g^{\kappa\nu}\, g^{\lambda\mu} \\
2^{-\omega}\mathrm{tr}\left(\gamma^\kappa\gamma^\lambda\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\right) &= g^{\kappa\lambda}\, g^{\mu\sigma}\, g^{\nu\rho} \;-\; g^{\kappa\lambda}\, g^{\mu\rho}\, g^{\nu\sigma} \;-\; g^{\kappa\mu}\, g^{\lambda\sigma}\, g^{\nu\rho} \\
&+ \; g^{\kappa\mu}\, g^{\lambda\rho}\, g^{\nu\sigma} \;+\; g^{\kappa\nu}\, g^{\lambda\sigma}\, g^{\mu\rho} \;-\; g^{\kappa\nu}\, g^{\lambda\rho}\, g^{\mu\sigma} \\
&+ \; g^{\lambda\mu}\, g^{\kappa\sigma}\, g^{\nu\rho} \;-\; g^{\lambda\mu}\, g^{\kappa\rho}\, g^{\nu\sigma} \;- g^{\lambda\nu}\, g^{\kappa\sigma}\, g^{\mu\rho} \\
&+ \; g^{\lambda\nu}\, g^{\kappa\rho}\, g^{\mu\sigma} \;-\; g^{\mu\nu}\, g^{\kappa\rho}\, g^{\lambda\sigma} \;+\; g^{\mu\nu}\, g^{\kappa\sigma}\, g^{\lambda\rho} \\
&+ \; g^{\kappa\nu}\, g^{\lambda\mu}\, g^{\rho\sigma} \;-\; g^{\kappa\mu}\, g^{\lambda\nu}\, g^{\rho\sigma} \;+\; g^{\kappa\lambda}\, g^{\mu\nu}\, g^{\rho\sigma} \\
\mathrm{tr}\left(\bar\gamma^\kappa\bar\gamma^\lambda\hat\gamma^\mu\hat\gamma^\nu\right) &= 2^\omega\, \bar g^{\kappa\lambda}\, \hat g^{\mu\nu} \\
\mathrm{tr}\left(\gamma_5\bar\gamma^\mu\bar\gamma^\lambda\bar\gamma^\rho\bar\gamma^\nu\right) &= -\, i\, 2^\omega \epsilon^{\mu\lambda\rho\nu} \\
\mathrm{tr}\left(\gamma_5\bar\gamma^\mu\bar\gamma^\lambda\bar\gamma^\rho\bar\gamma^\nu\bar\gamma^\sigma\bar\gamma^\tau\right) &= i\, 2^\omega \left(\epsilon^{\nu\sigma\tau\mu}\, \bar g^{\lambda\rho} + \epsilon^{\nu\sigma\tau\rho}\, \bar g^{\lambda\mu} + \epsilon^{\mu\lambda\rho\sigma}\, \bar g^{\nu\tau}\right) \\
&- \; i\, 2^\omega \left(\epsilon^{\nu\sigma\tau\lambda}\, \bar g^{\mu\rho} + \epsilon^{\mu\lambda\rho\nu}\, \bar g^{\sigma\tau} + \epsilon^{\mu\lambda\rho\tau}\, \bar g^{\nu\sigma}\right)
\end{aligned} \tag{1.62}
$$

Traces involving an odd number of Dirac's matrices do vanish.

Note : in an even integer dimension $d = 2\omega$, the standard representation of the Dirac's matrices has dimension 2^ω , whereas in the dimensional regularization the Dirac's matrices are infinite–dimensional. Nevertheless, if we set $\mathrm{tr}\mathbf{1} := f(\omega)$, it is not necessary to choose $f(\omega) = 2^\omega$. It is usually convenient to set $f(\omega) = f(2) = 4\,,\ \forall\omega \in \mathbf{C}$ [for further details see [7] p. 84]. It is also very important to clear a possible confussion. This definition is not Lorentz invariant on the full spacetime, since it is explicitly separated into the physical spacetime and the unphysical space. However this extended dimension d is not to be confused with those of higher dimensional theories, as in superstring theory nor in supergravity theories.

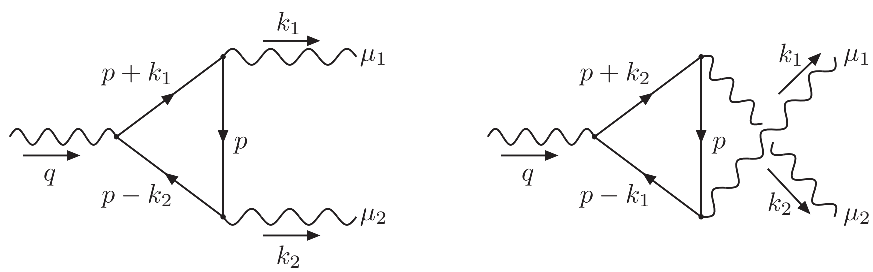
So let us see how $\overline{DR}$ of Feynman integrals is applied in cases involving γ_5. To this end and to complement the discussion about the axial anomaly, let us review this computation. In the main part of this thesis we will encounter a logarithmically divergent Feynman integral with a γ_5 term which will be regularized with $\overline{DR}$. As commented above the vector and Dirac algebra is modified introducing some subtleties that are not very of common usage. Therefore, we consider this example will serve a three-fold purpose, (a) Illustrate $\overline{DR}$ for cases involving γ_5 emphasizing its differences with DR of theories without fermions, (b) See the relation between the chiral anomaly and the $\overline{DR}$, yielding the correct result of the ABJ anomaly with the use of $\overline{DR}$ and (c) To serve as an example for our one-loop calculation of the photon self-energy diagram in the LIVQED model to be considered in the main part of this thesis.

Perturbative calculation of the Chiral Anomaly

We want to compute the divergence of the axial vector current $\partial_\lambda j_5^\lambda$ diagrammatically. Consider the amplitude of the axial vector current to create two real photons out of the vacuum, $\langle k_1, k_2 | j_5^\mu(x) | 0\rangle$, for which we must compute:

$$\int d^4x\, e^{-iq\cdot x} \langle k_1, k_2 | j_5^\lambda(x) | 0\rangle = (2\pi)^4 \delta^{(4)}(q - k_1 - k_2) \epsilon^*_{\mu_1} \epsilon^*_{\mu_2} \Gamma^{\lambda\mu_1\mu_2}(k_1, k_2). \tag{1.63}$$

To leading order, the contributions to $\Gamma^{\lambda\mu_1\mu_2}$ come from the diagrams shown below.



That is, the axial vector current annihilates into an electron-positron pair at point x. Each of these is annihilated by its corresponding antiparticle creating photons with momentum k_1 and k_2 and polarizations $\epsilon^*_{\mu_1}$ and $\epsilon^*_{\mu_2}$ respectively. According to the Feynman rules the expression for $\Gamma^{\lambda\mu_1\mu_2}$ is:

$$\begin{aligned}\Gamma^{\lambda\mu_1\mu_2} \quad &= \quad (-1)(-ie)^2 \int \frac{d^4p}{(2\pi)^4} \operatorname{tr}\left(\gamma^\lambda \gamma^5\, S_F(p - k_2)\, \gamma^{\mu_2}\, S_F(p)\, \gamma^{\mu_1}\, S_F(p + k_1)\right) \\ &\quad + \{k_1 \leftrightarrow k_2\, ;\, \mu_1 \leftrightarrow \mu_2\}\,. \end{aligned}\tag{1.64}$$

The -1 comes from the closed fermion loop and $S_F(p) = \frac{i}{\not p} = i\frac{\not p}{p^2}$ is the Feynman propagator of a chiral fermion of momentum p. This integral is linearly UV-divergent, and thus needs to be regularized. This will be done with the 't Hooft-Veltman-Breitenlohner-Maison dimensional regularization $\overline{DR}$ method described above. Recall that in it, γ^5 is defined as the product $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$, thus anticommuting with any γ^α if $\alpha = 0, 1, 2, 3$ and commutes for other values of α. Besides loop momentum are to be regarded as d-dimensional vector, *i.e.* :

$$p_\mu = \bar{p}_\mu + \hat{p}_\mu, \tag{1.65}$$

the first term having non-zero components for $\mu = 0, 1, 2, 3$ and the second for other values of μ.

So, computing the divergence of the axial vector current $\partial_\lambda j_5^\lambda$ amounts to computing $q_\lambda \Gamma^{\lambda\mu_1\mu_2}$. Thus we note that in the integrand we have the expression $q_\lambda \gamma^\lambda \gamma^5$. Since $q = k_1 + k_2$ we can conveniently write this term as:

$$q_\lambda \gamma^\lambda \gamma^5 = \not q \gamma^5 = (\not k_1 + \not k_2 + \not p - \not p)\gamma^5. \tag{1.66}$$

The subtlety coming form the fact that the loop momentum p is to live in $d = 2\omega$ dimensions, whereas k_1, k_2 live in 4-dimensions only. We can emphasize this as:

$$\not{k} = \bar{\not{k}} = \bar{\gamma}_\mu \bar{k}^\mu \,, \qquad \not{p} = \bar{\not{p}} + \hat{\not{p}} = (\bar{\gamma}^\mu \bar{p}^\mu + \hat{\gamma}^\mu \hat{p}^\mu). \tag{1.67}$$

Bearing this in mind, and the special commutation rules between γ^5 and $\bar{\gamma}^\mu, \hat{\gamma}^\mu$, expression (1.66) is written as:

$$\not{q}\gamma^5 = (\not{p} + \not{k}_1)\gamma^5 - \gamma^5(\not{k}_2 - \not{p}) - 2\gamma^5 \hat{\not{p}}. \tag{1.68}$$

Inserting this identity in the first term of (1.64) yields three terms:

$$\begin{aligned}
&(\not{p} + \not{k}_1)\gamma^5 \frac{(\not{p} - \not{k}_2)}{(p - k_2)^2} \gamma^{\mu_2} \frac{\not{p}}{p^2} \gamma^{\mu_1} \frac{(\not{p} + k_1)}{(p + k_1)^2} \\
&\quad - \gamma^5 (\not{k}_2 - \not{p}) \frac{(\not{p} - \not{k}_2)}{(p - k_2)^2} \gamma^{\mu_2} \frac{\not{p}}{p^2} \gamma^{\mu_1} \frac{(\not{p} + \not{k}_1)}{(p + k_1)^2} \\
&\qquad -2\gamma^5 \hat{\not{p}} \frac{(\not{p} - \not{k}_2)}{(p - k_2)^2} \gamma^{\mu_2} \frac{\not{p}}{p^2} \gamma^{\mu_1} \frac{(\not{p} + \not{k}_1)}{(p + k_1)^2} \,,
\end{aligned} \tag{1.69}$$

which are to be traced. Despite the fact that the loop momenta entails gamma-matrices in full d dimensional space, the cyclicity of the trace is still valid. Hence in the first term of (1.69) we move $(\not{p} + \not{k}_1)$ to the very right yielding:

$$\begin{aligned}
&\gamma^5 \frac{(\not{s} - \not{k}_2)}{(p - k_2)^2} \gamma^{\mu_2} \frac{\not{p}}{p^2} \gamma^{\mu_1} \mathbf{1} \\
&\quad - \gamma^5 (-\mathbf{1}) \gamma^{\mu_2} \frac{\not{p}}{p^2} \gamma^{\mu_1} \frac{(\not{p} + \not{k}_1)}{(p + k_1)^2} \\
&\qquad - 2\gamma^5 \hat{\not{p}} \frac{(\not{p} - \not{k}_2)}{(p - k_2)^2} \gamma^{\mu_2} \frac{\not{p}}{p^2} \gamma^{\mu_1} \frac{(\not{p} + \not{k}_1)}{(p + k_1)^2} \,.
\end{aligned} \tag{1.70}$$

Finally, in the second term of (1.70) we anticmmute γ^5 with γ^{μ_2} and pull the latter to the very right again by cyclicity of the trace. Meanwhile in the first term we shift the integration variable $p \to p + k_2$. Thus writing the first two terms only:

$$\gamma^5 \frac{\not{p}}{p^2} \gamma^{\mu_2} \frac{\not{p} + \not{k}_2}{(p + k_2)^2} \gamma^{\mu_1} \; - \; \gamma^5 \frac{\not{p}}{p^2} \gamma^{\mu_1} \frac{\not{p} + \not{k}_1}{(p + k_1)^2} \gamma^{\mu_2}, \tag{1.71}$$

we see that these two cancel exactly the corresponding two terms coming from the second diagram. Thus we are left with:

$$q_\lambda \Gamma^{\lambda \mu_1 \mu_2} = -e^2 \int \frac{d^d p}{(2\pi)^d} \frac{\mathrm{tr}\{-2\,\gamma^5\, \hat{\not{p}}\, (\not{p} - \not{k}_2)\, \gamma^{\mu_2}\, \not{p}\, \gamma^{\mu_1}\, (\not{p} + \not{k}_1)\,\}}{(p - k_2)^2\, p^2\, (p + k_1)^2} \,. \tag{1.72}$$

Next, to deal with the denominator we make use Feynman parameters. Recall that in the previous section (1.3.2) we showed in detail the validity of the analytic continuation needed for dimensional regularization, particularly after eq. (1.48) we commented that in practice the use of Schwinger and Feynman parameters to express the integrand of a given

Feynman integral is more straight-forward as done in the appendix (A). Equivalence of both methods is fully understood an we will proceed in such a way. Hence the denominator is written as:

$$\frac{1}{(p-k_2)^2}\frac{1}{p^2}\frac{1}{(p+k_1)^2} = \int_0^1 dx\, dy\, dz \delta(x+y+z-1)\frac{2}{\mathbb{D}^3}, \tag{1.73}$$

with $\mathbb{D} = (p^2 - \Delta(k_1^2, k_2^2, k_1 k_2; y, z))$, where we shifted the momentum as $p \to p + yk_1 - zk_2$. Since $k_{1,2}$ are defined in 4-dimensional space only, this shift does not pertain the term $\hat{\not p}$. Finally, we must consider in the numerator of the integrand the fact that:

$$\not p = \not p + y\not k_1 - x\not k_2 = \bar{\not p} + \hat{\not p} + y\bar{\not k}_1 - x\bar{\not k}_2. \tag{1.74}$$

Also, with the aid of the formulae of vector and Dirac algebra extended to d-dimensions (formulae (1.50) through (1.62)) we arrive at:

$$q_\lambda \Gamma^{\lambda\mu_1\mu_2} = -2e^2 \int_0^1 dx\, dy\, dz \delta(x+y+z-1) \int \frac{d^d p}{(2\pi)^d} \frac{\mathrm{tr}\{\, 2\,\hat{p}^2 \gamma^5\, \bar{\gamma}^\alpha\, \bar{\gamma}^{\mu_2}\, \bar{\gamma}^{\mu_1}\, \bar{\gamma}^\beta\, k_{1\alpha} k_{2\beta}\,\}}{\mathbb{D}^3}. \tag{1.75}$$

The trace involves only 4-dimensional gamma matrices yielding:

$$q_\lambda \Gamma^{\lambda\mu_1\mu_2} = -16e^2 \epsilon^{\mu_1\mu_2\alpha\beta} k_{1\alpha} k_{2\beta} \int_0^1 dx\, dy\, dz \delta(x+y+z-1) \int \frac{d^d p}{(2\pi)^d} \frac{\hat{p}^2}{\mathbb{D}^3}\,. \tag{1.76}$$

Then we must do the momentum integral. This is not too complicated but needs attention. In fact, the procedure to be exposed will be used in our computation of the vacuum polarization tensor in LIVQED in which case the integrand will be much more involved. However this will shed a light on how the computation goes.

Typically in a d-dimensional integral we do "symmetrical integration". That is, for example, under d-dimensional integration, we make $p^\mu p^\nu \to \frac{1}{d}\, g^{\mu\nu}\, p^2$, and so on. In this case, we can (a) perform the integral over $\hat{p}$ and then do the $d-4$ integrals with usual symmetrizations or (b) integrate at once d-dimensional p. We will do (b) and to do so we must take care in symmetrizing expressions of the kind:

$$\hat{p}^\mu \hat{p}^\nu p^\alpha p^\beta \cdots\,, \tag{1.77}$$

where $p^{\alpha(\beta)}$ are full d-dimensional p's. Equations (1.78) to (1.80) below will be meant to hold under d-dimensional integration. Bars or hats denote the range over which the vector's indices run: $\bar{V}_\mu$ is a d-dimensional vector with non negative components for $\mu = 0,1,2,3$, $\hat{V}_\nu$ similarly for $\nu \neq 0,1,2,3$. Thus, with the definitions for vector algebra in d-dimensions given in eqns. (1.56) through (1.61) we can write:

$$\begin{aligned} p^\alpha p^\beta &= \frac{1}{d}\, g^{\alpha\beta}\, p^2, \\ \to \hat{p}^\mu \hat{p}^\nu &= \frac{1}{d}\, \hat{g}^{\mu\nu}\, p^2 \\ \to \hat{p}^2 &= \frac{1}{d}\, \hat{g}^{\mu\mu}\, p^2 = \left(\frac{d-4}{d}\right) p^2\,. \end{aligned} \tag{1.78}$$

This is the property we need for the evaluation of the momentum integral for the divergence of the axial vector current. We list some other examples to clarify the point.

$$\begin{aligned}\bar{p}^\mu \bar{p}^\nu &= \frac{1}{d}\,\bar{g}^{\mu\nu}\,p^2 \\ \to \bar{p}^2 &= \frac{1}{d}\,\bar{g}^{\mu\mu}\,p^2 = \left(\frac{4}{d}\right)p^2\,. \end{aligned} \tag{1.79}$$

Another example would be:

$$\begin{aligned} p^\mu p^\nu p^\alpha p^\beta &= \frac{1}{(d^2+2d)}\,\mathtt{dd}^{\mu\nu\alpha\beta}\,(p^2)^2, \\ \to p^\mu p^\nu \hat{p}^2 &= p^\mu p^\nu p^\alpha p^\beta \hat{g}_{\alpha\beta} = \frac{1}{(d^2+2d)}\,\mathtt{dd}^{\mu\nu\alpha\beta}\,\hat{g}_{\alpha\beta}\,(p^2)^2 \\ &= \left(\frac{d-4}{d^2+2d}\right) g^{\mu\nu}(p^2)^2\,, \end{aligned} \tag{1.80}$$

The notation $\mathtt{dd}^{\mu\nu\alpha\beta} = g^{\mu\nu}g^{\alpha\beta}+g^{\mu\alpha}g^{\nu\beta}+g^{\mu\beta}g^{\nu\alpha}$ is based on the one used by the symbolic manipulation program FORM [30] with which several computations in this thesis were done. The last line in (1.80) is obtained considering the definition of $g^{\mu\nu}$ in d-dimensions, were it is crucial that for it to be different form 0, both indices have to correspond in 4- or $(d-4)$-dimensions. At first sight, such "symmetrizations" may seem spurious, since as it stands, it vanishes in the limit $d \to 4$. However, this is precisely a crucial point since we will encounter cases where the outcome of the momentum integral, which multiplies the combinatorial factors obtained in the symmetrizations as the one in round bracekts in (1.80), yield poles as $d \to 4$. Recall the essential formula that the 't Hooft-Veltman dimensional regularization method produced, eq. (1.48):

$$I_d = \Gamma\left(\frac{4-d}{2}\right)\tilde{I}_d \qquad \underbrace{\longrightarrow}_{d\to 4} \qquad \left(\frac{2}{4-d} - \gamma_E\right)\tilde{I}_d\,, \tag{1.81}$$

using the expansion of Euler's Gamma function for small arguments $\Gamma(x) \to 1/x - \gamma_E + \mathcal{O}(x)$ where $\gamma_E = \Gamma'(1) \approx 0,5772\ldots$ is the Euler-Mascheroni constant. Thus, the seemingly vanishing "combinatorial" term $d-4$ in the numerator of (1.80) results in a finite contribution when it hits a simple pole of the Gamma function $\sim \frac{1}{d-4}$. In this case this situation takes place in a simple manner resulting in a non-vanishing contribution for the ABJ anomaly:

$$\begin{aligned} q_\lambda \Gamma^{\lambda\mu_1\mu_2} &= -16e^2\epsilon^{\mu_1\mu_2\alpha\beta}k_{1\alpha}k_{2\beta}\int_0^1 \mathbb{F}_p \frac{d-4}{d}\int \frac{d^dp}{(2\pi)^d}\frac{p^2}{\mathbb{D}^3} \\ &= -16e^2\epsilon^{\mu_1\mu_2\alpha\beta}k_{1\alpha}k_{2\beta}\int_0^1 \mathbb{F}_p \left(\frac{d-4}{d}\right)\frac{(-1)^2 i}{(4\pi)^{d/2}}\left(\frac{d}{2}\right)\frac{\Gamma(2-d/2)}{\Gamma(3)}\Delta^{-2+d/2} \\ &= -16e^2\epsilon^{\mu_1\mu_2\alpha\beta}k_{1\alpha}k_{2\beta}\left(\frac{d-4}{d}\right)\frac{(-1)^2 i}{(4\pi)^{d/2}}\left(\frac{d}{2}\right)\frac{\Gamma(2-d/2)}{\Gamma(3)}\frac{1}{2}\Delta^{-2+d/2}\,. \end{aligned} \tag{1.82}$$

The integral over the Feynman parameter is represented by $\mathbb{F}_p$ and it yields $1/2$. The dependence of Δ on the Feynman parameters does not affect the $\mathbb{F}_p$ integral since the momentum integral makes $\Delta \to 1$ as $d \to 4$. The momentum integral is read off from appendix (A). A last bit of algebra yieds:

$$q_\lambda \Gamma^{\lambda\mu_1\mu_2} = \frac{ie^2}{4\pi^2}\epsilon^{\mu_1\mu_2\alpha\beta}k_{1\alpha}k_{2\beta}\,. \tag{1.83}$$

And since this expression is symmetric $\{k_1 \leftrightarrow k_2; \mu_1 \leftrightarrow \mu_2\}$, the second diagram contributes with equally, with which the final result for the ABJ anomaly using the tHVBM $\overline{DR}$ is:

$$q_\lambda \Gamma^{\lambda\mu_1\mu_2} = \frac{ie^2}{2\pi^2}\epsilon^{\mu_1\mu_2\alpha\beta}k_{1\alpha}k_{2\beta}\,. \tag{1.84}$$

When we get to the point of studying the LIVQED model in consideration, we will encounter an equally subtle but much more cumbersome loop integral to be regularized. The regularization method employed will be $\overline{DR}$. To this end we will use the program FORM [30]. In order to see that our programme's rationale is correct, we computed the chiral anomaly using similar algorithms in FORM. The code and the correct result can be found in the appendices [B].

Enough details about the chiral anomaly and the tHVBM $\overline{DR}$ method. Therefore we will not see how the demand of conservation of gauge invariance produces an anomalous non-conservation of the axial vector current by the amount just derived. But this is indeed so. The interested reader may consult [9, 10].

1.4 Quantum consistency of LIV

For the LIVQED model under consideration to be consistent, as for any QFT, there are other properties, (as important as gauge invariance and in some extent related to it), that one should demand. These are *unitarity, causality, stability and renormalizability.* For them to hold, Lorentz and CPT symmetries are crucial. In [1, 2, 31, 32, 33, 34], these issues have already been analyzed for general Lorentz and CPT violating scenarios.

Further on this thesis we will elaborate on how demanding these properties can restrict the specific model of Lorentz invariance violation under consideration. For completeness the meaning of these properties is briefly explained below.

1.4.1 Unitarity

Unitarity in QFTs is basically a property inherited from QM, stating that the S-matrix must be unitary. If it were not, negative lifetimes or cross-sections result, which is nonsensical, (just as negative probability amplitudes are in quantum mechanics). In fact, the

Ward identities are most relevant for proving unitarity and renormalizability of a theory, emphasizing the need to respect the Ward identities.

1.4.2 Causality

Before the advent of SR causality meant that cause must occur before the effect. Then it was realized that the notions of before and after depend on the observers' state of motion, and it turned out that causality was satisfied if no particle or signal had a speed[11] above the speed of light c, which is the same thing as saying that no particle or signal could propagate from spacetime point x to y if these are "separated" by a spacelike interval. And then in the context of QM causality meant not only that field quanta could not travel faster than light, rather that an observable measured at spacetime point x could not influence an observable measured at y if these points are separated by a spacelike interval.

Alternatively, causality can be expressed rather independent of Special Relativity and Quantum Mechanics by the principle of *locality*, which states that (in field theory) interactions occur at a (spacetime) point meaning that the interaction between two "objects" at different spacetime points must be mediated by a "force carrier" that transmits the interaction from one point to the other.

Nevertheless, the previous definition is more "operational" and will allow us to check whether the model of LIVQED we study is causal or not.

1.4.3 Stability

In QFTs it is well known that the properties of the vacuum of the theory are very important. Among these, is that it should be stable, which requires that the Hamiltonian of the theory be bounded from below (positivity of the energy). Thus, stability may be checked directly form the equations of motion seeing whether they give rise to solutions corresponding to imaginary frequencies or not.

Another important point related with this issue is whether in the presence of LIV, different vacua (as seen by two relatively boosted observers) are equivalent? This specific point, however, will not be dealt with here.

1.4.4 Renormalizability

Despite the relevance of the "renormalization programme" in QFT (understanding that without it QFT are both hard to interpret and of little predictive power) and all its

[11] We are talking of the speed as the magnitude of the group velocity.

intricacies, rather simple criteria for establishing whether a theory is renormalizable exists. We already mentioned that a theory is said renormalizable if only a finite number of counterterms (to cancel the divergencies of the predicted observables) were required for rendering the theory finite. Equivalently, and simpler, though, a theory is said renormalizable if the coupling constants are dimensionless. It would be rather mysterious that such a delicate matter as renormalizability could be guaranteed by so trivial a computation. In fact, to demonstrate renormalizability is far more subtle which is why the above criterion is only called *naive power counting* renormalizability.

In this thesis we are concerned with Lorentz (and CPT) violations by modifications to QED that are at renormalizable at least by power counting. However, we will only study one-loop LIV effects and to leading order in the Lorentz “breaking” parameter. Of course, a definite conclusion on the status renormalizability in the presence of LIV demands a thorough analysis on possible corrections coming form higher order terms. This point is also relevant for the other properties above (stability, causality, *etc*) because higher order quantum effects may induce unexpected violations. This issue is not definite a matter and certainly is relevant for future research.

Chapter 2

LIV - Frameworks, phenomenology and tests

In this chapter I will present some of the developed frameworks leading to LIV effects. Then I will briefly review the breadth of problems where these LIV effects have measurable implications. Special attention will be given to the Standard Model Extension of Colladay and Kostelecky [1, 2, 3] and also to a reduced sector of LIV in quantum electrodynamics. Next, a general kinematical framework of modified dispersion relations, tailor-made for probing LIV phenomenology at very high energies will be exposed. Here I will focus on those cases in which the phenomenology implied is similar to the one we will find in QED with a broken Lorentz symmetry[1]. In fact in many of these examples, the experiments or observations designed to scrutinize the validity of LIV are the same. Finally I will give an account on the most stringent experimental bounds that can be imposed on the quantities that parameterize the extent of departures from Lorentz symmetry, paying special attention to the bounds imposed on effects of LIV present in the modified QED that we will consider in this thesis.

2.1 LIV Frameworks

In this section I will review some frameworks where LIV emerges as an effective theory within various different approaches. Among these we encounter LIV in (a) String Theory [41], (b) Non-commutative field theory [44, 45, 46, 47, 48, 49], (c) Supersymmetry [57, 58, 59] and (d) Preferred reference frames a revival of the old aether idea [60].

[1] Obviously the list will not be exhaustive nor complete, and it will not follow an historical order on the searches for LIV. A brief historical account can be found in [35].

Regardless the framework form which LIV may be derived, it has been applied to numerous and diverse problems ranging from microphysical to cosmological issues. For example it has been considered as a possible explanation of the confinement problem [69] in non-commutative Yang-Mills theory. Also LIV effects have been applied to compute modifications for the Casimir force between two parallel conducting plates in vacuum [70]. LIV and CPT violation effects have also been considered in the neutral and charged Higgs boson decays, $H^0 \to f^+ f^-, H^0 \to Z\,Z, H^0 \to W^+ W^-$ and $H^+ \to W^+ H^0$, where f represents quarks and charged leptons [75, 76, 77] and also in finite temperature field theory [78, 79, 80].

Naturally, this list is not exhaustive. The wide variety of areas shown, gives a clear idea on the vast implications that eventual modifications of one of the cornerstones of theoretical physics, as Lorentz symmetry is, could have.

Next we review some other frameworks which are more related to this thesis.

2.1.1 Loop quantum gravity

In the framework of canonical quantum gravity in the loop representation, known as Loop Quantum Gravity (LQG), spacetime would be discrete. This discreteness would act as a dispersive medium for particles propagating on it, wherefrom unusual kinematics is expected. Gambini and Pullin [50] and also Alfaro, Morales-Técotl and Urrutia [51, 52, 53] devised a method for obtaining effective particle equations of motions from the expectation values of the quantum Hamiltonian of the theory. These expectation values are computed with respect to particular state which approximate the spacetime geometry to a flat one. This amounts to taking averages over a box of volume $\mathcal{L}^3$, with $\ell_P \ll \mathcal{L}$, *i.e.* at a scale $\mathcal{L}$ much bigger than the Planck scale $\ell_P = \sqrt{\frac{\hbar G_N}{c^3}} \sim 1,62 \times 10^{-35}$ m, particles are approximated by the expectation value of the corresponding particle field with a flat metric. In so doing they obtain modified particle dispersion relations which reveal the breaking of Lorentz symmetry. See also [54].

2.1.2 Spacetime discreteness and spacetime fluctuations

Other frameworks that yield Lorentz violating effects are those in which spacetime is considered as having a quantum structure in the sense that it is both discrete and also "fluctuating". For example Lieu and Hillman [61] consider that the very possibility of $t_P = (\hbar G_N/c^5)^{1/2}$ being a lower bound for any time measurement (or E_P a maximum attainable energy) implies uncertainties $\sigma_t \sim t_P$ for every time measurements and uncertainties $\sigma_{x_i} \equiv c\sigma_t$ for every spatial measurement as well. Likewise for E and p, therefore:

$$\frac{\sigma_E}{E} \approx \left(\frac{E}{E_P}\right)^{\epsilon} \approx \frac{\sigma_p}{p}, \text{ with } E \approx p. \tag{2.1}$$

Variations in the dispersion relation for massless particles if the variations of E and p are independent (as those of x and p are), take the form:

$$\begin{aligned} \delta(E^2 - p^2) &= 2E\delta E - 2p\delta p = 2\sqrt{2}E^2(E^2/E_P^2)^{\alpha}, \\ \text{thus} \quad E^2 - p^2 &= \pm 2\sqrt{2}E^2\left(\frac{E}{E_P}\right)^{\epsilon}, \end{aligned} \tag{2.2}$$

where ϵ is a small parameter to be determined by experiments. Note that this is nothing but a modified dispersion relation, to be discussed later in [2.2.1].

A different approach almost touching upon the ideas of quantum geometry is that envisaged by Alfaro [62, 63] . The basic idea is that quantum gravity would distort the spacetime metric by quantum fluctuations thereby deforming the integration measure of Feynman integrals. The deformation is characterized by a parameter α. This deformation would have sizeable effects at large momenta only (*i.e.* α is expected to be small) and is manifestly Lorentz non-invariant. To grasp the effect one could think of introducing a quantum gravity Lorentz asymmetric cut-off regulator in Feynman integrals, namely:

$$\int d^d p \rightarrow \int d^d p \; R\left(\frac{k^2 + \alpha k_0^2}{\Lambda^2}\right) \tag{2.3}$$

The function R is arbitrary and suitably normalized as $R(0) = 1$ and $R(\infty) = 1$. Alternatively and in the same spirit, an extension of Dimensional Regularization is proposed. It consists in proceeding in the usual manner for DR as discussed in [1.3.2], but with a general metric $g_{\mu\nu}$ containing a minuscule LIV proportional to $(d-4)\alpha$. Gamma matrix algebra is also as discussed before but with the general metric. Thus self-energies, vertices, modified dispersion relations and maximal attainable velocities, birefringent effects and modified reaction thresholds are obtained particularly for photons, electrons, neutrons and pions, LIV effects being parameterized by α. A particular feature of this model is that LIV effects, though different for each particle, they are all governed by the one parameter α.

Other approaches encountering eventual violations of Lorentz symmetry as a consequence of a discrete spacetime can be found in [64, 65, 66, 67, 68].

2.1.3 The Standard Model Extension - SME

The minimal Standard Model Extension[2] (mSME) [1, 2] is a general framework for studying LIV phenomenology suitable for both high- and low-energy physics. It consist in adding to the Standard Model Lagrangian all possible Lorentz-violating interaction terms that are: (*i*) power counting renormalizable (*i.e.* with mass dimension ≤ 4), (*ii*) $SU(3)\times SU(2)\times U(1)$ gauge invariant[3], (*iii*) observer Lorentz scalars, (*iv*) built out of fields in the SM.

The complete mSME can be written by explictly separating the usual Standard Model (SM) sector plus the Lorentz and CPT violating ones:

$$\mathcal{L}_{\text{mSME}} = \mathcal{L}_{\text{SM}} + \mathcal{L}_{\text{LIV}}\,, \tag{2.4}$$

where Lorentz and CPT violating piece is determined by the criteria explained above. We do not need to write all the terms of the mSME, these can be found in the reference cited before. Besides, we will be mostly concerned with the QED sector. To illustrate some terms of the mSME with the LIVQED sector restricted to a single fermion species we have:

$$\begin{aligned}\mathcal{L}_{\text{LIV}} &\supset \frac{1}{2} i\,\overline{\psi}\,\Gamma^{\mu}\,\overleftrightarrow{D_{\mu}}\,\psi - \overline{\psi}M\psi \\ &- \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}(k_F)_{\alpha\beta\mu\nu}F^{\alpha\beta}F^{\mu\nu} + \frac{1}{4}\,(k_{AF})^{\alpha}\,\epsilon_{\alpha\beta\mu\nu}A^{\beta}F^{\mu\nu}.\end{aligned} \tag{2.5}$$

where,

$$\Gamma^{\mu} = \gamma^{\mu} + c^{\mu\nu}\gamma_{\nu} + d^{\mu\nu}\gamma_5\gamma_{\nu} + e^{\mu} + if^{\mu}\gamma_5 + \frac{1}{2}g^{\mu\nu\lambda}\sigma_{\nu\lambda}, \tag{2.6}$$

$$M = m + a_{\mu}\gamma^{\mu} + b_{\mu}\gamma_5\gamma^{\mu} + \frac{1}{2}H^{\mu\nu}\sigma_{\mu\nu}. \tag{2.7}$$

Covariant derivatives are denoted by $D_{\mu} \equiv \partial_{\mu} + iqA_{\mu}$ and $A\overleftrightarrow{\partial_{\mu}}B \equiv A\partial_{\mu}B - (\partial_{\mu}A)B$. The tensors $a^{\mu}, b^{\mu}, c^{\mu\nu}, d^{\mu\nu}, e^{\mu}, f^{\mu}, g^{\mu\nu\lambda}, H^{\mu\nu}, k_F, k_{AF}$ determine the extent to which Lorentz and CPT symmetries are violated. It must be emphasized again that as it was pointed out at the beginning of this thesis, observer Lorentz symmetry is indeed preserved since $\mathcal{L}_{\text{LIV}}$ is independent of the choice of coordinates, but particle Lorentz symmetry is not, since the physical fields, ψ, A^{μ} transform accordingly with it, while the previously mentioned tensors remain unaltered. Furthermore, if for given values of

[2]This is only a subset of the original SME, which includes also LIV interaction between SM fields and gravitational ones as well which is not the purpose of study of this thesis.

[3]The $SU(3)\times SU(2)\times U(1)$ gauge invariance of the complete mSME is explicitly seen when the mSME Lagrangian is written in terms of the appropriate SM gauge multiplets plus the LIV couplings. For brevity, here we will only comment a reduced sector of the theory which proceeds from the mSME in an analogous fashion as QED does form the SM.

the components of the LIV tensors above, full quantum consistency (that associated with the issues of stability, causality, *etc*) is achieved, then observer Lorentz symmetry guarantees full quantum consistency in all other frames.

Also, not all the LIV terms shown must be present in the mSME, or at least have observable effects. For example, considering a single massive Dirac field $\psi(x)$ in four dimensions,

$$\mathcal{L}[\psi] = \mathcal{L}_{\text{Dirac}}[\psi] + \mathcal{L}_{\text{LIV}}[\psi], \tag{2.8}$$

where $\mathcal{L}_{\text{LIV}}$ represents some or all the terms coming form the gamma-matrix structure of (2.6) on the first line of (2.5) and $\mathcal{L}_{\text{Dirac}}$ is the usual Dirac Lagrangian with Lorentz and CPT symmetry. If only a_μ were different from zero, then under the redefinition $\psi' = \exp(ia_\mu x^\mu)\psi$ one can show that $\mathcal{L}[\psi = \exp(-ia_\mu x^\mu)\psi'] = \mathcal{L}_{\text{Dirac}}[\psi']$. Therefore the presumably LIV model for Dirac fermions is equivalent to the free and Lorentz invariant Dirac theory, *i.e.* without Lorentz or CPT breaking. At this point we must mention that the observable effects coming from the b_μ term cannot be eliminated by field redefinitions, even in the simple case of a single massive fermion.

To end this brief review of the SME, we mention that the complete mSME, includes similar terms involving general LIV couplings in the lepton, quark, gauge and Higgs sectors, as well as LIV Yukawa couplings between fermions and the Higgs field.

2.1.4 LIV in QED

The most relevant context for this thesis, where LIV is considered, is that of QED[4]. Research in this field was initiated by Carroll, Field and Jackiw (CFJ) [83] and still generates considerable attention. In the so-called CFJ model , the effects of a Lorentz violating modification of electrodynamics by the introduction of a kind of a CPT-odd Chern-Simons (CS) term is investigated,

$$\mathcal{L}_{\text{CFJ}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}p_\mu A_\nu \tilde{F}^{\mu\nu}. \tag{2.9}$$

The second term is a four dimensional analogue of a Chern-Simons term which couples the CS vector p_μ to the electromagnetic field and $\tilde{F}^{\alpha\beta} = \frac{1}{2}\epsilon^{\alpha\beta\mu\nu}F_{\mu\nu}$ is the dual field strength tensor. Under gauge transformations of the electromagnetic field $\delta A_\mu = \partial_\mu\Lambda$, the CS Lagrangian varies (up to a divergence) as:

$$\delta\mathcal{L}_{\text{CS}} = \frac{1}{4}\Lambda\tilde{F}^{\mu\nu}(\partial_\nu p_\mu - \partial_\mu p_\nu), \tag{2.10}$$

[4]Classical electrodynamics also presents LIV when considered as an effective theory. For example, see [81, 82].

therefore gauge invariance for arbitrary Λ requires $\partial_\mu p_\nu = 0$. If this holds in flat spacetime[5], it will vanish in all frames and p_μ could be regarded as a "constant" of Nature, picking out a preferred direction in space. This is why this model is said to break Lorentz symmetry. However, for the variation of $\mathcal{L}_{\text{CS}}$ to vanish identically it suffices that p_μ be the gradient of some scalar field $\partial_\mu\theta$. Thus such a modification, in spite of introducing a mass parameter, respects gauge symmetry at the expense of violating Lorentz symmetry. More about the possible interpretation of p_μ later.

By the way, such a CS term is also included in the mSME, $(k_{AF})_\mu$ being proportional to the CS vector p_μ. Readily it was realized that this CS term leads to measurable effects on the modified photodynamics, opening a window for probing eventual LIVs. Furthermore, it also became evident that this CS term could be induced by radiative corrections from other sectors of the theory. In fact, a LIV coupling in the fermion sector, does generate radiative corrections to CS term and this LIV coupling is the one we have commented before, namely $\sim\ \overline{\psi}\,\not{b}\,\gamma_5\,\psi$. In chapter [3] we will devote to the study of this issue and many of its intricacies. Particular attention will be payed to the quantum consistency of this theory, its phenomenology and the relation with experimental observations. For recent developments in this field see [5, 1, 2, 3, 31, 32, 33, 34, 83, 84, 85, 86].

In many of the frameworks mentioned before, the phenomenology implied is similar, therefore before going on deeper, let us review some advances in LIV phenomenology and on the experimental tests of Lorentz symmetry violation.

2.2 Modified Dispersion Relations

2.2.1 General features

A generally accepted means of probing Lorentz symmetry is the study of particle kinematics at high energies since most LIV frameworks result in modified dispersion relations. Inasmuch the same way that rotational symmetry $O(3)$ is encoded in the fact that vectors have rotational invariant squared norm, Lorentz symmetry $SO(1,3)$ is encoded in the invariance of the Minkowsky squared norm of four-vectors, particularly that of the energy-momentum four-vector p_μ, wherefrom free dispersion relations result $m^2 \equiv p_\mu p^\mu = p^2 = E^2 - \mathbf{p}^2$. We will therefore focus on the parameters involved in such modifications which will usually assume the form:

[5] In a curved spacetime as our Universe is, a constant vector should satisfy the covariant equation $\nabla_\mu p_\nu = 0$. However, a vector cannot have a vanishing covariant derivative everywhere on a curved mainfold, nevertheless, we are already assuming that p_μ picked out a preferred direction and therefore a preferred frame where $\partial_\mu p_\nu = 0$. If this is so, then $\partial_\mu p_\nu - \partial_\nu p_\mu = \nabla_\mu p_\nu - \nabla_\nu p_\mu = 0$ in any frame.

$$E^2 = m^2 + \mathbf{p}^2 + \Delta(E, \mathbf{p}). \tag{2.11}$$

The extent of violation of Lorentz symmetry is quantified by $\Delta(E, \mathbf{p})$ [6].

According to our discussion on the sense of LIV in section [1.1.2], this modification must be a Lorentz scalar. Therefore a plausible general ansatz for Δ is:

$$\Delta(E, \mathbf{p}) = F^{(1)}_{\mu_1} p^{\mu_1} + F^{(2)}_{\mu_1\mu_2} p^{\mu_1} p^{\mu_2} + \cdots + F^{(n)}_{\mu_1\mu_2\cdots\mu_n} p^{\mu_1} p^{\mu_2} \cdots p^{\mu_n} + \cdots \tag{2.12}$$

where the coefficients $F^{(n)}_{\mu_1\mu_2\cdots\mu_n}$ are dimensionful and arbitrary but presumably such that no modification whatsoever is implied at small energies. Some consequences of this ansatz were studied in [87].

Typically, rotational symmetry is preserved and furthermore in the expression above it is customary to introduce a model dependent energy scale, E_Λ, interpreted as the scale at which LIV effects appear and dimensionless constants, $f^{(n)}$, that could depend on the particle species, their energy E and $|\mathbf{p}|$. Thus the modified dispersion relation, MDR hereafter, reads:

$$E^2 = m^2 + \mathbf{p}^2 + \sum_n f^{(n)} \frac{|\mathbf{p}|^n}{E_\Lambda^{n-2}}. \tag{2.13}$$

These MDRs have various consequences the more readily noticeable being the lifting or lowering of the threshold energies for certain reaction/decays. This allows for immediately measuring its effects and therefore the analysis of well establish particles reactions/decays could serve as test of LIV.

Actually, on the basis that SR has been tested with outstanding accuracy in high energy particle accelerators, it has been a shared view that any trace of LIV must appear when the energies of the phenomena studied are extremely high, far beyond those accessible in particle accelerators. In this regime particles are ultrarelativistic and the MDR read:

$$E^2 = m^2 + \mathbf{p}^2 + \sum_n \frac{f^{(n)}}{E_\Lambda^{n-2}} E^n. \tag{2.14}$$

At low energies no departures form Lorentz symmetry is observed, therefore the “LIV scale” is typically very large and we see the LIV modification is highly suppressed. This is why the arena of ultra high energy astrophysical processes has been considered as one of the most promising areas in which LIV could be measured, though not the only one, (more about this later [2.2.3]). Particularly, ultra high energy cosmic rays (UHECRS)

[6] Naturally this $\Delta(E, \mathbf{p})$ function will depend on LIV parameters we encountered above, namely the SME tensors as those in eqn. (2.6) or the constant background axial vector b_μ.

whose energies climb ut to $\sim 10^{20}$ eV, are expected to reveal such MDRs in sign of a non exact Lorentz symmetry. In fact it was precisely these modified dispersion relations, that could give a possible explanation to long standing puzzles of astroparticle physics which we review below.

2.2.2 Old puzzles, new radical solutions

The GZK paradox

In 1966 Greisen [88] and Zatsepin & Kuzmin [89] noticed that particles traveling through the Cosmos may interact with the omnipresent Cosmic Microwave Background Radiation CMBR, a totally thermalized photon bath as discovered by Penzias and Wilson [90], with an almost perfect black body spectrum at $2,725°$K. They proposed that a highly energetic proton, typically of energies as large as $\sim 10^{20}$ eV, in its own reference frame, would see the CMBR photons as extremely energetic ones. If the proton is energetic enough, nuclear reactions of the kind

$$p + \gamma_{CMB} \rightarrow p + \pi^0 \tag{2.15}$$

may occur, decreasing the initial energy of the primary particle, an effect sometimes called *Cosmic friction*. Simple relativistic kinematics yields the threshold energy for such reaction, namely:

$$E_{th} = \frac{m_\pi^2 + 2m_\pi m_p}{4E_{\gamma_{CMB}}} \approx 4 \times 10^{19} \text{ eV}. \tag{2.16}$$

The previous equation implies no Cosmic rays with energies above $\sim 4 \times 10^{19}$ eV should be seen here on Earth. Later Stecker [91] refined this analysis due to the fact that not all photons of the CMBR have the same energy. Therefore the relevant piece of information was how long a given proton can travel in the photon bath of the CMBR if it starts with an energy of a typical Cosmic ray and the answer is that the maximum distance a proton with an energy $\gtrsim 1 \times 10^{20}$ eV could travel without decreasing its energy by the "Cosmic friction" is ~ 100 Mpc (1 Mpc $\approx 3.2 \times 10^6$ ly $\approx 3 \times 10^{22}$ m).

Then the natural question arose. Is this cut-off really seen, or do we receive the so called "Super GZK events", that violate it? In order to call a Cosmic ray as "super GZK" it does not suffice for it to have an energy above the cut-off but also to have traveled a distance larger than ~ 100 Mpc. Few years ago this was a controversial issue based on the lack of agreement between the two larger observational groups at the time. The High Resolution Fly's Eye Collaboration (HiRes) and the Akeno Giant Air Shower Array Collaboration (AGASA). The first group claim the cut-off has been observed [92]. More recently this same group claim to confirm their previous observation [95]. The second group, however, claim to have recorded events well above the cut-off [93, 94], however, no astronomical sources of such energetic events was known to lie closer than 100

Mpc, therefore these must have been produced beyond the GZK frontier, and therefore paradoxical, but why do they still reach our detectors, or do they?

If they did, then a LIV modification of the dispersion relation of protons were enough to up-shift the threshold energy for the photoproduction of pions and forbid the reaction, then these protons could arrive to the Earth's surface and solve the GZK paradox [97, 98, 99, 100, 101, 102, 103, 104, 105, 106, 107]. Now the Pierre Auger Collaboration [108] claim that the GZK-paradox is not such, *i.e.* they claim that although some events with energies above the GZK barrier are seen, the flux of these would indeed fall off as expected [96], concurring with the observations of HiRes. Low statistics and all the intricacies of the experimental accuracies may imply that these results are not definite still. Furthermore, if the GZK cut-off were in fact confirmed, still we would be left with the question as to which is the mechanism that produce such extremely energetic processes and what the sources are.

With time, and prior to PAO's outcomes, it was soon realized that LIV phenomenology was far more reaching than just explaining the GZK paradox, and the interest and relevance for LIV searches prevailed on its own[7].

TeV-photons

A similar (seemingly) paradoxical situation was found for photons from astrophysical sources. Photons of approximately 20 TeV coming from the BL Lac object Mrk 501, located at a distance of $\sim$ 150 Mpc form us have been observed. As in the case of protons interacting with the CMB photons, these photons should interact with the far infrared background radiation FIBR. According to [114], given the energies of the Mrk photons and those of the FIBR, this interaction should result in a pair-creation and therefore these TeV photons should not be observed, unless threshold energies were (LIV) modified. Nevertheless, this TeV photon cut-off violation was subject to criticism [115, 116], but still the idea of LIV motivated threshold modifications persisted.

2.2.3 "Low" energy phenomenology

Despite the potential of searching for LIV in the ultra-high energy sector, it is indeed as promising to search in low energy physics (the role of high energy is now played by high accuracy of "terrestrial" experiments). In fact, Lorentz and CPT symmetry are very accurately measured by experiments of atomic systems where the main interactions are those of QED. Their phenomenology is suitably described by the mSME Lagrangian

[7] For alternative explanations posed for the GZK paradox, see [109, 110, 111, 112, 113].

restricted to the QED sector (2.5) and the reason why these experiments in atomic systems serve as good tests of Lorentz and CPT symmetry is because atomic systems are very sensitive in detecting the slightest energy shifts, which as we have seen, are implied by LIV or CPT violation.

2.3 Experimental tests of Lorentz Invariance

Before reviewing the bounds imposed by different experiments on LIV, let us recall the following points:

- By LIV we do not mean a loss of covariance,
- LIV may be different and specific to different particle sectors,
- If LIV is generated at very high energies, say, by quantum gravity, it is most likely that a tiny LIV will be present in lower energies too.
- Lorentz symmetry has withstood all the experimental tests devised so far within the present day attainable energies,
- Any experimental trace of LIV in low- and high-energy physics must be minute. In low-energy physics (as compared to those in astrophysical processes) detectability is possible by the outstanding accuracies of the experiments performed. In (ultra) high-energy physics accuracy is not as good but, as expected, LIV effects scale with energy and hence allow for experimental detection,
- Typically, LIV in the low-energy sector is classified by the LIV tensors of the SME. In the (ultra) high-energy sector, LIV is generically classified by the MDRs above (2.14). If LIV is understood as an effective feature of quantum gravity then the natural LIV scale is the Planck energy $E_P = \sqrt{\frac{\hbar c^5}{G_N}} \approx 1,2 \times 10^{19}\, GeV$ which acts as a supression factor in the MDRs, nevertheless the huge energies of astrophysical observations allow for detection.

Thus, experimental test of Lorentz invariance translate on bounds on the parameters (the LIV tensors) of the mSME or or to bounds on the $f^{(n)}$ functions, altogether referred to as LIV parameters. Since LIV effects may be specific for different particles, the LIV parameters should be understood for each of these, typically denoted by a particle sub-index.

The above being said, let us review some of the experimental tests of Lorentz symmetry and some of the constraints on the LIV coefficients found so far. Further details can be found [117, 118, 119] and in references therein.

2.3.1 Experiments of LIV or CPT violation in QED

The list of experiments to probe Lorentz and CPT symmetry is indeed very large. In this brief review we could hardly do justice to them all by trying to include them here, each of which, by the way, is worth a serious and exclusive discussion. Many of these include earthly experiments pertaining different particle sectors including photons, electrons, neutrons, protons, mesons, neutrinos and others, see [118, 117] and references therein. However, among the most accurate experimental tests of Lorentz and CPT symmetry are those performed with photons and with particle or atomic systems, where the interactions are basically described by QED[8]. Let us mention but a few.

Atomic Clock-comparison experiments

In these experiments two high-precision atomic clocks composed by different atomic systems are compared as the Earth rotates. Typically the atomic clock frequencies are those emitted or absorbed on hyperfine or Zeeman transitions between the energy levels of the atomic system conforming the clock. These levels would be extremely sensitive to its orientation with respect to would-be Lorentz violating constant background vectors. Thus an atomic clock mounted on a satellite, for example, changes its orientation during the orbit, which in turns modifies the level spacing of the atom, producing a different frequency of light and therefore altering the ticking rate of the atomic clock.

Electrons and positrons in Penning traps

Lorentz and CPT violation experiments are also done with leptons in Penning Traps by comparing particles and antiparticles. A Penning trap is a device in which charged particles are trapped by electric and magnetic fields for long periods of time. The magnetic field confine the particles to move on a helix, while the electric field traps it in the longitudinal direction. While in the trap, the cyclotron frequency and the Larmor frequency can be measured accurately and their quotient determines the g-factor, $\omega_s/\omega_c = g/2$. Their difference defines the anomaly frequency $\omega_a = \omega_s - \omega_c$. The energy levels for a spin 1/2 particle in the trap is determined by the spin s, the quantum number n and the frequencies ω_s and ω_c. Under specific conditions two almost degenerate levels exists for which transitions can be induced by, for example, and oscillating external magnetic field, allowing to determine the value of ω_a. Ultimately the accurately measured anomaly frequency ω_a is related to the LIV parameters and thus provides a means to constrain the LIV parameters.

[8] The reason why these systems are so good at providing Lorentz or CPT test is that they are extremely sensitive to the slightest energy shifts which would be produced by LIV or CPT violations.

A wealth of very accurate and interesting experiments of the kinds above have been devised. They have allowed to constrain many parameters of the SME. The bounds obtained, though extremely relevant, are not presented here, leaving space for those we find more pertinent for our aims.

Spectral polarimetry experiments

This experimental technique is particularly interesting in this thesis and consist on the following. As we have already mentioned a general imprint of LIV is the modification of a given particle's dispersion relation. In many frameworks which incorporate LIV, once the photon dispersion relation are obtained one finds the phenomenon of birefringence of vacuum[9]. Namely, plane-wave solutions to the (modified) Maxwell equations with different energies/velocities for different polarization modes. Let us recall that in ordinary electrodynamics the polarization of light is given by the direction of the electric field which has 2 polarization modes, both travelling equally fast results in no change in the polarization of light (unless it travelled along a magnetic field, where the so-called Faraday effect may take place). If however, the modes travel at different speeds then the net polarization of light changes during its propagation.

If such a birefringent effects were due to LIV (or quantum gravity) they will most certainly be tiny ones. The point is that the net change in the polarization, as measured by the slight rotation of the polarization plane, is proportional to the distance travelled, hence looking further and further acts as a magnifier for such tiny effects.

In the CFJ model presented in (2.9) Lorentz and CPT violation was introduced by means of a constant background vector p_α coupled the the photon field A_β and the dual strength tensor $\tilde{F}^{\alpha\beta}$ to form a 4-dimensional Chern-Simons like term. Solving the modified Maxwell equations and assuming plane-wave solutions they arrive at the photon dispersion relations. Experience confirms $p^2 \ll k^2$, where $k_\alpha = (k_0 = E, \mathbf{k})$ is the photon's four-vector, thus the dispersion relation takes the form:

$$|\mathbf{k}| = E \mp \frac{1}{2}(p_0 - |\mathbf{p}|\cos\theta), \qquad \cos\theta = \frac{\mathbf{p}\cdot\mathbf{k}}{|\mathbf{p}||\mathbf{k}|}\,. \tag{2.17}$$

The $-(+)$ive branch corresponding to left(right)-polarized photons. Since the phase change of circularly polarized light travelling over a distance L is $\phi = kL$, the rotation of the polarization plane is $\Delta\phi\frac{1}{2}(\phi_L - \phi_R) = -\frac{1}{2}(p_0 - |\mathbf{p}|\cos\theta)L$.

Detailed observations of the plane of polarization of light form distant galaxies where used by CFJ without observing any substantial rotation of the polarization plane due to

[9] This feature will be found in our LIVQED model (3.4.3)!

the CS Lorentz violating vector p_α, wherefrom a bound for the CS vector was obtained To this end one typically assumes $L \sim t$ with universal $c = 1$. Of course that in this context this will not be so, but taking different c's for different modes would result in higher order effects[10] .The thorough analysis[11] of nearly 150 galaxies lead them to consider a rotation of the polarization plane $\Delta\phi \leq 6.0°$, resulting in the very stringent bound:

$$p_0 - |\mathbf{p}| \cos\theta \leq 1.7 \times 10^{-42} h_0\, GeV. \tag{2.18}$$

where h_0 is the Hubble constant in unit of $100\,\mathrm{km}\,(\sec \mathrm{Mpc})^{-1}$. With the current date observation data $0.5 \lesssim h_0 \lesssim 1.0$, a crude estimate yields:

$$p_{CS} < \times 10^{-31}\, eV. \tag{2.19}$$

Recent searches [142] and more recently using WMAP and BOOMERANG data [143] have found an additional (to the expected Faraday) rotation of the plane of polarization that could be due to LIV or CPT violation of order:

$$|\Delta\phi| < 6.0° \pm 4.0°. \tag{2.20}$$

This time however the accuracies were improved and the objects considered were much farther, the last group for example, taking objects at $z \approx 20$. The bounds the the LIV vector p_α are consequently more stringent. Nevertheless, in section [2.3.4] below we will consider these bounds cautiously when it comes to constraining the LIV vector b_μ of the particular model considered in this thesis.

2.3.2 Test of LIV induce by a constant background axial vector

Now we will focus on those experimental test that have relation with the phenomenology of the particular LIVQED model to studied in chapter [3], namely, a model in which LIV is induced by a constant background axial vector b^μ coupled to fermions in QED:

$$\mathcal{L}_{LIV} = \overline{\psi} b^\mu \gamma_\mu \gamma^5 \psi\,. \tag{2.21}$$

This coupling will result in modification of fermion and photon dynamics. As will be shown in chapter [3], this coupling will produce one-loop radiative corrections in the photonic sector yielding:

$$\begin{aligned} \mathcal{L}_\gamma \;\; &= \;\; -\frac{1}{4}\left(1 + \xi\,\frac{b^2}{m_e^2}\right) F^{\nu\lambda} F_{\nu\lambda} + \xi\,\frac{b_\nu b^\rho}{2m_e^2}\, F^{\nu\lambda} F_{\rho\lambda} \\ &\quad -\frac{1}{2}\,\zeta\, b_\nu A_\lambda \widetilde{F}^{\nu\lambda} + \frac{1}{2}\, m_\gamma^2\, A_\nu A^\nu + B\, \partial_\nu A^\nu\,, \end{aligned} \tag{2.22}$$

[10] Also the evolution of a matter-dominated Universe is taken into account *i.e.* the time during the emission of light at red-shift z and the moment we observe it is $t = t_0 \left[1 - (1+z)^{3/2}\right]$ with $t_0 = 2/3H_0$ where t_0 is the present age of the Universe and $H_0 = (\dot{R}/R)_0$ is the Hubble constant.

[11] This of course, takes into account the effect of Faraday rotation of the observed radiation as it traverses magnetic fields during its propagation. Such an effect is "subtracted" in order to associate the cosmological birefringence solely to the LIV effect.

where the induced coupling constants ξ, ζ result of order α, the fine structure constant and are computed in the next chapter in section [3.2.2]. Bounds from the analysis of such a LIV term in the purely fermionic sector are essentially different from those of the photon sector. In either cases experimental bounds stem from the plane wave solutions to the equations of motion derived from Lagrangians above and the corresponding dispersion relations obtained. Among the phenomenological implications of this model in the photon sector, we will find:

- variation of the speed of light,
- birefringence effects,
- different times of flight for different energies and
- an induced photon mass.

Thus we will focus on the experimental bounds that have been imposed on (a) the speed of light c, (b) the photon mass m_γ and (c) the LIV vector b^μ.

2.3.3 Bounds on the speed of light c

Naturally speeds of light different from c must be minute, otherwise these would have been detected experimentally long ago. This can be written as:

$$c^2 = 1 + \epsilon, \tag{2.23}$$

where ϵ is much smaller than 1[12]. Such a speed of light can be obtained from a modified photon dispersion relation as those of eqn. (2.11) with a LIV function $\Delta = \epsilon \mathbf{p}^2$ where ϵ is a dimensionless constant and possibly dependent on the LIV model. In particular, we will see that the one-loop induced photon Lagrangian above yields such a modification *i.e.* we will find a similar ϵ dependent on the LIV vector b^μ. In ref. [151], the following bound was obtained:

$$|1 - c^2| = |\epsilon| < 6 \times 10^{-22}\,. \tag{2.24}$$

This was done by considering that such a modification $c^2 = 1 + \epsilon$ breaks Lorentz invariance yet translational and rotational symmetries are preserved (only) in a so-called "preferred reference frame". If this preferred frame is that in which the CMB is isotropic, then minuscule anisotropies in laboratory experiments should appear. Yet high-precision spectroscopic experiments failed to find such anisotropies, wherefrom the bound above was obtained. In [152] Coleman and Glashow elaborated further on this analysis and extended it to the case of particles having maximal attainable velocities different to the speed of light is analyzed as indicative of LIV. After their work, the idea of writing

[12] Here we are using the notation that the (usual) speed of light in vacuo, c is 1.

kinematical models for studying LIV phenomenology in connection to ultra-high energy processes gained considerable attention.

2.3.4 Birefringence effects

The bound for the LIV vector p_α obtained by Carroll, Field and Jackiw is indeed very stringent. However, the CFJ model considers a bare CS term which is intrinsically CPT-odd whereas we will focus on the CPT-even part of the one-loop radiatively induced photon sector by the coupling $\overline{\psi} b^\mu \gamma_\mu \gamma^5 \psi$ in the fermion Lagrangian. A CPT-odd part is also induced but that has already been studied in [136]. Furthermore, in our case it will turn out that birefringence will be dependent on the angle between the LIV vector and the wave vector. Also, there is not certainty as to when in the evolution of the Universe did any CPT-odd violation takes places. This together with the above may cause a total smearing of any trace of birefringence as due to a CPT-odd effect if the distances involved are extremely large, say corresponding to very early stages of the Universe. Thus in order to obtain a more pertinent bound for the LIV and CPT-even vector b_μ considered in this thesis we shall rely more upon bounds related to CPT-even terms and also those that consider nearer sources as compared to cosmological ones.

2.3.5 Times of flight effects

Although there is no definite consensus on the proper definition of velocity in a LIV scenario, it is generally assumed that velocity is given by:

$$v = \frac{\partial E}{\partial p} \qquad p = |\mathbf{p}|. \tag{2.25}$$

If this is the case two possibilities for experimental tests emerge.

Energy dependent velocity

If we focus on LIV as seen by the modified dispersion relation of the kind (2.14) we will find that the velocity of a photon is given by:

$$c = 1 + \frac{(n-1) f_\gamma^{(n)} E^{n-2}}{E_\Lambda^{n-2}}\,, \tag{2.26}$$

This has been noted by [154] in the context of Quantum Gravity motivated LIV, therefore the natural LIV scale would be the Planck energy scale ($E_P \sim 1,2 \times 10^{19}\, GeV$). We see that if $n \neq 2$ the velocity is energy dependent and two photons emitted simultaneously at a distant source, with slightly different energies, will not arrive together. For given energies and distance travelled, the time difference is a function of the LIV function f. Present day devices allow for extremely accurate measurements of the difference in the arrival of light signals to detectors. Observations of the rapid flares of the blazar Markarian 421 at $z = 0.03$, revealed the emission of highly correlated photons in the 1-2 TeV range on a timescale of 280 s. Assuming the emission is simultaneous and due to the same event at the source, allows to estimate that a delay in the time of arrival of 1 and 2 TeV photons should be less than 280 s. In so doing they obtained following bound for the LIV function $|f_\gamma^{(3)}| < 128$. Going back to the equation (2.26) we see a modification of the speed of light of the form:

$$c^2 = 1 + \epsilon' = 1 + 2f_\gamma^{(3)} \frac{E}{E_P} \,. \tag{2.27}$$

Introducing the numerical values of the energies involved and the bound for $f_\gamma^{(3)}$ we find $|\epsilon'| < 1.3 \times 10^{-9}$. Although this is a very interesting approach form the experimental point of view, as a means of probing LIV it is not good enough to compete with the bound on modifications of the speed of light cited in eqn. (2.24).

Polarization mode dependent velocity

The other possibility for testing LIV related with the light velocity is by its dependence on the polarization mode. In [120] the authors write the photon dispersion relation as:

$$E_\pm = k(1 + \rho \pm \sigma) \,, \qquad k = |\mathbf{k}|. \tag{2.28}$$

where the LIV contributions ρ, σ are related to the LIV and CPT-even part of the purely photon sector of the mSME, in other words, σ and ρ are ultimately related with the LIV vector that we consider in this thesis, b_μ. With the above definition of light velocity, the difference in the velocities of the different polarization modes is:

$$\Delta v = v_+ - v_- = 2\sigma \,, \tag{2.29}$$

with possible measurable consequences over the times of flight and polarization of radiation. From the analysis of (some) relatively near sources $L \lesssim 10\,\text{kpc}$ the velocity constraint (similar to the one obtained above), written in terms of our LIV vector b_μ is:

$$|\vec{b}_e| < 10^{-18}\, eV \,. \tag{2.30}$$

2.3.6 Bounds on the photon mass m_γ

The photon mass has been experimentally constrained by diverse experiments ranging form $m_\gamma < 10^{-7}\, eV$ to $m_\gamma < 10^{-32}\, eV$. These extreme values are arguable though and the commonly accepted value for the bound of the photon mass as published by the Particle Data Group [149] is:

$$m_\gamma < 6 \times 10^{-17}\, eV. \tag{2.31}$$

2.3.7 Bounds on the LIV vector b^μ

Form the above mentioned measurements let us consider other indirect bounds can be posed on the LIV vector b^μ. These will be obtained by analyzing how the speed of light, the birefringence effect and the photon mass specifically depend on the LIV vector b^μ. For this we need to know explicitly the one-loop induced photon action which will be done in section [3.2.2] and the photon dispersion relations derived form it, in different cases. The nature of these will be explained in [3.4.3][13], here we just present the different cases.

Light-like b^μ (with no bare photon mass)

In this case the dispersion relations are:

$$b^\mu = (|\vec{b}|, \vec{b})\,, \quad \mu_\gamma = 0 \quad \Rightarrow \begin{cases} p^0_+ + |\,\vec{b}\,| = \pm\sqrt{\left(\vec{p}+\vec{b}\right)^2 + m_f^2} & \\ p^0_- - |\,\vec{b}\,| = \pm\sqrt{\left(\vec{p}-\vec{b}\right)^2 + m_f^2} & \text{for fermions} \\ \hline k_0 \simeq |\,\vec{k}\,|(1+\delta c_\theta) \mp \zeta\,|\,\vec{b}\,|\,\sin^2\theta/2 & \text{for photons} \end{cases}$$

Where $\cos\theta \equiv \frac{\vec{b}\cdot\vec{k}}{|\,\vec{b}\,|\,|\,\vec{k}\,|}$ is the angle between the LIV vector and the wave vector and $\delta c_\theta \equiv \frac{2\xi}{m_e^2}\,|\,\vec{b}\,|^2\,\sin^4\theta/2$. We will determine the induced coupling constants ξ, ζ and these will be of order α.

If we focus on the photon dispersion relation to contrast with the measurements above, as anticipated, we find a modification of the speed of light and a birefringent effect. The latter has already been commented above. If we take the bound $|1-c^2| < 6\times 10^{-22}$ form [151, 152] and use it to constrain δc_θ, we find (of course this is a crude estimate for it depends on the angle θ, but we don't need to be very precise in this point):

$$|\vec{b}_\gamma| < \times 10^{-18}\text{eV}\,, \tag{2.32}$$

which is of the same order of magnitude of the bound obtained above.

Time-like b^μ (with bare photon mass)

In this case we will also focus on photodynamics only. As will be emphasized in sec. [3.4.3] the possibility of having a time-like b^μ was ruled out theoretically since it leads to tachyonic massive photon and instability of photodynamics. However in ref. [5] we noted that if one allows for a bare photon mass, together with the LIV induced, the consistency

[13] There we will also see how yet more conditions can be imposed on b_μ regarding its spacetime nature, by studying quantum consistency issues.

problems in the photonic sector by a time-like b^μ could by amended. With $b^\mu = (b^0, \vec{0})$ and $m_\gamma^2 = \mu_\gamma^2 + \delta m_\gamma^2$ the photon dispersion relation reads:

$$k_0^2 \approx \kappa + m_\gamma^2 - \frac{b_0^2 \zeta^2}{4}\,, \tag{2.33}$$

where κ is a positive quantity. With the obtained value for the induced coupling constant $\zeta = 16\frac{\alpha}{\pi}$[14]. we see that if $m_\gamma \geq \zeta b_0/2 = 8\alpha b_0/\pi$, then the photon energy remains real for all wave vector $\vec{k}$ and the problems of photon instabilities are no longer present, thus the possibility for a time-like b^μ is not precluded any more. Finally using the commonly accepted bound on the photon mass mentioned above [2.3.6], $m_\gamma < 6 \times 10^{-17}$ eV we obtain another bound for (a time-like) LIV vector:

$$b_0 < 3 \times 10^{-15}\text{eV}\,. \tag{2.34}$$

We cannot finish this chapter without emphasizing again that the list of experimental test of Lorentz invariance violation presented is just a small part of all the experiments done on the subject, focusing on those that are most related with the theoretical and phenomenological searches addressed in this thesis.

[14]As mentioned in eqn. (3.37) this is obtained by analyzing the CPT-odd contribution to the photon Lagrangian as done in ref. [136].

Chapter 3

Lorentz and CPT Symmetry breaking in QED

This chapter is based on the results reported in [5] which constitute a novel and independent contribution for the thorough understanding of the field of LIV in QED. However as we saw in chapter [2], Lorentz and CPT symmetry can be violated in many different ways and as a consequence of radically different effects. Here we will be concerned with a very particular kind of LIV in QED which consist on the study of how fermions coupled to a constant background axial-vector induce radiative corrections to the Maxwell Lagrangian. These induced effects break Lorentz symmetry in the sense described after equation (2.10) and they contain both CPT-even and CPT-odd contributions. We will focus on the former, the latter, leading precisely to an induced Chern-Simons like term as in the CFJ model, was studied in [136]. To this aim we computed the exact modified fermion propagator and thus the radiatively induced modifications in the photonic sector to second order in the coupling constant e by means of the one-loop vacuum polarization tensor $\Pi_2^{\mu\nu}(b,k)$. For this calculation we made use of the 't Hooft-Veltman-Breitenlohner-Maison dimensional regularization scheme $\overline{DR}$ [6, 25, 26]. As commented on section [1.3] of chapter [1], this regularization scheme is perfectly justified for this context. Our results do indeed preserve gauge invariance, contrary to what has been reported in [121].

I will begin by presenting the LIV model under consideration supported by some motivations, then I will present some other works on the field, stating the difference with ours. Next the calculation of the induced LIV effects on the photonic sector is presented, and the corresponding modified particle kinematics for fermions and photons. Finally I will discuss our results, by a critical examination concerning the consistency of our analysis, related to the issues of stability, causality, to then discuss the phenomenology implied and its relation with experimental observations commented at the end of the previous chapter.

3.1 Fermions coupled to a constant axial-vector background

Our starting point will be the classical spinor Lagrange density for a given fermion species f of electric charge eq_f and mass m_f coupled to a background axial-vector b_f^μ and also (minimally) to photons. In principle we allow for each fermion species to couple to different LIV vectors.

$$\mathcal{L}_f = \overline{\psi}_f \Big(i\not{\partial} - m_f - b_f^\mu \, \gamma_\mu \gamma_5 + eq_f \not{A} \Big) \psi_f \, , \tag{3.1}$$

3.1.1 The modified fermion propagator

As usual, the fermionic Feynman's propagators in the four dimensional momentum space is obtained as the inverse of the quadratic term in the free spinor Lagrangian density (first three terms in eqn. (3.1)), namely:

$$S_F^f(p;b) = \frac{i}{\not{p} - m_f - \not{b}\gamma_5}, \tag{3.2}$$

which rationalized according to:

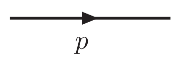
$$\begin{aligned} S_F^f(p;b) \quad &= \quad \left(\frac{i}{\not{p} - m_f - \not{b}_f\gamma_5} \right) \times \\ & \left(\frac{\not{p} + m_f + \not{b}_f\gamma_5}{\not{p} + m_f + \not{b}_f\gamma_5} \right) \left(\frac{\not{p} + m_f - \not{b}_f\gamma_5}{\not{p} + m_f - \not{b}_f\gamma_5} \right) \left(\frac{\not{p} - m_f + \not{b}_f\gamma_5}{\not{p} - m_f + \not{b}_f\gamma_5} \right), \end{aligned} \tag{3.3}$$

and considering the usual $i\varepsilon$ prescription to displace the poles, can be cast in the form:

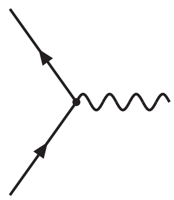
$$S_F^f(p;b) = i \left(\gamma^\nu p_\nu + m_f + b_f^\nu \, \gamma_\nu \gamma_5 \right) \times \frac{p^2 + b_f^2 - m_f^2 + 2 \left(p \cdot b_f + m_f \, b_f^\lambda \, \gamma_\lambda \right) \gamma_5}{\left(p^2 + b_f^2 - m_f^2 + i\varepsilon \right)^2 - 4 \left[\, (p \cdot b_f)^2 - m_f^2 \, b_f^2 \, \right]} . \tag{3.4}$$

3.1.2 Feynman Rules of LIVQED

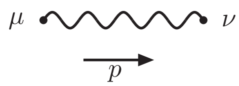
For simplicity the superscript f on the momentum space Feynman propagator of fermions will be omitted. Thus, the Feynman rules for this LIVQED model, due to fermions coupled to a constant background axial-vector as in eqn. (3.1) are:



$$= \quad S_F(p;b) \quad = \frac{i}{\not{p} - m - \not{b}\gamma_5}, \tag{3.5}$$

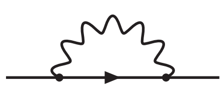


$$= \quad -i(eq_f)\gamma_\mu, \tag{3.6}$$

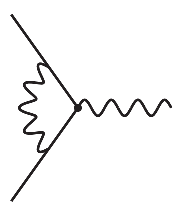


$$= \quad \Delta_F^0(p)_{\mu\nu} = \frac{-ig_{\mu\nu}}{p^2 + i\epsilon}. \tag{3.7}$$

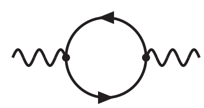
The 0 superscript in the last diagram represents the 0-loop contribution to the photon propagator. With the rules above, the 1-loop contributions to the theory are given by the diagrams below corresponding to (a) the electron self-energy, (b) vertex correction and (c) vacuum polarization, respectively. Diagrammatically these are:



(a) Electron self-energy



(b) Vertex correction

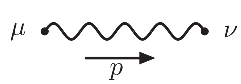


(c) Vacuum Polarization

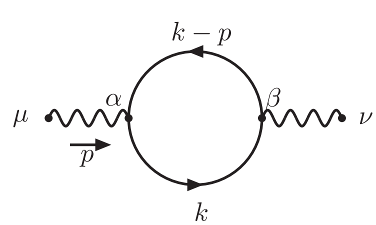
3.2 One-loop induced effective action of photon sector

To compute the one-loop induced effective action of the photon sector, we need to consider only the contribution form the last diagram corresponding to the vacuum polarization, which produces a correction to the photon propagator.

$\Delta_F^2(p)_{\mu\nu} \quad =$



$+$



$$= \quad \Delta_F^0(p)_{\mu\nu} \quad + \quad \Delta_F^0(p)_{\mu\alpha}\Pi_2^{\alpha\beta}(p;b,m)\Delta_F^0(p)_{\beta\nu}\,. \tag{3.8}$$

The superscript 2 means to second order in e and the b label in $\Pi^2(p;b,m)$ is to emphasize that the vacuum polarization tensor is to be computed with the modified fermion propagators obtained above which depend on the background axial-vector b^μ. To see how the modification of the photon propagator induces a one-loop effective action in the photon sector, first recall the relation between the zeroth-order propagator and the action. As mentioned in eqn. (1.18) let us consider the free photon action, written in momentum space:

$$S[A] = \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\widetilde{A}^\mu(k)\widetilde{\Theta}^0_{\mu\nu}(k)\widetilde{A}^\nu(-k)\,. \tag{3.9}$$

The photon Feynman propagator is defined as the inverse of the quadratic term in the Lagrangian. Formally:

$$\widetilde{\Theta}^0_{\mu\nu}(k) \equiv i(\Delta^0_F)^{-1}_{\mu\nu} = (-k^2 g_{\mu\nu} + k_\mu k_\nu)\,, \tag{3.10}$$

which as noted in sec. [1.2.4], thus written it is not properly defined. Now we want to find the one-loop induced effective action, namely:

$$\mathcal{W}[A] = \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\widetilde{A}^\mu(k)\widetilde{\Theta}^2_{\mu\nu}(k)\widetilde{A}^\nu(-k) \tag{3.11}$$

with

$$\begin{aligned}
\widetilde{\Theta}^2 &= i(\Delta^2_F)^{-1} \\
&= i[\Delta^0_F + \Delta^0_F\Pi_2\Delta^0_F]^{-1} \\
&\approx i(\mathbf{1} - \Pi_2\Delta^0_F)\;(\Delta^0_F)^{-1} \\
&= i[(\Delta^0)^{-1}_F \;-\; \Pi_2].
\end{aligned} \tag{3.12}$$

Thus the one-loop induced photon action reads:

$$\mathcal{W}[A] = \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\widetilde{A}^\mu(k)[(-k^2 g_{\mu\nu} + k_\mu k_\nu) - \Pi^2_{\mu\nu}(k;b,m)]\widetilde{A}^\nu(-k)\,. \tag{3.13}$$

Equivalently, the latter can be written as:

$$\mathcal{W}[A] = \frac{1}{2}\int d^4x A^\mu(x)[(+\partial^2 g_{\mu\nu} - \partial_\mu\partial_\nu) - \Pi^2_{\mu\nu}(k;b,m)]A^\nu(x)\,, \tag{3.14}$$

where it is understood that in $\Pi^2_{\mu\nu}$, k_μ is to be replaced by $i\partial_\mu$. The Lagrangian density for the photon sector is read off immediately from the above. The first term in round brackets yields the usual free Maxwell Lagrangian density $\mathcal{L}^0_{\text{free}-\text{Maxwell}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$. To this aim, and to deal with the contributions from Π^2 also, it is most convenient to consider the following expressions:

$$\begin{aligned}
A_\alpha\partial_\mu\partial_\mu A_\alpha &= \partial_\mu(A_\alpha\partial_\mu A_\alpha) - (\partial_\mu A_\alpha)(\partial_\mu A_\alpha)\,, \\
A_\mu\partial_\mu\partial_\nu A_\nu &= \partial_\nu(A_\mu\partial_\mu A_\nu) - (\partial_\nu A_\mu)(\partial_\mu A_\nu)\,,
\end{aligned} \tag{3.15}$$

since the first terms in each line yield boundary-term contributions to the action and therefore are spurious, assuming of course the appropriate behaviour of the fields at infinity.

To complete the computation of the one-loop induced photon action, we must consider the Π^2 term.

3.2.1 Vacuum Polarization

The one–loop vacuum polarization tensor is formally determined to be

$$\Pi_2^{\nu\sigma}(k\,;b_f,m_f) = -ie^2 q_f^2 \int \frac{d^4p}{(2\pi)^4}\ \mathrm{tr}\,\{\gamma^\nu\, S_f(p)\,\gamma^\sigma\, S_f(p-k)\}\ . \tag{3.16}$$

The above formal expression for $\Pi_2^{\nu\sigma}$ exhibits, by power counting, ultraviolet divergencies. In the presence of LIV due to the background axial–vectors b_f^μ, the physical cutoff in fermion momenta does emerge as a result of fermion–antifermion pair creation at very high energies. To this concern, it has been proved [136] that the calculations with such a physical cutoff do actually give the same results of a Lorentz invariance violating dimensional regularization scheme (LIVDRS)

$$\int \frac{d^4p}{(2\pi)^4}\ \longrightarrow\ \mu^{4-2\omega}\ \int \frac{d^{2\omega}p}{(2\pi)^{2\omega}}\,, \tag{3.17}$$

suitably tailored in order to strictly preserve the residual Lorentz symmetry. This LIVDRS coincides with the conventional one, with 't Hooft–Veltman–Breitenlohner–Maison algebraic rules for gamma–matrices, when it is applied to the integrand (3.16) with fermion propagators (3.4). The general structure of the regularized polarization tensor turns out to be

$$\mathtt{reg}\Pi_2^{\nu\sigma}\ =\ \mathtt{reg}\Pi_{2,\,\mathrm{even}}^{\nu\sigma}\ +\ \mathtt{reg}\Pi_{2,\,\mathrm{odd}}^{\nu\sigma}\,. \tag{3.18}$$

Thus the relevant part of the induced photon action reads:

$$\mathcal{W}[A] = \frac{1}{2}\int d^4x A_\mu(x)[(+\partial^2 g^{\mu\nu} - \partial^\mu\partial^\nu) - \mathtt{reg}\Pi_{2,\,\mathrm{even}}^{\mu\nu}\ -\ \mathtt{reg}\Pi_{2,\,\mathrm{odd}}^{\mu\nu}]A_\nu(x)\,. \tag{3.19}$$

The odd part

The regularized CPT-odd part has been unambiguously evaluated in [136] with the help of LIVDRS. For small $|b^\mu| \ll m_e$ reads

$$\mathtt{reg}\Pi_{2,\,\mathrm{odd}}^{\nu\sigma} = 2\,i\,\left(\frac{\alpha}{\pi}\right)\,\epsilon^{\,\nu\sigma}{}_{\rho\lambda}\,k^{\,\lambda}\sum_f q_f^2\,b_f^{\,\rho}\,. \tag{3.20}$$

We can check that such a term indeed induces radiatively a Chern-Simons like term as in the CFJ model. In fact, in we consider the same background axial-vector for all fermion species, $b_f^\rho = b^\rho$ and calling:

$$2\left(\frac{\alpha}{\pi}\right)\sum_f q_f^2 b_f^\rho \equiv \zeta b^\rho, \tag{3.21}$$

the CPT-odd induced photon Lagrangian, to second order in e reads:

$$\mathcal{L}_{2,\mathrm{odd}} = \frac{1}{2}\zeta b^\rho A^\mu \epsilon_{\mu\nu\rho\lambda}\partial^\lambda A^\nu, \tag{3.22}$$

where the replacement $k \to i\partial$ is understood in Π^2. Furthermore, by the antisymmetry of the Levi-Civita symbol we can write $\epsilon_{\mu\nu\rho\lambda}\partial^\lambda A^\nu = -\frac{1}{2}\epsilon_{\lambda\nu\rho\mu}F^{\lambda\nu} = -\widetilde{F}_{\rho\mu}$, whereform the contribution of the odd part to the photon Lagrangian is indeed:

$$\mathcal{L}_{2,\mathrm{odd}} = -\frac{1}{2}(\zeta b^\rho)A^\mu \widetilde{F}_{\rho\mu}. \tag{3.23}$$

The even part

With the help of the LIVDRS [136], the even part of the vacuum polarization tensor can be also found unambiguously. The complete computation of $\mathtt{reg}\Pi^{\mu\nu}_{2,even}(p;b,m)$, reported in [5] is rather involved and for the sake of clarity will be presented in the appendices [C], so as not to deter from the discussion. In this thesis we focus our attention on the LIV deviations of free photons on mass shell $k^2 \sim 0$. The latter ones are expected to be really small $\Delta k^2 \ll m_e^2$ and therefore it makes sense to retain only leading orders in k^2 and b^μ. Correspondingly this part of the polarization tensor takes the form,

$$\mathtt{reg}\Pi^{\nu\sigma}_{2,\mathrm{even}} = (k^2 g^{\nu\sigma} - k^\nu k^\sigma)\,\Pi_{\mathrm{div}} + \frac{2\alpha}{3\pi}\sum_f q_f^2\left\{b_f^2\, g^{\nu\sigma} - m_f^{-2}\, S_f^{\nu\sigma}\right\}, \tag{3.24}$$

in which we have set

$$S_f^{\nu\sigma} \equiv g^{\nu\sigma}\Big[(b_f\cdot k)^2 - b_f^2\, k^2\Big] - (b_f\cdot k)\left(b_f^\nu\, k^\sigma + b_f^\sigma\, k^\nu\right) + k^2\, b_f^\nu\, b_f^\sigma + b_f^2\, k^\nu k^\sigma\,. \tag{3.25}$$

In eqn. (3.24) the first term Π_{div} is logarithmically divergent and does renormalize the electric charges in a conventional way. In fact it has no dependence on k which is solely in the round brackets expression, which definitively has the structure as in normal QED and the divergent part has the appropriate behaviour $\sim \frac{\alpha}{3\epsilon}$ of charge renormalization as well (see the comments in appendix [C]). The second term in curly brackets has two parts. The first one is independent of k and produces an induced photon mass, δm_γ. In fact the contribution from this term to the photon Lagrangian reads:

$$\begin{aligned}\mathcal{L}_{\mathrm{mass\,term}} &= \frac{1}{2}A^\mu\left\{-\left(\frac{2\alpha}{3\pi}\right)\sum_f q_f^2 b_f^2 g_{\mu\nu}\right\}A^\nu \\ &= \frac{1}{2}\left\{-\left(\frac{2\alpha}{3\pi}\right)\sum_f q_f^2 b_f^2\right\}A_\mu A^\mu\,, \end{aligned} \tag{3.26}$$

wherefrom the (induced) squared mass term is read off immediately,

$$\delta m_\gamma^2 = -\left(\frac{2\alpha}{3\pi}\right)\sum_f q_f^2 b_f^2\,. \tag{3.27}$$

Finally we must work out the contributions form the $S_f^{\nu\sigma}$ term. Calling $(\frac{2\alpha}{3\pi})\sum_f \frac{q_f^2}{m_f^2} = \mathcal{K}$ we can write the even contribution to the photon Lagrangian as:

$$\begin{aligned}
\mathcal{L}_{2,S} &= \frac{\mathcal{K}}{2} A^\mu \{S_1^{a,b} + S_2 + S_3 + S_4\}_{\mu\nu} A^\nu\,, && (3.28)\\
\text{where}\quad S_{1\,\mu\nu}^{a,b} &= -g_{\mu\nu}\left[b_\alpha b_\beta \partial_\alpha\partial_\beta - b^2\partial_\alpha\partial_\alpha\right]\,, && (3.29)\\
S_{2\,\mu\nu} &= \left[b_\alpha b_\mu \partial_\alpha \partial_\nu + b_\alpha b_\nu \partial_\alpha \partial_\mu\right]\,, && (3.30)\\
S_{3\,\mu\nu} &= (-1) b_\mu b_\nu \partial_\alpha\partial_\alpha\,, && (3.31)\\
S_{4\,\mu\nu} &= (-1) b^2 \partial_\mu \partial_\nu\,. && (3.32)
\end{aligned}$$

where the fermion species subscript has been omitted for simplicity. After putting together terms S_1^b and S_4 and a brief manipulation (also using the eqns. (3.15)) we get the following contribution:

$$\begin{aligned}
\mathcal{L}_{b^2} &= -\frac{1}{4}\left[\frac{b^2}{m_e^2}\left(\frac{2\alpha}{3\pi}\right)\sum_f q_f^2 (\frac{m_e}{m_f})^2\right] F_{\mu\nu}F^{\mu\nu}\\
&= -\frac{1}{4}\xi\left(\frac{b}{m_e}\right)^2 F_{\mu\nu}F^{\mu\nu}\,, && (3.33)
\end{aligned}$$

where once more we have assumed a universal b_f, and ξ is determined as:

$$\xi \equiv \frac{2\alpha}{3\pi}\sum_f q_f^2\left(\frac{m_e}{m_f}\right)^2\,. \tag{3.34}$$

Finally when taking together the terms S_1^a, S_2, S_3 and after similar manipulations and indices gymnastics as above, we arrive at:

$$\mathcal{L}_{b_\lambda b_\rho} = \frac{1}{2m_e^2} b_\lambda b_\rho\, \xi\, F^\lambda_{\ \sigma} F^\rho_{\ \sigma}\,. \tag{3.35}$$

Altogether, the relevant LIVQED Lagrange density in the photon sector reads

$$\begin{aligned}
\mathcal{L}_\gamma = &-\frac{1}{4}\left(1+\xi\,\frac{b^2}{m_e^2}\right) F^{\nu\lambda}F_{\nu\lambda} + \xi\,\frac{b_\nu b^\rho}{2m_e^2}\,F^{\nu\lambda}F_{\rho\lambda}\\
&-\frac{1}{2}\,\zeta\, b_\nu A_\lambda \widetilde{F}^{\nu\lambda} + \frac{1}{2}\,m_\gamma^2\,A_\nu A^\nu + B\,\partial_\nu A^\nu\,, && (3.36)
\end{aligned}$$

with the photon mass assembling both a bare and an induced one, $m_\gamma^2 = \mu_\gamma^2 + \delta m_\gamma^2$. The reason for allowing a bare photon mass at this point will be explained further in sec [3.4.3], but we anticipate that it proves convenient when analyzing the quantum consistency of the theory. As the gauge invariance is broken by the photon mass, we have suitably introduced the Stückelberg's auxiliary field $B(x)$, (see sec. (1.2.5))together with the usual dual field tensor $\widetilde{F}^{\nu\lambda} \equiv (1/2)\,\epsilon^{\nu\lambda\rho\sigma}F_{\rho\sigma}$ and the electron mass m_e.

3.2.2 Induced coupling constants

As shown explicitly above, the induced coupling constants in the photon Lagrangian eqn. (3.36) stem form the computation of the regularized vacuum polarization tensor, for which LIVDRS, which coincides with $\overline{DR}$ of 't Hooft-Veltman-Breitenlohner-Maison, is perfectly justified. For completeness they are collected below:

$$\begin{aligned} \zeta b^\mu &\equiv 2\left(\frac{\alpha}{\pi}\right)\sum_f q_f^2 b_f^\mu\,, & (3.37)\\ \xi &\equiv \frac{2\alpha}{3\pi}\sum_f q_f^2\left(\frac{m_e}{m_f}\right)^2, & (3.38)\\ \delta m_\gamma^2 &\equiv -\left(\frac{2\alpha}{3\pi}\right)\sum_f q_f^2 b_f^2\,. & (3.39)\end{aligned}$$

thus completing the computation of the one-loop induced photon Lagrangian.

3.3 LIV dispersion laws

The dispersion relations implied by this LIVQED model are important for analyzing the theory's quantum consistency. From them, for example, we will see how the theory's stability or causality may be jeopardized by the LIV effects according to the space, time or light-like nature of the LIV vector b^μ. This has been done already for the fermionic sector of the theory [31, 32, 33, 34, 83, 84, 128] therefore here we will only present and comment those results. However, since the CPT-even part of the LIVQED had never been analyzed before, we will pay closer attention to the photon sector.

3.3.1 Fermionic dispersion relations

Let us begin by presenting the fermion dispersion relations. From the free-fermion Lagrangian in eqn. (3.1), we can read the equations of motion readily in momentum space (the fermion species subscript f omitted for convenience):

$$\begin{aligned} \Big(\gamma_\mu\, p^\mu - m - \gamma_5\,\gamma_\mu\, b^\mu\Big)\psi &= 0\,,\\ \widetilde{\mathcal{D}^b}_{--}\psi &= 0\,, \end{aligned} \qquad (3.40)$$

in reference to the signs of the mass term and the b^μ term in the Dirac operator in momentum space. Thus, the free fermion dispersion relation is obtained "squaring twice" the EoM resulting in a diagonal equation in spinor space:

$$\begin{aligned} \widetilde{\mathcal{D}^b}_{--}\widetilde{\mathcal{D}^b}_{-+}\widetilde{\mathcal{D}^b}_{++}\widetilde{\mathcal{D}^b}_{+-}\,\psi &= 0\,,\\ \Big[\big(p^2+b^2-m^2\big)^2 + 4\,b^2\,m^2 - 4\,(b\cdot p)^2\Big]\,\psi &= 0\,. \end{aligned} \qquad (3.41)$$

Therefore the free continuous spectrum of fermions in a constant axial-vector background is governed by the expression

$$\Big[\big(p^2+b^2-m^2\big)^2+4\,b^2\,m^2-4\,(b\cdot p)^2\Big]=0\,. \tag{3.42}$$

3.3.2 Photon dispersion relations

The modified Maxwell's equations are the Euler-Lagrange eqns. of the Maxwell Lagrangian of eqn. (3.36):

$$\begin{aligned}&\left(1+\xi\,\frac{b^2}{m_e^2}\right)\partial_\lambda F^{\,\lambda\nu}-\frac{\xi}{m_e^2}\left(b^{\,\rho}b_\lambda\partial_\rho F^{\,\lambda\nu}-b^\nu b_\lambda\partial_\rho F^{\,\lambda\rho}\right)\\ &+m_\gamma^2\,A^\nu-\;\zeta\,b_\lambda\widetilde{F}^{\,\nu\lambda}\;=\;\partial^{\,\nu}B\,, \qquad (3.43)\\ &\partial_\nu A^\nu=0\,. \qquad (3.44)\end{aligned}$$

After contraction of eq. (3.43) with ∂_ν we find

$$\partial^{\,2}B(x)=0\,, \tag{3.45}$$

whence it follows that the auxiliary field is a decoupled massless scalar field, which is not involved in dynamics.

After using eq. (3.44) one can rewrite the field equations in terms of the gauge potential, *i.e.*,

$$\begin{aligned}&\left(1+\xi\,\frac{b^2}{m_e^2}\right)\partial^{\,2}A^\nu-(\,\xi/m_e^2)\,\Big[(b\cdot\partial)^2A^\nu-\partial^{\,\nu}(b\cdot\partial)(b\cdot A)+b^\nu\partial^{\,2}(b\cdot A)\Big]\\ &+\;\;m_\gamma^2\,A^\nu-\;\zeta\,\epsilon^{\,\nu\lambda\rho\sigma}\,b_\lambda\partial_\rho A_\sigma\;=\;0\,. \qquad (3.46)\end{aligned}$$

After contraction of eq. (3.46) with $b^{\,\nu}$ we get

$$\Big(\partial^{\,2}+m_\gamma^2\Big)(b\cdot A)=0\,, \tag{3.47}$$

for the special component $b\cdot A$ of the vector potential. Thus, for this polarization, we actually find the ordinary dispersion law of a real massive scalar field, whereas the two further components with polarizations orthogonal to both k_ν and b_ν are affected by the fermion induced LIV radiative corrections.

Going to the momentum representation, the equations of motion take the form

$$K^{\,\nu\sigma}\,\mathrm{A}_\sigma(k)=0\,,\qquad k^\sigma\mathrm{A}_\sigma(k)=0\,, \tag{3.48}$$

where

$$K^{\,\nu\sigma}\quad\equiv\quad(k^2-m_\gamma^2)\,g^{\nu\sigma}-k^\nu k^\sigma-(\xi/m_e^2)\,S^{\nu\sigma}+i\,\zeta\,\epsilon^{\,\nu\lambda\rho\sigma}\,b_\lambda k_\rho\,. \tag{3.49}$$

This can be readily seen from the one-loop induced photon action written in momentum space eqn. 3.13. Where $S^{\mu\nu}$ is given by eqn. (3.25), the coupling constants take the values obtained above and both the CPT-even and CPT-odd parts have been considered. Also a bare photon mass is allowed as well and of course assuming that renormalization carries along in the usual way in order to remove the Π_{div} term, as it is indeed the case.

At this point some observations are in order. (i), as we have shown above, A^μ field is transverse to k^μ, and the component longitudinal to b^μ has a rather trivial dispersion law, therefore we are really interested in the dispersion law of the polarizations orthogonal to both k^μ and b^μ, (ii) The operator $\mathrm{e}^{\mu\nu} \equiv (1/\mathtt{D})\, S^{\mu\nu}$, with $\mathtt{D}$ as defined below, turns out to be exactly a projector onto the two-dimensional hyperplane orthogonal to b^μ and k^ν, (a property that comes as no surprise since our result for the LIV part of the vacuum polarization tensor does indeed respect the Ward identities with respect to k^μ and b^μ.), (iii) The $K^{\mu\nu}$ operator is Hermitian wherefrom $\mathbf{K}^t(p) = \mathbf{K}^*(p) = \mathbf{K}(-p)$, all of them having the same eigenvalues of $\mathbf{K}(p)$. Thus left– and right–handed polarizations can be constructed in the following way.

In order to pick out the two independent field degrees of freedom, we have introduced the quantity

$$\mathtt{D} \equiv (b \cdot k)^2 - b^2 k^2 \tag{3.50}$$

and the projector onto the two–dimensional hyperplane orthogonal to b_ν and k_ν ,

$$\mathrm{e}^{\nu\sigma} \equiv g^{\nu\sigma} - \frac{b \cdot k}{\mathtt{D}} \left(b^\nu k^\sigma + b^\sigma k^\nu\right) + \frac{k^2}{\mathtt{D}} b^\nu b^\sigma + \frac{b^2}{\mathtt{D}} k^\nu k^\sigma \,. \tag{3.51}$$

One can always select two real orthonormal four–vectors corresponding to the linear polarizations in such a way that

$$\mathrm{e}_{\nu\sigma} = -\sum_{a=1,2} \mathrm{e}_\nu^{(a)} \mathrm{e}_\sigma^{(a)} \,, \qquad g^{\nu\sigma} \mathrm{e}_\nu^{(a)} \mathrm{e}_\sigma^{(b)} = -\,\delta^{ab} \,. \tag{3.52}$$

It is also convenient to define another couple of four–vectors, in order to describe the left– and right–handed polarizations: in our case, those generalize the circular polarizations of the conventional QED. To this aim, let us first define

$$\epsilon^{\,\nu\sigma} \equiv \mathtt{D}^{-1/2}\, \epsilon^{\,\nu\lambda\rho\sigma} b_\lambda k_\rho \,. \tag{3.53}$$

Notice that we can always choose $\mathrm{e}_\lambda^{(a)}$ to satisfy

$$\epsilon^{\,\nu\sigma} \mathrm{e}_\sigma^{(1)} = \mathrm{e}^{(2)\nu} \,, \qquad \epsilon^{\,\nu\sigma} \mathrm{e}_\sigma^{(2)} = \, - \, \mathrm{e}^{(1)\nu} \,. \tag{3.54}$$

Let us now construct the two orthogonal projectors

$$P_{\nu\sigma}^{(\pm)} \equiv \frac{1}{2} \left(\mathrm{e}_{\nu\sigma} \pm i \epsilon_{\nu\sigma}\right) . \tag{3.55}$$

and set, *e.g.*,

$$\varepsilon_\nu^{(L)} \equiv \frac{1}{2} \left(\mathrm{e}_\nu^{(1)} + i\, \mathrm{e}_\nu^{(2)}\right) = P_{\nu\sigma}^{(+)}\, \mathrm{e}^{(1)\,\sigma} \,, \tag{3.56}$$

$$\varepsilon_\nu^{(R)} \equiv \frac{1}{2} \left(\mathrm{e}_\nu^{(1)} - i\, \mathrm{e}_\nu^{(2)}\right) = P_{\nu\sigma}^{(-)}\, \mathrm{e}^{(1)\,\sigma} \,. \tag{3.57}$$

We remind that as noted in [136], actually the left– and right–handed (or chiral) polarizations only approximately correspond to the circular ones of Maxwell QED. In the presence of the CS kinetic term, the field strengths of electromagnetic waves are typically not orthogonal to the wave vectors.

Once the physical meaning of polarizations has been suitably focused, one can readily find the expression of the dispersion relations for the doubly transversal photon modes. From the equation of motion of the doubly transversal photon modes, eq. (3.48) and from the properties mentioned above we can write:

$$\mathbf{K}^T(-k)\cdot\mathbf{K}(k)\,\mathrm{A}^{\perp}(k) \quad = \quad 0 \tag{3.58}$$

yielding the condition:

$$\Big\{\big(k^2-\frac{\xi}{m_e^2}\mathtt{D}-m_\gamma^2\big)^2-\zeta^2\mathtt{D}\Big\}\mathrm{A}^{\perp}(k) \quad = \quad 0. \tag{3.59}$$

and the consequent dispersion relation for the doubly transversal photon modes:

$$\left\{k^2-\frac{\xi}{m_e^2}\Big[\,(b\cdot k)^2-b^2k^2\,\Big]-m_\gamma^2\right\}^2-\ \zeta^2\Big[\,(b\cdot k)^2-b^2k^2\,\Big]=0\,. \tag{3.60}$$

Evidently real solutions exist only iff

$$\mathtt{D}=(b\cdot k)^2-b^2k^2\geq 0\ ,$$

Notice that on the photon mass shell, deviations off the light–cone are of order $|\,b_\nu|^2$. As a consequence, the on–shell momentum dependence of the polarization tensor (3.18) is dominated by the lowest order $k^2=0$, whereas the higher orders in k^2 do represent simultaneously higher orders in b_ν, which are neglected in the present analysis.

3.4 LIVQED consistency

3.4.1 Renormalizability

The LIVQED model studied in this thesis has proven to be renormalizable (at least at one-loop level). In fact only one counterterm is needed to cancel the divergencies and this is the same as in ordinary QED since the LIV effects do not generate new divergencies.

Actually renormalizability is expected to be satisfied for the general class of Lorentz and CPT violating models encompassed in the SME, of which our model is a particular case . Therefore, no inconsistency is expected from this point of view. Of course this would deserve a detailed inspection if one is interested in higher order effects, which is not addressed in this thesis.

3.4.2 Unitarity, causality and stability

In sec. [1.4] we stressed the importance of the properties of stability, causality, and unitarity. These issues have already been studied in the literature [31, 32, 33, 34, 84, 128, 138, 1] and considerable attention has been given to the CPT-odd and fermionic sector of the theory implied by the constant background axial-vector coupling in eq. (3.1).

However, the one-loop CPT-even induced effects computed for the first time in [5] that stem from the fermionic Lagrangian in consideration imply that the issues of stability, causality and unitarity need to be checked. Furthermore we must see whether or not demanding these properties for a consistent photon quantization is in conflict with the conclusions drawn so far in the references cited above for a consistent fermion quantization, this will be addressed in [3.4.3].

Unitarity

Unitarity also seems to be "naively" preserved in our model. Our starting point was the fermion Lagrangian 3.1. The one-loop induced photon action resulted gauge invariant and governed by an Hermitian operator 3.13 in a sense as a consequence of the above. In fact, the fermionic equations of motion 3.40 can be derived from a Schrödinger equation governed by the Hermitian Hamiltonian [153]:

$$H_b = \gamma^0 \left(-i\vec{\gamma} \cdot \vec{\nabla} + m_e + \gamma_5 \not{b} \right) . \tag{3.61}$$

Hence, no unitarity violations are expected in this LIVQED model [1] at least in the fermionic sector. However, we still need to check explicitly whether or not the photon sector is hampered by imaginary energies due to the constant background axial-vector b^μ, which will be done in [3.4.3].

Causality

As in the case of unitarity, the spacetime nature of b^μ also determines the violation of causality. In rigor we should check for microcausality, which has also been thoroughly studied for the fermionic sector [1, 32, 33, 136, 139] in terms of (anti)-commutativity of local operators separated by a space–like interval, as well from the proper behaviour of gauge-field propagators. Also a microcausality criterion can be established in terms of the velocity of signal propagation.

In our case it will be necessary to study the implications on the one-loop induced photon sector. To this aim, in section [3.4.3] we will focus on the (phase, group and front) velocities of light derived from the dispersion relations obtained.

Stability

In sec. [1.4.3] we already mentioned that stability is typically addressed form demanding energy positivity. This is so because whenever Lorentz symmetry is valid, for free QFTs the above requirement of energy positivity in a given frame of reference, implies the stability of vacuum for all reference frames, allowing for a meaningful quantum theory. In our case, although we have cleared up that we are dealing with particle Lorentz symmetry breaking, and observer Lorentz covariance intact, we need to be cautious. Thus we need an operational means of check whether the vacuum of our theory is stable or not. In fact, the above implication of *energy positivity* $\rightarrow$ *stability of "all" vacua* relies, among others, on the fact that the 4-momenta of all one-particle states in the particular frame of reference where energy is positive, are either time–like or light–like, with nonnegative 0th components. We will adopt this criterion in order to check for the stability of our theory. Of course, the fermionic sector has also been studied in this respect, and demanding it on the photon sector will leads us in section [3.4.3], to the study of the implications of the b^μ vector on the photon dispersion relations.

3.4.3 Consistency examination

In section 3.3 we obtained the general dispersion relations for fermions and photons in our LIVQED model. Having led the basis for consistency analysis discussed above, let us consider the dispersion relations for fermions and photons in particular cases of interest, depending on the spacetime nature of b^μ. Naturally the conditions that the consistency of the theory imposes on b^μ coming form fermion dynamics and from photodynamics must coincide, thus we must address them simultaneously.

If LIV occurs spontaneously in QED, due to some vector–like condensate, then the related low–energy effective action is actually dominated by the classical gauge invariant Maxwell–Chern–Simons modified electrodynamics [122], with a Chern–Simons (CS) fixed vector η_μ, and if LIV manifests itself as a fundamental phenomenon in the large–scale Universe, it is quite plausible that LIV is induced universally by different species of fermion fields coupled to the very same axial–vector b^μ , albeit with different magnitudes depending upon flavors. Then both LIV vectors become [135, 136] collinear, *i.e.* $\eta^\mu = \zeta b^\mu$. Meantime it has been found [128, 138, 139, 140] that a consistent quantization of photons just requires the CS vector to be space–like, whereas for the consistency of the spinor free field theory a space–like axial–vector b^μ is generally not allowed but for the pure space–like case which, however, is essentially ruled out by the experimental data [141], thus for a consistent fermion quantization a timelike axial–vector b^μ is required [136, 139, 140].

Nonetheless, it has been found that in the lack of a bare photon mass and/or a bare

CS vector of different direction a time–like vector b^{μ} just leads to a tachyonic massive photon that means to imaginary energies for the soft photons, implying an instability of the photodynamics [138, 139].

Two possibilities, so far unattended, emerge:

Light–like axial–vector $b^{\mu} = (|\vec{b}|, \vec{b})$.

If we adopt classical photons in Lorentz invariant QED to be massless $\mu_{\gamma} = 0$ then, in such a situation, a fully induced LIV appears to be flawless only for *light–like* axial–vectors b^{μ}. In particular, for a light–like universal axial–vector $b^{\mu} = (|\vec{b}|, \vec{b})$, from the fermion dispersion relation with $b^2 = 0$,

$$\left[\left(p^2 + b^2 - m^2\right)^2 + 4\,b^2\,m^2 - 4\,(b \cdot p)^2\right] = 0,$$

after taking the square root and writing $\pm 2(b \cdot p) = -(p \mp b)^2 + p^2 + b^2$, we get:

$$\begin{aligned}
(p^2 - m_f^2) &= -(p \mp b)^2 + p^2 + b^2 = 0, \\
(p_0 \mp |\vec{b}|)^2 - (\vec{p} \mp \vec{b})^2 &= m_f^2, \\
(p_0 \mp |\vec{b}|) &= \pm\sqrt{(\vec{p} \mp \vec{b})^2 + m_f^2},
\end{aligned} \tag{3.62}$$

wherefrom we find the dispersion relations for the LIV 1–particle states of a fermion species f that read

$$p^0_{+} + |\vec{b}| = \pm\sqrt{\left(\vec{p} + \vec{b}\right)^2 + m_f^2}\,, \tag{3.63}$$

$$p^0_{-} - |\vec{b}| = \pm\sqrt{\left(\vec{p} - \vec{b}\right)^2 + m_f^2}\,. \tag{3.64}$$

Now, it turns out that the requirement $p^2_{\pm} > 0$ for the LIV free 1–particle spinor physical states just drives to the high momenta cut–off $|\vec{p}| \leq m_e^2/4\,|\vec{b}|$ which is well compatible with the LIVDRS treatment of fermion loops. Thus the model is consistent at the quantum level in the fermionic sector.

Then one can use the induced values of the LIV parameters (3.21), (3.34) in the case $b^2 = 0$ and the dispersion law for photons (3.60) is reduced [1] to

$$k^2 - (\xi/m_e^2)\,(b \cdot k)^2 \pm \zeta\, b \cdot k = 0\,. \tag{3.65}$$

This can be written as:

$$\begin{aligned}
k_0^2 - \vec{k}^2 - \xi/m_e^2(|\vec{b}|k_0 - \vec{b} \cdot \vec{k})^2 \pm \zeta|\vec{b}|k_0 \mp \zeta\vec{b} \cdot \vec{k} &= 0, \\
k_0^2\left(1 + Ax + Bx^2\right) - \vec{k}^2\left(1 + Cx + Dx^2\right) &= 0,
\end{aligned} \tag{3.66}$$

[1] At this stage it is not necessary to consider the bare photon mass, μ_{γ}.

where $x \sim \frac{|\vec{b}|}{|\vec{k}|} \sim \frac{|\vec{b}|}{k_0} \ll 1$, and the constants A, B, C, D determined as follows:

$$
\begin{aligned}
A &= \pm\zeta, \\
B &= \frac{\xi|\vec{k}|^2}{m_e^2}(2\cos\theta - 1), \\
C &= \pm\zeta\cos\theta, \\
D &= \frac{\xi}{m_e^2}|\vec{k}|^2\cos^2\theta.
\end{aligned}
\tag{3.67}
$$

After some Taylor gymnastics, for photon momenta $|\,\vec{k}\,| \gg |\,\vec{b}\,|$ one approximately finds the relationship for positive energies (frequencies)

$$
\begin{aligned}
& k_0 \simeq |\,\vec{k}\,|(1 + \delta c_\theta) \mp \zeta\,|\,\vec{b}\,|\,\sin^2\theta/2\,, \\
& \cos\theta \equiv \frac{\vec{b}\cdot\vec{k}}{|\,\vec{b}\,|\,|\,\vec{k}\,|}\,, \quad \delta c_\theta \equiv \frac{2\xi}{m_e^2}\,|\,\vec{b}\,|^2\,\sin^4\theta/2,
\end{aligned}
\tag{3.68}
$$

and a similar expression for negative energies (frequencies). One can see clearly that LIV entails an increment δc_θ of the light velocity, which makes it different from its decrement generated by quantum gravity in the leading order [51, 52, 53, 62, 133]. This effect, however, depends upon the direction of the wave vector $\vec{k}$ and actually vanishes in the direction collinear with $\vec{b}$.

Genuine time–like axial–vector b^μ

Another way to implement the LIV, solely by fermion coupling to an axial–vector background, is to start with the Maxwell's photodynamics supplemented by a bare and *Lorentz invariant* photon mass μ_γ, so that

$$
m_\gamma^2 = -\frac{2\alpha}{3\pi}\sum_f q_f^2\, b_f^2 + \mu_\gamma^2\,,
\tag{3.69}
$$

where the first term is the radiatively induced mass computed in (3.27). Then, for a genuine time–like $b^\mu = (\sum_f q_f\, b_f^0\,, 0, 0, 0)$ one finds from eq. (3.60) and after similar expansions as in the previous case:

$$
k_0^2 = \left(1 + \frac{\xi\, b_0^2}{m_e^2}\right)\left(|\,\vec{k}\,| \pm \frac{1}{2}\,\zeta\, b_0\right)^2 + m_\gamma^2 - b_0^2\left\{\frac{1}{4}\,\zeta^2 + O(b_0\,|\,\vec{k}\,|/m_e^2)\right\}.
\tag{3.70}
$$

Hence, if $m_\gamma \geq \zeta\, b_0/2$ then the photon energy keeps real for any wave vector $\vec{k}$ and LIVQED happens to be free of instabilities. From the computed induced couplings in eq. (3.37), the latter condition implies

$$
m_\gamma = 8\alpha\, b_0/\pi\,.
\tag{3.71}
$$

From this relation and the experimental bounds for the photon mass in section [2.3.6], we will obtain in the next section a bound for b_0.

3.5 Phenomenological Implications and bounds on b^μ

In the two cases derived above we can distinguish different phenomenology, wherefrom experimental test presented in sec.[2.3] can used to constrain b^μ.

3.5.1 Lightlike b^μ phenomenology

Eq. (3.68) implies an increment of the light velocity, contrary to the decrement typically generated by quantum gravity in the leading order (for example, see [133, 52, 53, 62, 63]) . Both the variation in the light velocity and the birefringence effect caused by a phase shift between left– and right–polarized photons – alternate signs in (3.68) – depend upon the direction of the wave vector $\vec{k}$. Both effects do vanish in the direction collinear with $\vec{b}$. Thus the compilation of the UHECR data in search for deviations of the speed of light must take into account this possible anisotropy of photon spectra.

This is also true for the compilation of the data on polarization plane rotation for radio waves from remote galaxies. The earlier search for this effect [122, 142, 143] led to the very stringent upper bound on values of $|\vec{b}|$:

$$|\,\vec{b}\,| < \times 10^{-31}\,\mathrm{eV}. \tag{3.72}$$

However, in addition to the previous remark on the photon spectrum anisotropy, we would like to give more arguments in favor of a less narrow room for the possibility of LIV and CPT breaking in the Universe. Indeed one must also take into account the apparent time variation of an anisotropic CS vector , when its origin derives from the v.e.v. of a parity–odd quintessence field [145] very weakly coupled to photons. That v.e.v. may well depend on time and obtain a tiny but sizeable value in the later epoch of the Universe evolution [146], just like the cosmological constant [147] might get. As well a non-vanishing CS vector may be induced also by the non-vanishing v.e.v. of a dark matter component if its coupling to gravity is CPT odd . Eventually it means that, for large distances corresponding to earlier epochs in the Universe, one may not at all experience this kind of LIV and CPT breaking. Conversely, in a later time such a CS term may gradually rise up. Then, the earlier radio sources – galaxies and quasars with larger Hubble parameters – may not give any observable signal of birefringence, whereas the individual evidences from a nearest radio source may be of a better confidence. So far we cannot firmly predict on what is an actual age of such CPT odd effects and therefore, to be conservative, one has to rely upon the lab experiments and meantime pay attention to the data from quasars of the nearest Universe. Thus one may certainly trust to the estimations [124] performed in the laboratory and the nearest Universe observations. So far the most conservative value of the LIV parameter from [124, 141] arises from hydrogen maser experiments: namely,

$$|\,\vec{b}_e\,| < 10^{-18}\,\mathrm{eV} \tag{3.73}$$

for electrons.

3.5.2 Timelike b^{μ} phenomenology

The timelike case also exhibits birefringence effects as can be verified form the dispersion relation. For photon momenta $|\vec{k}| \gg b_0$ the dispersion relation reads (for positive energies/frequencies):

$$k_0 = \sqrt{1+\xi b_0^2/m_e^2}\left(|\vec{k}| \pm \frac{\zeta b_0}{2}\right). \tag{3.74}$$

Thus we can see that the presence of a (timelike) constant background axial-vector turns the vacuum into a birefringent media. Nevertheless, the experimental bounds on the photon mass can give a best estimate on a bound for b_0 than those from the rotation of the polarization plane of astrophysical sources.

The present day very stringent experimental bound on the photon mass [148], $m_\gamma < 6\times10^{-17}$ eV, allow us to put a bound on b_0. From eq. (3.71) and the previous experimental bound for m_γ, we get:

$$b_0 < 3\times10^{-15}\ \text{eV}\,. \tag{3.75}$$

To conclude this chapter a few more comments on estimates for the LIV vector components are in order.

1. There are no better bounds on b^{μ} coming from the UHECR data on the speed of light for photons. This is because the increase of the speed of light depends quadratically on components of b_μ . Thus, for example, the data cited in [123, 124] do imply less severe bounds on $|\,\vec{b}\,|$ or b_0 than those ones above mentioned.

2. For the LIVQED examined in the present paper, the typical bounds on LIV and CPT breaking parameters in the context of quantum gravity phenomenology are not good enough to compete with the laboratory estimations. They are, in fact, of a similar order of magnitude as other LIV effects in the high energy astrophysics.

3. An interesting bound on deviations of the speed of light is given in [150] where, in the spirit of quantum gravity phenomenology, space–time fluctuations are addressed to produce modifications of the speed of light and, as well, of the photon dispersion relations exhibiting helicity dependent effects. Using an interferometric technique, the authors of ref. [150] were able to estimate $\Delta c < 10^{-32}$. However this estimation does not imply a better bound for a LIV vector b_μ , as it actually gives $b_0 < 10^{-12}$ eV , which is certainly in agreement with the more stringent bounds discussed above .

Chapter 4

Conclusions

In this thesis we have studied some specific aspects of Lorentz symmetry breaking in high energy physics and more particularly in QED. Also the role of breaking other symmetries such as CPT symmetry or gauge symmetry have also been emphasized. Simultaneously, we have shown and succinctly described some models in diverse contexts which exhibit LIV as a common and maybe necessary feature of a would-be more fundamental theory that could supersede both the Standard Model and General Relativity.

The first important point to remark is that when we speak of Lorentz symmetry breaking or Lorentz invariance violation we mean particle Lorentz violation and not coordinate invariance violation, otherwise rendering the theory nonsensical as Nature would be different for different inertial observers.

The second point is that no clear-cut evidence in favour of a mechanism that would produce such a LIV in Nature, at a fundamental level, exist. So far, only motivations for such a scenario can be put forward. For example it might arise in several ways reviewed in [125, 126]. In particular, spontaneous symmetry breaking [127] may cause LIV after condensation of massless axion–like fields [128, 129] and/or of certain vector fields [130] (maybe, of gravitational origin [131]),as well as short–distance space–time asymmetries may come from string [132] and quantum gravity effects [133, 52, 53, 62, 63] and non–commutative structure of the space–time [134].

If LIV occurs spontaneously in QED, due to some vector–like condensate, then the related low–energy effective action is actually dominated by the classical gauge invariant Maxwell–Chern–Simons modified electrodynamics [122], with a Chern–Simons (CS) fixed vector η_μ. In addition, the low–energy effective action has to be indeed supplemented by a Lorentz–invariant bare photon mass μ_γ and take into account the contribution from the one–loop radiative corrections [135, 136, 137] induced by the fermion sector, in which some constant axial–vector $b_\mu = \langle B_\mu \rangle = \langle \partial_\mu \theta \rangle$ does appear. The latter one might represent the vacuum expectation value of a vector field $B_\mu(x)$, such as some torsion field of a cosmological nature, or of a gradient of some axion field or quintessence field $\theta(x)$, or anything else. Regardless of the mechanism responsible for such a LIV, its empirical parameterization does not represent a tedious and difficult task, as presented in (2.2), see refs. [123, 124] too. However, the consistent unraveling of its dynamical origin is far more subtle and involved. Also the quantum consistency of LIVQED must be thoroughly checked. In this respect, the consistency of a LIVQED model as due to fermions coupled to constant background axial-vector has already been done for the CPT-odd part. However, in order to complete the consistency study for such a theory the CPT-even part must also be studied.

To this aim we computed [5] the one-loop induced parity even part of the photon effective action resulting from the fermion Lagrangian (3.1). This highly non-trivial result (see appendix C) had never been obtained before and without it a full consistent quantization cannot be done. In this respect we must stress some crucial points of this computation:

1. From the fermion Lagrangian (3.1), we obtained the exact fermion propagator (3.4).
2. In order to compute the one-loop induced (parity-even) photon action, we used the suitably justified 't Hooft-Veltman-Breitenlohner-Maison dimensional regularization scheme, specially endowed with specific rules to treat γ_5 terms in higher dimensions.
3. Our results explicitly show that the LIV effects considered in this thesis, do no entail any divergent contributions, *i.e.* the (regularized) even part of the vacuum polarization tensor, to first order in α has the usual logarithmically divergent piece of Lorentz covariant QED responsible of renormalizing the electric charge. The LIV part (3.24,3.25) is finite and fully determined.
4. Contrary to what has been claimed in [121], our result is indeed gauge invariant, as it respect the Ward identities. Furthermore the non-diagonal part of the vacuum polarization tensor (3.25) is both transverse to the external photon momenta k^μ and also to the LIV vector b^σ. This in turn fixes the doubly-transverse structure of the one-loop induced photon propagator and the photon equations of motion, wherefrom the corresponding photon dispersion relations are obtained. The consequence of the latter is the different behaviour of the different polarization states of the A^μ field. The polarization longitudinal to b^μ exhibits the usual dispersion law of a real massive scalar field, and two more components with polarizations orthogonal

to k^μ and b^ν result, both affected by the induced LIV radiative corrections. Thus the dispersion relations for the doubly-transverse photon modes were obtained, (3.60).

Regarding the quantum consistency of this LIVQED model we must mention that new results are implied by our analysis. In fact, two new possibilities allowing for a consistent quantization are put through. Let us recall that the presence of the LIV constant background axial-vector b^μ introduces among others, modifications to the dispersion relations of fermions and photons, and the spacetime nature of b^μ is crucial for the theory's consistency. As has been checked in sections [3.4.2] [3.4.3] and in the refs. therein the theory's renormalizability, unitarity and causality remain unaltered. This last point may be the most controversial, since in two cases addressed in this thesis, light exhibits superluminal behaviour, nonetheless, we argued that in principle this pose no difficulty since superluminality criteria is valid for fully Lorentz covariant theories, and no satisfactory definition of velocity has been given for these LIV contexts. Meanwhile, from the point of view of the stability two new possibilities have been found. Summarizing the status of the consistency of LIVQED prior to our results, we can say that (a) A space-like b^μ is not allowed for a consistent free spinor field theory, except for the pure spacelike case. However, this last possibility is ruled out by experimental data, [141], (b) Thus a timelike b^μ is required for consistency, however, in lack of a bare photon mass and/or a bare CS vector of different direction than that of b^μ, leads to imaginary energies and instabilities of photodynamics.

Thus in absence of a bare photon mass $\mu_\gamma = 0$, there remains the possibility for a lightlike b^μ. In which case we found that, (i) From the fermion dispersion relations (3.63),the requirement $p_\pm^2 > 0$ for the LIV free 1-particle spinor physical states just drives to the high momenta cut-off $|\vec{p}| \leq m_e^2/4|\vec{b}|$, which is well compatible with the LIVDRS treatment of fermion loops, *i.e.* the previous bound is the natural cut-off regulator, equivalent to the LIVDRS regularization employed here [136], thus ensuring the consistency condition $p_\pm^2 > 0$. (ii) On the other hand, the corresponding modified photodynamics results in real energies for photons, regardless the values of the modifications parameters. (iii) Anisotropic birefringent effects from this case are obtained as well, which, however, depend on the direction of the wave vector $\vec{k}$. We argue that this behaviour does not allow us to employ the polarimetry studies done on the observations of radio waves from extremely distant galaxies to constrain $|\vec{b}|$, thus we adhere to the more conservative bound coming form laboratory experiments and from astronomical observations of near sources [124, 141].

On the other hand, from the one-loop induced photon Lagrangian obtained in eq. (3.36) we see that the photon mass term has a radiative contribution and a bare one as well, which is in fact allowed, (recall the role of the gauge symmetry restoring Stueckelberg field $B(x)$ 1.2.5). Thus there may be room for a consistent quantization for a timelike b^μ, otherwise forbidden. In this case, the most relevant result is yet another bound (very stringent indeed) on the timelike b_0 stemming from the demand that photon energies keep real for all values of the photon wave vector and using the experimental bounds on the photon mass [148], $m_\gamma < 6 \times 10^{-17}$ eV. Thus ensuring the theory's stability, obtaining:

$$b_0 < 3 \times 10^{-15} \text{ eV}.$$

Appendix A

Dimensional Regularization of Feynman Integrals

A.1 The basic loop integral

Having justified the dimensional regularization for Feynman integrals in presence of a γ_5-matrix in 1.3.2, I will present a detailed calculation of a typical expression. In fact due to the aforementioned "symmetrization" properties under the momentum integral (which hold true in $\overline{DR}$),

$$p^\mu p^\nu \to \frac{1}{d} p^2 g^{\mu\nu} \tag{A.1}$$

$$p^\mu p^\nu p^\rho p^\sigma \to \frac{1}{d(d+2)} (p^2)^2 (g^{\mu\nu} g^{\rho\sigma} + g^{\mu\rho} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\rho}), \tag{A.2}$$

and similar expressions for the cases of more powers of loop momentum in the numerator, all Feynman integrals can be written in terms of an integral of the kind:

$$I_{s,n} = \int \frac{d^d p}{(2\pi)^d} \frac{p^{2s}}{(p^2 - \Delta + i\epsilon)^n}. \tag{A.3}$$

For which we momentarily consider the Euclidean expression:

$$I^E_{s,n} = \int \frac{d^d p}{(2\pi)^d} \frac{p_E^{2s}}{(p_E^2 + \Delta)^n}, \tag{A.4}$$

Henceforth the subscript E will be omitted in the loop momenta. Using the integral representation of the causal propagator (sometimes called the Schwinger parametrization) in momentum space

$$\frac{1}{a^n} = \frac{1}{\Gamma(n)} \int_0^\infty dt\ t^{n-1} e^{-ta}\,, \tag{A.5}$$

we write the previous integral as:

$$\begin{aligned} I^E_{s,n} &= \frac{1}{\Gamma(n)}\int_0^\infty dt t^{n-1}\int\frac{d^dp}{(2\pi)^d}\; p^{2s}e^{-t(p^2+\Delta)} \\ &= \frac{1}{\Gamma(n)}\int_0^\infty dt t^{n-1}e^{-t\Delta}\int\frac{d^dp}{(2\pi)^d}\; p^{2s}e^{-tp^2} \\ &= \frac{(-1)^s}{\Gamma(n)}\int_0^\infty dt t^{n-1}e^{-t\Delta}\frac{d^s}{dt^s}\int\frac{d^dp}{(2\pi)^d}\; e^{-tp^2}, \end{aligned} \tag{A.6}$$

where we have used: $p^{2s}e^{-tp^2} = (-1)^s\frac{d^s}{dt^s}e^{-tp^2}$. Now the Gaussian integral can be done as usual, recalling that in Euclidean space the components of the momentum vector can take any value in the real line, therefore,

$$\begin{aligned} \int\frac{d^dp}{(2\pi)^d}\; e^{-tp^2} &= \left(\frac{1}{2\pi}\right)^d\int d^dpe^{-tp^2}, \quad p\in(-\infty,\infty) \\ &= \left(\frac{1}{2\pi}\right)^d\left(\int dye^{-ty^2}\right)^d \\ &= \left(\frac{1}{2\pi}\right)^d\left(\frac{\pi}{t}\right)^{\frac{d}{2}} = \left(\frac{1}{4\pi t}\right)^{\frac{d}{2}}, \end{aligned} \tag{A.7}$$

therefore our integral is:

$$I^E_{s,n} = \frac{(-1)^s}{\Gamma(n)}\frac{1}{(4\pi)^{d/2}}\int_0^\infty dt t^{n-1}e^{-t\Delta}\frac{d^s}{dt^s}\left(\frac{1}{t}\right)^{d/2}, \tag{A.8}$$

and the derivative evaluates easily,

$$\frac{d^s}{dt^s}\left(\frac{1}{t}\right)^{d/2} = \frac{(-1)^s}{2^s}\prod_{j=0}^{s-1}(d+2j)t^{-(d/2+s)}. \tag{A.9}$$

therefore,

$$I^E_{s,n} = \frac{(-1)^s}{\Gamma(n)}\frac{1}{(4\pi)^{d/2}}\left[\frac{(-1)^s}{2^s}\prod_{j=0}^{s-1}(d+2j)\right]\int_0^\infty dt t^{(n-d/2-s)-1}e^{-t\Delta}. \tag{A.10}$$

making the change $t\Delta = z \Rightarrow dt = \Delta^{-1}dz$, and $t^{\alpha-1} = \Delta^{(1-\alpha)}z^{\alpha-1}$, we get for the last integral:

$$\int_0^\infty dt t^{\alpha-1}e^{-\Delta t} = \Delta^{-\alpha}\int_0^\infty dz z^{\alpha-1}e^{-z} = \Delta^{-\alpha}\Gamma(\alpha). \tag{A.11}$$

where $\alpha = n - d/2 - s$ yielding the integral in z in terms of a Gamma function. Therefore the final result for the basic integral in Euclidean space is:

$$\begin{aligned} I^E_{s,n} &= \int\frac{d^dp_E}{(2\pi)^d}\frac{p_E^{2s}}{(p_E^2+\Delta)^n} \\ &= \frac{1}{\Gamma(n)}\frac{1}{(4\pi)^{d/2}}\left[\frac{1}{2^s}\prod_{j=0}^{s-1}(d+2j)\right]\Delta^{(-n+d/2+s)}\Gamma(n-d/2-s). \end{aligned} \tag{A.12}$$

We must stress that in this derivation, the expression in square brackets yields 1 when $s = 0$. However we are really interested in the Minkowsky space integral, namely:

$$I_{s,n} = \int \frac{d^d p}{(2\pi)^d} \, \frac{p^{2s}}{(p^2 - \Delta + i\epsilon)^n} \qquad \text{with } p \text{ Minkowsky}, \tag{A.13}$$

where the poles are conveniently displaced above and below the real line for negative and positive p^0, respectively. Therefore, assuming the appropriate analytic behaviour of the integrand, *i.e.* that for large values of $|p^0|$ it goes to zero fast enough, then we can safely make an anticlockwise rotation of the contour of integration in the p^0 complex plane to the imaginary axis making, $p^0 \to ip_E^0$ and split the momentum integral as

$$\int d^d p = \int d^{d-1} p \int dp^0 = \int d^{d-1} p \, (i \int dp_E^0). \tag{A.14}$$

Also, $p^2 = (p^0)^2 - \vec{p}^{\,2} = -(p_E^0)^2 - \vec{p}^{\,2} \equiv -p_E^2$, thus $p^{2s} = (-1)^s p_E^{2s}$, which implies and overall factor $(-1)^s$. And similar thing happens in the denominator:

$$(p^2 - \Delta + i\epsilon)^n = (-p_E^2 - \Delta)^n = (-1)^n (p_E^2 + \Delta)^n. \tag{A.15}$$

Altogether we get as prefactors $i(-1)^{s-n}$, thus $I_{s,n} = i(-1)^{s-n} I_{s,n}^E$. Finally the basic momentum integral in Minkowsky space is:

$$\int \frac{d^d p}{(2\pi)^d} \, \frac{p^{2s}}{(p^2 - \Delta + i\epsilon)^n} \;=\; \frac{i}{\Gamma(n)} \frac{(-1)^{s-n}}{(4\pi)^{d/2}} \left[\frac{1}{2^s} \prod_{j=0}^{s-1} (d + 2j) \right] \Delta^{(-n+d/2+s)} \Gamma(n - d/2 - s) \tag{A.16}$$

With this formula we can readily obtain the loop integrals that can be found in most QFT textbooks[1],namely:

$$\begin{aligned}
I_{0,n} &= \int \frac{d^d l}{(2\pi)^d} \, \frac{1}{(l^2 - \Delta + i\epsilon)^n} = \frac{(-1)^n i}{(4\pi)^{d/2}} \frac{\Gamma(n - \frac{d}{2})}{\Gamma(n)} \Delta^{(-n+\frac{d}{2})} \\
I_{1,n} &= \int \frac{d^d l}{(2\pi)^d} \, \frac{l^2}{(l^2 - \Delta + i\epsilon)^n} = \frac{(-1)^{n-1} i}{(4\pi)^{d/2}} \frac{d}{2} \frac{\Gamma(n - \frac{d}{2} - 1)}{\Gamma(n)} \Delta^{(-n+\frac{d}{2}+1)} \\
I_{2,n} &= \int \frac{d^d l}{(2\pi)^d} \, \frac{(l^2)^2}{(l^2 - \Delta + i\epsilon)^n} = \frac{(-1)^n i}{(4\pi)^{d/2}} \frac{d(d+2)}{4} \frac{\Gamma(n - \frac{d}{2} - 2)}{\Gamma(n)} \Delta^{(-n+\frac{d}{2}+2)}.
\end{aligned} \tag{A.17}$$

Furthermore, by the symmetrical integration, using A.1, we can obtain, for example:

$$\begin{aligned}
\int \frac{d^d p}{(2\pi)^d} \frac{(a \cdot l)(b \cdot l)}{(p^2 - \Delta + i\epsilon)^n} &= a_\mu b_\nu \int \frac{d^d p}{(2\pi)^d} \frac{l^\mu l^\nu}{(p^2 - \Delta + i\epsilon)^n} \\
&= a_\mu b_\nu \frac{g^{\mu\nu}}{d} \int \frac{d^d p}{(2\pi)^d} \frac{l^2}{(p^2 - \Delta + i\epsilon)^n} \\
&= \frac{a \cdot b}{d} I_{1,n}(\Delta).
\end{aligned} \tag{A.18}$$

[1] A common reference on the subject is the book by Peskin and Schroeder [8]. The formulas commented appear on the appendix of this textbook.

And also,

$$\begin{aligned}
\int \frac{d^d l}{(2\pi)^d} \frac{l^\mu l^\nu}{(l^2 - \Delta + i\epsilon)^n} &= \frac{g^{\mu\nu}}{d} \int \frac{d^d l}{(2\pi)^d} \frac{l^2}{(l^2 - \Delta + i\epsilon)^n} \\
&= \frac{g^{\mu\nu}}{2} \frac{(-1)^{n-1} i}{(4\pi)^{d/2}} \frac{\Gamma(n - \frac{d}{2} - 1)}{\Gamma(n)} \Delta^{(-n+\frac{d}{2}+1)}, \qquad \text{(A.19)} \\
\int \frac{d^d l}{(2\pi)^d} \frac{l^\mu l^\nu l^\rho l^\sigma}{(l^2 - \Delta + i\epsilon)^n} &= \frac{g^{\mu\nu} g^{\rho\sigma} + g^{\mu\rho} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\rho}}{d(d+2)} \int \frac{d^d l}{(2\pi)^d} \frac{l^4}{(l^2 - \Delta + i\epsilon)^n} \\
&= \frac{g^{\mu\nu} g^{\rho\sigma} + g^{\mu\rho} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\rho}}{4} \times \\
&\quad \frac{(-1)^n i}{(4\pi)^{d/2}} \frac{\Gamma(n - \frac{d}{2} - 2)}{\Gamma(n)} \Delta^{(-n+\frac{d}{2}+2)}. \qquad \text{(A.20)}
\end{aligned}$$

In these last integrals no prefactors on the RHS depending on d appear, as for $I_{1,n}$ or $I_{2,n}$, since these cancel with equal terms coming from the momentum symmetrization of the integrals.

A.2 Isolating the divergencies

Now that we have evaluated the basic integral, we must isolate the divergent pieces. However, the physical integral is a 4-dimensional one. As explained in 1.3.2, the power of dimensional regularization is to consider instead of the original 4-dimensional integral, the equivalent expression but where formerly there was a 4 putting a parameter d,

$$\int \frac{d^4 p}{(2\pi)^4} \longrightarrow \int \frac{d^d p}{(2\pi)^d}, \qquad \text{(A.21)}$$

where d is now a parameter[2] . It is important to keep the parameter d more abstract than just the dimension of spacetime. In fact, d may not be integer and actually complex. The crucial point is that the previously obtained formulas for Feynman integrals are analytic in d, (except for some values of d where they diverge), and therefore one can define them in terms of the expression for those values of d where it is indeed well defined and from it make an analytic continuation to the whole complex plane to define their value even on those regions were it was originally divergent.

Taking the case of a UV-divergent 4-dimensional Feynman integral, with the knowledge of dimensional regularization we can take $d = 4 - 2\varepsilon$ and make ε tend to zero at the end of the calculation. (It is only a this step that recover our physical quantity). Thus we can keep track of ("isolate") the divergencies by analyzing what happens when $\varepsilon \to 0$ with every factor in the previous results for $I_{s,n}$ that depends on d.

[2] As commented in 1.3.2, of course that the norm of a vector in the integrand, should also be considered as a norm "in d-dimensions". And similarly this extends for Dirac-matrices' algebra.

First, since $d = 4 - 2\varepsilon \Rightarrow d/2 = 2 - \varepsilon$, then:

$$\frac{\Delta^{d/2+s-n}}{(4\pi)^{d/2}} = \frac{\Delta^{2+s-n}}{(4\pi)^2}\left(\frac{\Delta}{4\pi}\right)^{-\varepsilon}, \tag{A.22}$$

for which we use the expansion $a^x = e^{x \ln a} = 1 + x\ln a + \cdots, \quad -\infty < x < \infty$, therefore,

$$\frac{\Delta^{d/2+s-n}}{(4\pi)^{d/2}} = \frac{\Delta^{2+s-n}}{(4\pi)^2}\left[1 - \varepsilon \ln\left(\frac{\Delta}{4\pi}\right)\right]. \tag{A.23}$$

We must also remember that when doing dimensional regularization, we must introduce a mass scale μ. This is evident because Δ is dimensionful therefore we need a dimensionless combination to evaluate the log. On more general grounds, we must be certain that at all stages in "d-dimension", the action remains dimensionless, or which amounts to the same, that the Lagrangian has mass dimension d. Thus from the QED Lagrangian we can verify that $[\psi] \to [M]^{\frac{d}{2}-\frac{1}{2}}$ and $[A_\mu] \to [M]^{\frac{d}{2}-1}$ wherefrom, $[e] \to [M]^{2-\frac{d}{2}}$. Therefore, the relevant coupling constant in our calculations is $e^2 \to e^2\mu^{4-d}$, where e is now to be understood as dimensionless and μ has dimensions of mass. Of course, this mass scale as it depends on the parameter d will also contribute with divergent pieces as $\epsilon \to 0$ and the equation A.23 now reads:

$$\mu^{4-d}\frac{\Delta^{(d/2+s-n)}}{(4\pi)^{d/2}} = \frac{\Delta^{(2+s-n)}}{(4\pi)^2}\left[1 - \frac{\epsilon}{2}\ln\left(\frac{\Delta}{4\pi\mu^2}\right)\right] + \mathcal{O}(\epsilon^2), \tag{A.24}$$

i.e. the mass scale is naturally incorporated in the log, rendering its argument dimensionless. Also we have to analyze:

$$\begin{aligned}
\frac{1}{2^s}\prod_{j=0}^{s-1}(d+2j) &= \frac{1}{2^s}\prod_{j=0}^{s-1}(4-2\varepsilon+2j) = \frac{1}{2^s}\left[(4-2\varepsilon)(4-2\varepsilon+2)(4-2\varepsilon+4)\cdots\right] \\
&= \frac{1}{2^s}[\overbrace{2(2-\varepsilon)2(2-\varepsilon+1)2(2-\varepsilon+2)\cdots}^{s \text{ terms}}] = (2-\varepsilon)(3-\varepsilon)(4-\varepsilon)\cdots \\
&= (2-\varepsilon)(3-\varepsilon)(4-\varepsilon)\cdots((1+s)-\varepsilon) \\
&= [(1\cdot 2\cdot 3\cdots)] \\
&\quad -\varepsilon[(\cancel{2}\cdot 3\cdot 4\cdots) + (2\cdot\cancel{3}\cdot 4\cdots) + \cdots + (2\cdot 3\cdots\cancel{s}\cdot(s+1)) + (2\cdot 3\cdots s\cdot(\cancel{s+1}))] \\
&\quad +\mathcal{O}(\varepsilon^2) \\
&= (s+1)! - \varepsilon[\frac{(s+1)!}{2} + \frac{(s+1)!}{3} + \cdots + \frac{(s+1)!}{(s+1)}] + \mathcal{O}(\varepsilon^2) \\
&= (s+1)!\left[1 - \varepsilon(\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{s+1})\right] + \mathcal{O}(\varepsilon^2) \\
&\approx (s+1)!\left[1 - \varepsilon\sum_{t=1}^{s}\frac{1}{t+1}\right].
\end{aligned} \tag{A.25}$$

Finally there will also be some divergencies coming from the Gamma function, which will depend on the exponents of our fundamental integral, which takes the form, recalling that $\Gamma(n) = (n-1)!$ since $n \in \mathbb{N}$.

$$\begin{aligned}\int \frac{d^dp}{(2\pi)^d} \frac{p^{2s}}{(p^2-\Delta+i\epsilon)^n} &= i(-1)^{s-n}\frac{(s+1)!}{(n-1)!}\frac{\Delta^{2+s-n}}{(4\pi)^2}\left[1-\varepsilon\texttt{lnDelta}\right] \\ &\quad \left[1-\varepsilon C(s)\right]\Gamma(n-2-s+\varepsilon), \end{aligned} \tag{A.26}$$

where `lnDelta` has been defined above and $C(s) = \sum_{t=1}^{s} \frac{1}{t+1}$. This expression is supposed to incorporate the mass scale μ necessary in dimensional regularization. Our calculations in the text do not show it explicitly as it carries along in the same manner as in ordinary QED.

Finally we must see what happens with the Gamma function above, when $\varepsilon \to 0$, since the $\Gamma(n)$ has isolated poles at $n = 0, -1, -2, \cdots$. For this we use the definition of the Gamma function for negative arguments.

$$\Gamma(n) = \frac{\Gamma(n+1)}{n} \qquad \text{for} \qquad n < 0, \tag{A.27}$$

therefore:

$$\begin{aligned}\Gamma(-1+\varepsilon) &= \frac{\Gamma(\varepsilon)}{-1+\varepsilon} \\ &\approx -\Gamma(\varepsilon)(1+\varepsilon) = -\Gamma(\varepsilon) - \varepsilon\Gamma(\varepsilon) = -\Gamma(\varepsilon) - \Gamma(1+\varepsilon) \\ &\approx -\Gamma(\varepsilon) - 1 \\ &\approx -1 - \frac{1}{\varepsilon} - \Gamma'(1) \\ &\approx -1 - \frac{1}{\varepsilon} + \gamma_E + \mathcal{O}(\varepsilon). \end{aligned} \tag{A.28}$$

where the the limit $\varepsilon \to 0$ in $\Gamma(1+\varepsilon)$ is straightforward. And to evaluate $\Gamma(\varepsilon)$ we simply Taylor expand about ε. Similarly we get

$$\Gamma(-2+\varepsilon) \approx \frac{1}{2\varepsilon} + \frac{3}{4} - \frac{1}{2}\gamma_E + \mathcal{O}(\varepsilon), \quad \text{and so on.} \tag{A.29}$$

Alternatively one could make use of the infinite product representation of the gamma fuction.

Appendix B

FORM computation of the axial anomaly using $\overline{DR}$

```
***********************************************
* This program computes the ABJ anomaly using the
* 't Hooft-Veltman-Breitenlohner-Maison Dimensional Regularization.
* (For additional comments see LSBQED notebook2, pp. 276)
***********************************************
* DECLARATIONS

Symbol e,h,LK,L,LP,ABC,N,[lnM^2],j,j2,s1,s2,s3,h,n1,n2,n3,ABC,fx,fy,fz,
        c0,lh2,[lnDelta],g,lnM2,ep,c0,ct,pi,DD;
Vector q,p,k,l,lh,l,v,o;
Function A,SLP,SLK,SL,Gh;
AutoDeclare Indices m;
CFunction cc,B,[Gamma],C;
Indices m,n,r;
Dimension N;
*Unittrace N;
Off Statistics;

* THE CODE

Local qTD= e^2*( -2*g5_(1)*Gh(lh)* SLK * g_(1,m2) * SL * g_(1,m3) * SLP);
Print;
.sort;
id SLK = (g_(1,l) + Gh(lh) - g_(1,k))/LK;   * THESE ARE THE FERMIONIC PROGATORS.
id SL = (g_(1,l) + Gh(lh))/L;               * Gh(lh) IS \hat \gamma_\mu \hat l^\mu.
```

```
id SLP = (g_(1,l) + Gh(lh) + g_(1,p))/LP;   * lh is part of l living in N-4-dims.
.sort
trace4,1;                                   * TRACES CAN BE DONE AND Gh DEALT
.sort;                                      * WITH LATER.
id Gh(p?)*Gh(p?)=lh2;                       * lh2 = \hat l_\mu \hat l^\mu
.sort                                       * NO NEED TO DEFINE FURTHER ALGEBRA
id lh2 = lh.lh;                             * FOR Gh. ONLY Gh^2 SQUARED TERMS.
.sort
id lh.lh=c0*l.l;                            * SYMMETRIZING COEF. REFERRED TO IN
.sort                                       * THE TEXT. EQN \gl{eq:chiral16}
                                            * AND DEFINED BELOW IN TERMS OF
                                            * THE SPACETIME DIMENSION
id l = j*l + (fx*k - fy*p);                 * fx, fy FEYNMAN PARAMETERS
.sort;
id j^2=j2;                                  * COUNTER FOR SYMMETRICAL INTEGRATION
.sort                                       * ONLY EVEN POWERS SURVIVE.
id j=0;
.sort
id j2 = 1;
.sort

* B() IS THE PRODUCT OF GAMMA FNS. IN THE USE OF FEYNMAN PARAMETERS
* ABC IS THE COMMON DENOMINATOR OF THE 3 PROPAGATOR'S DENOMINATORS.
id LK^-n1?*L^-n2?*LP^-n3? = B(n1,n2,n3)*fx^(n1-1)*fy^(n2-1)*fz^(n3-
1)*ABC^(n1+n2+n3);
.sort
id B(s1?,s2?,s3?)=fac_(s1+s2+s3-1)/(fac_(s1-1)*fac_(s2-1)*fac_(s3-1));
.sort
id c0 = (N-4)/N;
.sort;

* THE MOMENTUM INTEGRAL USING THE GENERAL RESULT OBTAINED BY DIM
* REG. [Gamma] IS THE Gamma FN. DD IS THE TYPICAL \Delta IN THE
* DR'ed MOMENTUM INTEGRAL.
id l.l^s1?*ABC^s2? = ((-1)^(s2-1)*i_*N)/(2*(4*pi)^(N/2))*
                    (([Gamma](s2-s1-2+ep))/(fac_(s2-1)))
                    *DD^(2-ep+1-s2);
.sort
repeat;
id N = 4 - 2*ep;
endrepeat;
.sort;
```

```
id [Gamma](ep)=-g+1/ep;                * EXPANSIONS OF Gamma FN FOR SMALL ARGS.
id [Gamma](1+ep)=1;
.sort;
id N = 2*(2 - ep);
id ep = 0;                             * "EVANESCENT" TERMS VANISH.
id DD^-ep = 1;
.sort;
Local finalresult = qTD/2;             * THE INTEGRAL OVER FEYNMAN PARAMETERS
Print +s finalresult;                  * NEEDS TO BE DONE YET. IT IS:
.sort;                                 * \int_0^1 D(xyz) \delta(x+y+z-1).1=1/2.
.end
```

The output of the program reads:

```
finalresult =
       + 4/((4*pi)^(1/2*N))*e_(p,k,m2,m3)*i_*e^2.
```

Considering the SCHOONSCHIP notation for contraction of indices employed by FORM and the fact that it uses the Pauli metric for the Levi-Civita symbol, this result which computes the contribution to the divergence of the axial vector current of the first triangle diagrams considered in section (1.3.2) is the same as the standard result.

Appendix C

Explicit calculation of vacuum polarization with FORM, using $\overline{DR}$

In this appendix we present the computation of the vacuum polarization tensor of the LIVQED model under consideration. The computation was done in FORM using the 't Hooft-Veltman-Breitenlohner-Maison dimensional regularization method already described. The computation algorithm implemented is similar to the one that was built in order to find the correct perturbative value for the divergence of the axial vector current in QED, exhibited in appendix (B).

```
Symbols M,M2,h,e,P2;
Vector p,b,q,p2,b2,k2,q2,bp,bq,bpk,p4,b4,ph,qh,q4,k4,ph2,k,PH;
Functions g0,g1,g2,g3,g,G5,GP,GQ,Gh;
Indices mu,nu,rho,sigma,lambda;
Indices m,n;
Symbol N,y,Y;
Off Statistics;

Local [Spnum] = i_*( (p2 + b2 - M2) + 2*( M*y*g_(1,b) + bp)*G5 )*(GP + M + y*g_(1,b)*G5 );
Local [Sp-knum] = i_*( (q2 + b2 - M2) + 2*( M*y*g_(1,b) + bq)*G5 )*(GQ + M + y*g_(1,b)*G5 );

id M2 = M^2;                    * p and q are loop momenta
id b = b4;                      * q=p-k, k is the external incoming momentum
id p2 = p.p;                    * Thus, qh = ph
id b2 = b4.b4;                  * k lives in 4D, hence kh = 0.
id bp = b4.p4;                  * p.q = (p4,ph)[g](q4,qh)
```

```
id GP = y*g_(1,p4) + y*Gh(ph);  * [g] is block diagonal [g] = diag([g4],[gh])
id G5 = y*g5_(1);
id GQ = y*g_(1,p4) - y*g_(1,k4) + y*Gh(ph);
                                        * q2 =  qb2 - qh2 = (p4 - k4)^2 - ph^2 =
id q2 = p.p - 2*p4.k4 + k4.k4;          *       p4.p4 + k4.k4 -2*p4.k4 - ph.ph =
.sort                                   *       p4.p4 - ph.ph - 2p4.k4 + k4.k4 =
                                        *       p.p - 2*p4.k4 + k4.k4
id bq = b4.p4 - b4.k4;
.sort;
Local [PolTensorNum] = h*y*((g_(1,mu)*[Spnum]*y*g_(1,nu)*[Sp-knum]));
id h = (-i_*e)^2*(-1);
.sort;
id y^2 = Y;                   * A counter of the order of gamma matrices using y.
.sort                         * Terms with even number of g matrices (g5 in-
cluded) survive
id y = 0;                     * not only upon tracing but also for CPT even contrib
.sort
repeat;                       *   Remove the counter.
id Y = 1;
endrepeat;
.sort

repeat;                             * g5 commutes with all gamma matrices of N-
4 dims.
id Gh(p?)*g5_(1)=g5_(1)*Gh(p);
endrepeat;
repeat;                             * Gh anticommute with gamma matrices of 4D.
id Gh(p?)*g_(1,q?)=-g_(1,q)*Gh(p);
id Gh(p?)*g_(1,m?)=-g_(1,m)*Gh(p);
endrepeat;
.sort
id Gh(p?)*Gh(p?)=ph2;
.sort
id Gh(p?)=0;
.sort
trace4,1;
.sort
id ph2 = ph.ph;
.sort
*************************************************************************
* Now we proceed with the Denominators of each propagator.
*************************************************************************
```

```
Symbol xp,xq,Dp,Dq;
*   These are the denominators of each propagator, expanded up to second order
*   in b.

Local [Dp] = (1 - 2*b4.b4/xp + 4*((b4.p4)^2 - b4.b4*M^2)/xp^2)/xp^2;
Local [Dq] = (1 - 2*b4.b4/xq + 4*((b4.p4 - b4.k4)^2 - b4.b4*M^2)/xq^2)/xq^2;
.sort;
Local [PolTensor] = [PolTensorNum]*[Dp]*[Dq];

*   and the full PolTensor is also expanded to second order in b.
if (count(b4,1)>2) discard;
.sort;

id p.p = p2;
.sort;
*******************************************************
**   here we make the shift in the momentum and
**   the replacements of the propagators' denominator
**   in term of one single denominator, by introducing
**   Feynman's Parameters, which we call $fx$.
**   B is the product of gamma fns. G(m+n)/(G(m)*G(n))
**   AB is the one single denominator.
*******************************************************
Symbol N,[lnM^2],j,j2,s1,s2,s3,h,n1,n2,AB,fx,c0,ct,K,ep,k42,k4v;
AutoDeclare Symbol c;
Indices m,n,r;
CFunction B,cc,Op;
Vector l,v,o,u,w;
Set F: k,b;
Tensor t;
Dimension 4;

* The Momentun shift to use Feynman parameter fx!
id p4 = p4 + k4*(1-fx);    * k  in real space only, thus ph is not shifted
.sort;
*   However, p2 = p.p is!
id p2 = p.p + 2*p4.k4*(1-fx) + k4.k4*(1-fx)^2;
.sort;
id p4.p4 = p.p + ph.ph;         * To leave everything in terms of p and ph only.
.sort;                          * Use the "general" formulae
id p4 = p - ph;
.sort;
```

```
*   there where however terms p4.b4 -> p.b4 - ph.b4, and so with k4,
*   and since they are physical have no projection to the "rest" of space.
id k4.ph = 0;
id b4.ph = 0;
id ph.p4 = 0;
.sort

*   RECALL THAT k4.p = k4.p4 and so with b.
*   Eliminate odd powers of j, thus of p for symmetrical integration
id p = j*p;
id ph = j*ph;
.sort;

id j^2=j2;
.sort
id j=0;
.sort
id j2 = 1;
.sort

*   The shift above also generates terms linear in ph
*   These should not survive.

id ph = j*ph;
.sort
id j^2=j2;
.sort
id j=0;
.sort
id j2 = 1;
.sort;

*   Bringing both denoms into a single one and below the products of Gamma fns.
id xp^-n1?*xq^-n2? = B(n1,n2)*fx^(n1-1)*(1-fx)^(n2-1)*AB^(n1+n2);
.sort

id B(s1?,s2?)=fac_(s1+s2-1)/(fac_(s1-1)*fac_(s2-1));
.sort

*   And simply to write less,
repeat;
id k4 = k;
```

```
id b4 = b;
endrepeat;
.sort;

*   The output (in mc1_patterns.tx, printout #999) yielded terms
*   with ph(mu), ph(nu). However, the indices mu, nu are
*   external/physical ones. Thus this terms must vanish

id ph(mu) = 0;
id ph(nu) = 0;
.sort;

************************************************************
*   here we use symmetry properties under the momentum integral
*   such as: p(m)*p(n) = (1/n)*d_(m,n)*p.p
******************************************
* Identifications must be done from "highest powers"
* of p to lowers, in order not to produce identifs
* prior to what is needed.
* To this aim, halted the program previous to the
* identifs. Printed and factored out ph. This
* produces the tensor structure of the vacuum polarization
* with all the terms to symmetrize. Printed out in Printout # 100.
* simplified in Printout # 101, whereform the terms are identified.
*****************************************************************
Symbol C8,C6,C4,C2;
AutoDeclare Symbols K;
AutoDeclare Symbols F;
AutoDeclare Indices m;
Function f2,f4,f6;
********************************
* ph.ph terms
*******************************

* pattern 0                                        * O(p^8) terms
id p.p^2*p.b^2*ph.ph = p.p^4*b.b*K0;
.sort;

* pattern 1                                        * O(p^6) terms
id ph.ph*p.p^2 = p.p^3*K1;
.sort;
* pattern 2
```

```
id ph.ph*b.p^2*p.k^2 = p.p^3*(b.b*k.k + 2*b.k^2)*K2;
.sort;
* pattern 5
id ph.ph*p.p*p.b^2 = p.p^3*b.b*K3;
.sort;
* pattern 3
id ph.ph*p.p*p.b*p.k = p.p^3*b.k*K4;
.sort;

* pattern 8                                          * O(p^4) terms
id ph.ph*p.p = p.p^2*K5;
.sort;
* pattern 6
id ph.ph*b.p^2 = p.p^2*b.b*K6;
.sort;
* pattern 7
id ph.ph*k.p^2 = p.p^2*k.k*K7;
.sort;
* pattern 4
id ph.ph*b.p*p.k = p.p^2*b.k*K8;
.sort;



* pattern 9                                          * O(p^2) terms
id ph.ph = p.p*K9;
.sort;

* The K coefficients
id K0 = (N-4)*(N^2 + 10*N + 24)*C8;
id K1 = (N-4)*(N^2 + 6*N + 8)*C6;
id K2 = (N - 4)*C6;
id K3 = (N^2 - 16)*C6;
id K4 = (N^2 - 16)*C6;
id K5 = (N^2 - 2*N - 8)*C4;
id K6 = (N-4)*C4;
id K7 = (N-4)*C4;
id K8 = (N-4)*C4;
id K9 = (N-4)*C2;
.sort;

* pattern XX                                         * O(p^6) terms without ph
id p(m1?)*p(m2?)*(p.b)^2*(p.k)^2 = C6*f6(m1,m2,m3,m4,m5,m6)*b(m3)*b(m4)*k(m5)*k(m6)*p.p^3;
```

```
* pattern 20                                          * O(p^4) terms without ph
id p.b*p.k^3 = C4*f4(m1,m2,m3,m4)*b(m1)*k(m2)*k(m3)*k(m4)*p.p^2;
* pattern 21
id p.b^2*p.k^2 = C4*f4(m1,m2,m3,m4)*b(m1)*b(m2)*k(m3)*k(m4)*p.p^2;
* pattern 22
id p(m1?)*p(m2?)*(p.b)*(p.k) =  C4*f4(m1,m2,m3,m4)*b(m3)*k(m4)*p.p^2;
* pattern 23
id p(m1?)*p(m2?)*(p.b)^2 = C4*f4(m1,m2,m3,m4)*b(m3)*b(m4)*p.p^2;
* pattern 24
id p(m1?)*p(m2?)*(p.k)^2 = C4*f4(m1,m2,m3,m4)*k(m3)*k(m4)*p.p^2;
* pattern 25
id p(m1?)*(p.b)*(p.k)^2 =  C4*f4(m1,m2,m3,m4)*b(m2)*k(m3)*k(m4)*p.p^2;
* pattern 26
id p(m1?)*(p.b)^2*(p.k) =  C4*f4(m1,m2,m3,m4)*b(m2)*b(m3)*k(m4)*p.p^2;

* pattern 33                                          * O(p^2) terms without ph
id p.b^2 = C2*f2(m1,m2)*b(m1)*b(m2)*p.p;
.sort;
* pattern 34
id p.k^2 = C2*f2(m1,m2)*k(m1)*k(m2)*p.p;
.sort;
* pattern 32
id p.b*p.k = C2*f2(m1,m2)*b(m1)*k(m2)*p.p;
.sort;
* pattern 27
id p(m1?)*(p.b) = C2*f2(m1,m2)*b(m2)*p.p;
.sort;
* pattern 28
id p(m1?)*(p.k) = C2*f2(m1,m2)*k(m2)*p.p;
.sort;
** pattern 29
*id p(m1?)*b(m2?)*(p.b) = C2*f2(m1,m3)*b(m2)*b(m3)*p.p;
*.sort;
** pattern 30
*id p(m1?)*b(m2?)*(p.k) = C2*f2(m1,m3)*b(m2)*k(m3)*p.p;
*.sort;
* pattern 31
id p(m1?)*p(m2?) = C2*f2(m1,m2)*p.p;
.sort;

id f2(m1?,m2?) =dd_(m1,m2);
```

```
id f4(m1?,m2?,m3?,m4?) =dd_(m1,m2,m3,m4);
id f6(m1?,m2?,m3?,m4?,m5?,m6?) =dd_(m1,m2,m3,m4,m5,m6);
.sort;

*******************************************
*   Now we replace (p.p)^s/[AB]^n by the momentum
*   integral with tha Gamma funs, the i_ and so on.
*   Below, C is the products of (d/2)(d/2+1)...
*   There is a overall factor (4Pi)^2 that we'll leave
*   outside.
*   lnDelta is Ln(Delta/4Pi)
*   and Gamma(s2) is (s2-1)! since s2 is integer
*******************************************
Symbol s1,s2,d,[lnDelta],gE,[Delta],yy,lnM2;
*gE is the Euler Macheroni constant
CFunction [Gamma],C;

id p.p^s1?*AB^s2? = i_*(-1)^(s1-s2)*fac_(1+s1)*(1-ep*C(s1))*[Delta]^(2+s1-s2)*
    (1-ep*[lnDelta])*([Gamma](s2-2-s1+ep)/fac_(s2-1));
.sort

id C(0)=0;
.sort
id C(s1?) = sum_(s2, 1, s1, (1+s2)^-1 );
.sort

id [Gamma](-1+ep)=-1-gE-1/ep;
id [Gamma](ep)=gE+1/ep;
id [Gamma](1+ep)=1;
id [Gamma](2+ep)=1;
id [Gamma](3+ep)=2;
id [Gamma](4+ep)=6;
.sort

*   See file ccoefs.frm and then put N = 4 - 2 ep , the expand to first
*   order in ep. See file C:\...\Results in LSB QED\ccoefs.nb
*   symmetrization coeffs expanded in terms of ep
*   terms quadratic in ep do no harm but kept just in case.

id C2 =1/4 + ep/8 + ep^2/16;
id C4=1/24 + 5*ep/144 + 19*ep^2/864;
id C6=1/192 + 13*ep/2304 + 115*ep^2/27648;
```

```
id C8 = 1/1920 + 77*ep/115200 + 3799*ep^2/6912000;
.sort;

id N=4-2*ep;
.sort

*   Following normalqed.frm here we compute to second order in k

id [Delta]=-k.k*fx*(1-fx)+M^2;
.sort
id [Delta]^-s1?=M^-(2*s1)*(1+s1*y+s1*(s1+1)*y^2/2);
id [lnDelta]=lnM2-y+y^2/2;
.sort
id y=k.k*fx*(1-fx)/M^2;
.sort

*id fx^s1?=(1+s1)^-1;
*.sort
*   FORM gets dizzy with this integration but it is no harm to do it by hand

id fx^16 = (17)^-1;
id fx^15 = (16)^-1;
id fx^14 = (15)^-1;
id fx^13 = (14)^-1;
id fx^12 = (13)^-1;
id fx^11 = (12)^-1;
id fx^10 = (11)^-1;
id fx^9 = (10)^-1;
id fx^8 = (9)^-1;
id fx^7 = (8)^-1;
id fx^6 = (7)^-1;
id fx^5 = (6)^-1;
id fx^4 = (5)^-1;
id fx^3 = (4)^-1;
id fx^2 = (3)^-1;
id fx^1 = (2)^-1;
.sort

* to compute to second order in k:
if (count(k,1)>2) discard;
.sort
```

```
id ep=0;
.sort

id gE = 0;
id [lnDelta] = 0;
.sort;

id gE = 0;
id lnM2 = 0;
.sort;

* For taking only the part quadratic in b,
if (count(b,1)<2) discard;
.sort

B ep;
Print +s [PolTensor];
.end
```

And the output of this program yields:

```
[PolTensor] =
    - 8/3*b(mu)*b(nu)*k.k*i_*M^-2*e^2
       + 8/3*b(mu)*k(nu)*b.k*i_*M^-2*e^2
       + 8/3*b(nu)*k(mu)*b.k*i_*M^-2*e^2
       - 8/3*k(mu)*k(nu)*b.b*i_*M^-2*e^2
       + 8/3*d_(mu,nu)*b.b*k.k*i_*M^-2*e^2
       + 8/3*d_(mu,nu)*b.b*i_*e^2
       - 8/3*d_(mu,nu)*b.k^2*i_*M^-2*e^2.
```

This constitutes one of the main results of this thesis, presented in (3.25) and reported in [5].

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