Université d'Aix-Marseille
Faculté de Sciences
Ecole Doctorale Physique et Sciences de la Matière
Centre Physique Théorique de Luminy

THÈSE

Thèse presńtée pour l'obtention du diplôme

Docteur d'Aix-Marseille Université

Mention: Physique Théorique et Mathématique

par Ernesto Frodden

Sur les Propriétés Thermodynamiques et Quantiques des Trous Noirs

---

Composition du jury:

Jorge Alfaro, Universidad Católica de Chile.	Examinateur
Fernando Barbero, Instituto de Estructura de la Materia, CSIC España.	Rapporteur
Karim Noui, APC Université Diderot, Paris 7.	Rapporteur
Alejandro Pérez, Centre de Physique Théorique, Aix-Marseille.	Directeur
Jorge Zanelli, Centro de Estudios Científicos del Sur, Chile.	Examinateur

Université d'Aix-Marseille
Faculté de Sciences
Ecole Doctorale Physique et Sciences de la Matière
Centre Physique Théorique de Luminy

THESIS

A thesis in candidacy for the degree of

Doctor of Philosophy

Mention: Theoretical and Mathematical Physics

by Ernesto Frodden

On the Thermodynamic and Quantum Properties of Black Holes

Thesis jury:

Jorge Alfaro, Universidad Católica de Chile.	Examiner
Fernando Barbero, Instituto de Estructura de la Materia, CSIC España.	Referee
Karim Noui, APC Université Diderot, Paris 7.	Referee
Alejandro Pérez, Centre de Physique Théorique, Aix-Marseille Université.	Advisor
Jorge Zanelli, Centro de Estudios Científicos del Sur, Chile.	Examiner

CONTENTS

Résumé en Français

Les trous noirs sont une des prédictions les plus étonnantes de la Relativité Générale. Au niveau théorique, les principaux concepts révolutionnaires introduits par la Relativité Générale comme l'espace-temps, la courbure, ou la causalité jouent un rôle fondamental pour comprendre les solutions de trous noirs. Les trous noirs font, dans le même temps, parti des solutions non triviales les plus simples, autant conceptuelement que mathématiquement, de la théorie. Par conséquent, ils constituent un espace naturel pour comprendre la Relativité Générale, ainsi qu'un domaine privilégié dans lequel toute modification ou extension de cette théorie peut être vérifié. D'autre part, au niveau de l'astrophysique, les trous noirs sont tout autant étonnants. Ils sont la seule explication pour le devenir de certaines étoiles massives. Par conséquent, ils sont fondamentales pour comprendre l'évolution des corps célestes de notre univers.

Invraisemblablement, en elles-mêmes, les solutions mathématiques des trous noirs annoncent également les limites de la Relativité Générale. Elles souffrent du même problème que la solution de particule ponctuelle dans l'Électromagnétisme: Une région avec une densité de charge infinie est un non-sens physique. Dans ce dernier cas, une théorie complètement différente, l'Électrodynamique Quantique, était nécessaire pour résoudre le problème et donner une description précise de la façon dont se comportent effectivement les champs à de courtes échelles, et, au cours de ce voyage, elle change aussi le concept de particule et leurs interactions. Dans les trous noirs, la présence des singularités dans leurs description mathématique, une région où la densité de la matière est infinie, est également un non-sens physique. Elles mettent en évidence la nécessité d'aller au-delà de la Relativité Générale et de chercher une nouvelle théorie qui permet une meilleure compréhension.

Une question plus intéressante et fructueuse concernant les trous noirs est le résultat théorique que, déjà au niveau classique, les trous noirs stationnaires imitent les lois de la thermodynamique. Par exemple, la masse d'un trou noir stationnaire peut être considéré comme son contenu énergétique. La variation de la masse est naturellement liée à la variation d'autres propriétés intrinsèques du trou noir comme la charge, la vitesse angulaire et l'aire géométrique de l'horizon du trou noir. Ce rapport est réalisé d'une manière très particulière qui est complètement analogue à *la première loi de la thermodynamique*. De même, le théorème de l'aire de Hawking qui affirme que dans tout processus classique l'aire de l'horizon peut seulement grandir, cela est aussi en analogie avec *la deuxième loi de la thermodynamique*, si l'aire de l'horizon est identifié avec l'entropie du système de trou noir. *La loi zéro de la thermodynamique*, dit qu'en équilibre thermique la température

des systèmes physiques doit être uniforme. Dans les trous noirs, *la gravité de surface* est uniforme sur l'horizon, c'est encore en analogie avec la loi zéro, et suggère en plus que la gravité de la surface doit être identifié avec la température du trou noir. Enfin, il y a aussi une analogie avec l'une des formulations de *la troisième loi de la thermodynamique* qui nous dit qu'il est impossible pour un système thermodynamique d'atteindre la température de zéro absolu par un processus physique fait d'étapes finies. Les trous noirs dits *extrêmes* sont des solutions particulières, caractérisé par une gravité de surface nulle. Pour ces derniers, nous avons exactement la même situation que dans la troisième loi: il est impossible de parvenir à un trou noir extrême depuis un trou noir non extrême par un processus comportant un nombre fini d'étapes. Cette analogie thermodynamique pour les trous noirs stationnaires a été dévoilée dans les années 70-80, et aujourd'hui, elle est considérée comme une acquis résumé dans les soi-disant *lois de la mécanique des trous noirs.* Ces lois sont intitulées *mécaniques* pour mettre en évidence le fait que elles sont des résultats purement classiques et, de plus, qu'elles ne fournissent pas une compréhension complète des trous noirs comme des systèmes thermodynamiques.

Le calcul semi-classique de Hawking [1] permet de comprendre le trous noirs, qui jusqu'alors étaient uniquement comparés à des systèmes thermodynamiques, comme de vrais systèmes thermodynamiques. En tenant compte des champs de matière quantique autour des trous noirs classiques, l'analogie thermique entre les trous noirs et les systèmes thermodynamiques standards peuvent être exact. Les grands trous noirs produits par effondrement gravitationnel se comportent comme des corps noirs parfaits à une *température* proportionnelle à la gravité de la surface une fois qu'ils ont atteint leur état d'équilibre stationnaire. Du point de vue classique, ce phénomène prédit est surprenant. à la base les trous noirs sont considérés comme des objets très simples dont la caractéristique principale est qu'ils avalent la matière et ne la laisse jamais s'échapper. Mais, du point de vue de la Théorie Quantique des Champs, son rayonnement peut être considéré comme tout à fait naturel. Il s'agit d'une propriété générique que les potentiels localisés qui perturbent des champs quantiques sans masse, dans un espace-temps plat, induiraient un rayonnement thermique (voir 14.2 dans [2]). En fait, c'est en utilisant cette idée que la température, ou plus précisément l'effet de rayonnement de Hawking, a été trouvée.

Comme ce fut le cas pour le problème de la singularité, la thermodynamique des trous noirs a besoin aussi d'une compréhension plus profonde de la Relativité Générale au niveau quantique. Aujourd'hui, il est largement admis qu'un champ quantique interagissant avec un trou noir produit un spectre thermique des particules rayonnantes. En outre, grâce à la première loi de la mécanique des trous noirs, il est possible d'associer à chaque trou noir une *entropie* proportionnelle à l'aire de l'horizon. Par conséquent, la température, et notamment l'entropie, apparaissent comme des propriétés génériques de véritables trous noirs. Avec ces éléments, la compréhension des trous noirs comme des systèmes thermodynamiques est pleinement accomplie. Cependant, une question naturelle se pose: est-ce que la thermodynamique des trous noirs peut être comprise du point de vue de la mécanique statistique? La réponse est inconnue. Mais, comme dans la plupart des systèmes thermodynamiques cette réponse devrait être fournie par une théorie quantique. On s'attend à ce que *les degrés de liberté microscopiques de la gravitation quantique* soient en mesure d'expliquer l'entropie trouvé indirectement par la méthode semi-classique. En ce sens, la température du trou noir et l'entropie peuvent être considérés comme des éléments forts qui poussent la recherche à explorer les degrés de liberté quantiques et microscopiques associés aux trous noirs. Et, dans un sens plus général, comme une motivation pour étudier les extensions quantiques de la Relativité Générale.

Pour résoudre le problème une théorie quantique de la gravitation est nécessaire. Comme il n'y a pas une telle théorie, cela est un problème difficile. Néanmoins, beaucoup de travail a été fait à partir de perspectives très différentes, principalement à partir de la Théorie des Cordes et de la Gravitation Quantique à Boucles. Ces sont les candidats de gravité quantique les plus largement étudié de nos jours. Cependant, les réels progrès réalisés pour résoudre le dilemme du trou noir quantique sont très subtiles parce que ces théories ne sont pas complètes. Par conséquence, pour calculer des applications physiquement intéressantes plusieurs hypothèses doivent être faites. D'ailleurs, il s'agit là d'une façon standard de procéder dans un contexte où la théorie n'est pas complète.

Cette thèse aborde le problème des trous noirs quantiques dans différents niveaux: depuis un cadre classique des lois thermodynamiques jusqu'aux modèles quantiques microscopiques qui fournissent des degrés de liberté pour expliquer leurs propriétés thermiques.

Pour traiter avec le régime de la gravité quantique nous adoptons l'approche de la **Gravitation Quantique à Boucles**. C'est une théorie relativement nouvelle, pas encore achevée, mais déjà assez bien développée au point qu'une description physique raisonnable des trous noirs puisse être construite. Nous nous servons de cette théorie au niveau mathématique à travers sa structure mathématique spécifique. Mais, nous l'utilisons aussi comme une source d'intuition pour guider les hypothèses en cas de besoin.

En outre, en tant que principe directeur général pour aider à la compréhension des trous noirs, nous utilisons une **perspective quasilocal**. Cela signifie que nous faisons particulièrement attention à étudier le problème des trous noirs en recherchant une description classique et quantique dans le voisinage de l'horizon du trou noir, là où la nature géométrique de la gravitation et l'analogie thermodynamique est fortement révélée.

Du point de vue quasilocal le concept de trou noir lui-même est reformulé. Au tout début, nous introduisons une définition quasilocal de trou noir qui est basée sur l'utilisation d'Horizons Isolés. Cette définition sera le point de départ d'une partie du calcul présenté dans cette thèse.

Dans le même esprit quasilocal, nous explorons la première loi de la thermodynamique des trous noirs dans une perspective nouvelle sur la base des observateurs proches de l'horizon. Il s'avère que la première loi peut être reformulée, quasilocalement, tout simplement en termes de variations de l'aire de l'horizon du trou noir. Le rôle prépondérant de l'aire est très pratique dans le contexte de la Gravitation Quantique à Boucles comme on le fera remarquer plus tard.

Pour les mêmes motifs, la perspective quasilocal basée sur des observateurs proche de l'horizon est utilisée pour examiner l'approche de la Gravitation Quantique Euclidienne. Cette approche permet la construction d'une fonction de partition de trous noirs dans une approximation semi-classique. Nous montrons que le cadre quasilocal est compatible à la fois avec la entropie de Bekenstein-Hawking et avec la première loi quasilocal mentionné précédemment.

L'approche quasilocal des trous noirs quantiques a déjà été adoptée dans des travaux précédents en utilisant les Horizons Isolés. Dans cette thèse, nous explorons également, à partir d'une analyse de *la structure symplectique*, l'extension de la quantification du modèle d'Horizon Isolé sphériquement symétrique vers le cas plus général de l'Horizon Isolé axialement symétrique.

Enfin, dans le cadre de cette thèse, nous explorons aussi, à travers une analyse statistique, l'entropie du trou noir des modèles quantiques microscopiques basés sur l'approche discrète de la Gravitation Quantique à Boucles. Un accent particulier est mis sur le modèle quantique de trou noir en rotation. Les résultats ne sont pas concluants parce que plusieurs hypothèses doivent être faites au fur et à mesure de la réalisation des calculs. Cependant, la perspective est encore prometteuse du fait que certains résultats semi-classiques, concernant notamment l'entropie, puissent être reproduits.

Maintenant, prenons un peu de recul pour expliquer les motivations générales de ce travail.

Le problème de la gravitation quantique.
Du point de vue théorique, la formulation d'une théorie cohérente qui repro-

duise naturellement tant la Théorie Quantique des Champs que la Relativité Générale dans certaines limites, est un problème ouvert principal en physique aujourd'hui. Cela est appelé aussi le problème de l'unification puisqu'une telle théorie devrait pouvoir expliquer les quatre forces présentes dans la nature. Une démarche moins ambitieuse vers l'unification est la reformulation de la Relativité Générale comme une théorie quantique. Aujourd'hui, ce problème ouvert se pose comme une pierre angulaire des débats qui ont cours. Le domaine de recherche qui abouti à cette reformulation théorique est appelé Gravitation Quantique.

En dépit des solides arguments théoriques pour l'existence de la gravitation quantique et la révolution conceptuelle de la vision du monde qu'elle impliquerait, l'identification a priori du domaine d'application d'une telle théorie est un problème très délicat. En outre, les données expérimentales auxquelles nous pouvons accéder pour guider la recherche sont très rares. Ces problèmes mettent la recherche en physique théorique dans une grave crise. La plupart des progrès de la recherche, au sein de l'unification ou dans la gravitation quantique, restent actuellement circonscrits au domaine de la physique mathématique, mettant de ce fait sur la touche le domaine expérimental. De ce fait, toutes les conséquences surprenantes que ces nouvelles théories proposent aujourd'hui (dimensions supplémentaires, de nouvelles particules, discrétisation de l'espace-temps, etc) ne peuvent pas être directement corroborée.

Cependant, le défi théorique de formuler une telle théorie est en soit une tâche difficile, et comme nous l'argumenterons maintenant, un principe directeur. On pourrait penser qu'en raison de l'absence d'expériences (dans le régime critique de la gravité quantique), on a beaucoup de théories en compétition qui sont d'accord avec le régime expérimental auquel nous pouvons accéder aujourd'hui. La situation est en fait complétement le contraire. Aujourd'hui, il n'y a même pas une théorie mathématiquement cohérente capable de satisfaire les critères basiques d'être une théorie quantique et en même temps reproduire la Relativité Générale dans son limite classique. En ce sens, au stade actuel, la simple formulation cohérente d'une telle théorie peut être considérée comme un premier défi.

De ce point de vue une stratégie historiquement fructuese pour comprendre les théories elles-mêmes, en restant toujours dans le domaine théorique, a été la formulation des *expériences pensées*, c'es-à-dire, d'imaginer et de tenter de résoudre des situations physiques à l'aide d'outils purement théoriques. C'est là que les trous noirs pourraient jouer un rôle important dans la construction de la gravité quantique et aussi d'une théorie unificatrice de toutes les forces.

Pourquoi les trous noirs?
La découverte du rayonnement de Hawking est l'un des phénomènes concrets où la gravité quantique devrait jouer un rôle. Étonnamment, les trous noirs ont une entropie qui ne peut pas être expliquée simplement par les degrés de liberté correspondant à des fluctuants classiques autour d'une solution stationnaire (même situation que pour la température des trous noirs). Cela suggère que cette entropie, jusque là calculée par une méthode indirecte, est un phénomène intrinsèquement quantique qui devrait en fait être liée aux degrés de liberté quantiques auquels le trou noir peut accéder. Cette image est un simple analogie avec le rayonnement d'un corps gris dont le spectre est expliqué par la structure quantique de la matière qui le compose. Dans ce cas, l'entropie pertinente pour les processus thermodynamiques du système est l'entropie totale calculée sur la description de sa structure atomique. Et évidemment, ce n'est pas l'entropie calculée sur la simple perturbation de la forme du système macroscopique (par exemple les modes de vibration d'un objet solide).

Le problème général est d'identifier les degrés de liberté qui sont excités dans les processus thermodynamiques. Comme indiqué dans [3] "Les propriétés thermiques d'un état macroscopique sont capturées par l'expression de son entropie en fonction de variables macroscopiques. L'interprétation physique de l'entropie, c'est qu'elle est une mesure du volume de l'espace de phase microscopique de l'ensemble d'états qui partagent les mêmes variables macroscopiques. Autrement dit, il s'agit d'une mesure de la perte d'information dans la caractérisation macroscopique de l'etat." En ce sens, il semble que dans la thermodynamique des trous noirs les degrés de liberté de la gravitation quantique sont indispensables pour expliquer les informations du système tenu en compte par l'entropie. Ainsi, nous observons que l'étude des trous noirs est une voie permettant d'obtenir une meilleure idée de ce à quoi une théorie de la gravitation quantique devrait ressembler, et en même temps, un lieu où toute tentative de théorie de la gravitation quantique doit être appliquée.

C'est en raison de ce rapport entre les trous noirs et la gravitation quantique que nous choisissons d'étudier les trous noirs dans le cadre fourni par l'un des candidats à la théorie: la Gravitation Quantique à Boucles.

Pourquoi la Gravitation Quantique à Boucles?
La Relativité Générale nous dit que la structure de l'espace est dynamique et qu'elle ne peut être comprise que si elle s'inscrit dans une structure plus complexe appelée espace-temps. En contraste avec le reste des théories physiques, la Relativité Générale n'est pas construite au sommet d'une structure de l'espace-temps sous-jacente, c'est la théorie de l'espace-temps elle-même. En

conséquence, sa formulation ne peut *a priori* pas s'appuyer sur aucun espace-temps. C'est un fait simple mais profond, généralement désigné comme *l'indépendance du fond*, nous l'adoptons ici comme un principe fondamental pour formuler des théories physiques.

La Théorie Quantique des Champs, et en particulier le Modèle Standard, ignore complètement ce fait en utilisant comme fond un espace-temps fixe, normalement l'espace-temps de Minkowski, lorsque tous les champs sont définis. Mais, en dépit de la réussite expérimentale fantastique de ces théories, et précisément en raison de sa dépendance du fond, nous savons que le Modèle Standard ne peut pas être le fin mot de l'histoire. Une nouvelle théorie indépendante du fond est alors nécessaire.

La Gravitation Quantique à Boucles est une tentative pour construire une version quantique *seulement* de la Relativité Générale qui soit indépendante du fond. L'espoir est que si elle est finalement couronnée de succès, le formalisme mathématique et des idées similaires pourront être utilisés pour résoudre le problème complet de comprendre l'unification de toutes les interactions. La façon de procéder est de reformuler la Relativité Générale en termes de nouvelles variables de connexion, appelées variables Ashtekar-Barbero, et de construire une représentation de l'espace de Hilbert: la représentation à boucles. L'introduction de ces variables peut être accomplie si l'on considère un feuilletage temporel de l'espace-temps, et effectuer un choix de jauge partielle: la jauge temporelle. On pourrait penser que ce choix détruit l'indépendance du fond. Cependant, le foliation temporelle est totalement arbitraire et en fait, ce n'est rien d'autre que le point de départ standard de la procédure pour une quantification canonique.

Le principal défi ouvert pour le programme de la Gravitation Quantique à Boucle est reproduire la Relativité Générale. En ce sens, elle doit être prise comme une théorie en progrès. En dépit de cela, les particulières développement de la théorie sur la nature discrétisée des quantités géométriques sont simples et méritent déjà d'être testées dans des situations physiques concrètes. Les trous noirs sont l'un des phénomènes naturels dans lequels ces tests peuvent être effectués. L'espoir ambitieux est que nous pouvons apprendre des choses dans les deux directions, c'est-à-dire, sur de la nature quantique des trous noirs, mais aussi sur la Gravitation Quantique à Boucles comme une théorie incomplète.

Ensuite, nous résumerons le contenu de chaque chapitre.

Tout d'abord, dans l'introduction, Section 1.1, nous mettons en place la définition quasilocal de trou noir: **Horizon Isolé**. En termes simples, l'Horizon Isolé représente l'ensemble des propriétés minimales que nous demandons à n'importe quel espace-temps afin de s'assurer qu'il contient un trou noir qui est en fait en équilibre.

Le **Chapitre 2** est consacré à discuter une nouvelle **version quasilocal de la première loi de la mécanique des trous noirs**. Le résultat est simple: en considérant une famille d'observateurs proches de l'horizon, l'échange d'énergie pertinente du trou noir avec son entourage est mesurée simplement par les changements de l'aire de l'horizon lui-même. L'idée est pleinement développé dans la Section 2.2. En préambule, trois dérivations de la première loi standard sont présentées. Un lecteur familier avec les lois de la mécanique des trous noirs pourrait les omettre. Les nouveaux résultats de ce chapitre peuvent être également trouvés dans [arXiv:1110.4055/gr-qc] et ont été publiés dans [4].

Dans le **Chapitre 3** est présenté la première incursion de ce travail sur le problème de la gravitation quantique en discutant l'approche semi-classique de la **gravitation quantique Euclidienne**. Il se trouve que l'approche semi-classique Euclidienne peut être adaptée à un cadre quasilocal de telle sorte que la température locale, l'énergie quasilocal, et la loi de l'entropie de Bekenstein-Hawking sont récupérées. Encore une fois, un lecteur entièrement familier avec la function de partition Euclidienne pourrait aller directement à la fin du chapitre, Section 3.3, où les nouvelles idées sont présentées. Toutefois, dans ce cas, pour bien comprendre la logique du raisonnement, le lecteur est encouragé à lire le chapitre entier.

Le **Chapitre 4** présente en détail le programme de la **quantification du trou noir**. Il est divisé en deux sections principales: dans la Section 4.1 nous allons revoir la procédure de quantification de l'Horizon Isolé sphériquement symétrique. Dans la Section 4.2 nous effectuons la généralisation du programme de quantification pour le cas de l'Horizon Isolé axialement symétrique. Nous identifions les principaux obstacles à un premier traitement naïf, de nouvelles idées pour les éviter sont proposées, et un nouveau modèle de quantification pour le cas de symétrie axiale est esquissée. En raison de sa complexité, certains calculs techniques ont été relégués aux Annexes D et E pour faciliter la lecture. Les principaux résultats de ce chapitre peuvent être trouvés dans [arXiv:1212.5166/gr-qc].

Dans le **Chapitre 5**, nous effectuons plusieurs calculs pour l'**entropie** du trou noir. Ils diffèrent dans la manière dont le moment angulaire est incorporé à l'échelle microscopique. En outre, différents outils mathématiques sont utilisés pour calculer l'asymptotique du nombre d'etats. Les résultats de l'entropie sont résumés dans la table de la page 115, qui permet d'exclure certains modèles microscopiques. Les travaux décrits dans ce chapitre sont en cours et seront publiés en temps voulu [5].

1. INTRODUCTION

Black holes are one of the most astonishing predictions of General Relativity. At the theoretical level, all the main revolutionary concepts introduced by General Relativity such as spacetime, curvature, or causality play a fundamental role to understand the black hole solutions. Black holes are, at the same time, among the most mathematically simple and conceptually non-trivial solutions of the theory. Therefore, they constitute a natural arena to understand General Relativity, as well as a preferential place where to check any tentative modification or extension of that theory. On the other hand, at the astrophysical level, black holes are also astonishing. Black holes are the only explanation for the fate of certain massive stars. Therefore, they are fundamental to understand the evolution of the celestial bodies in our universe.

Incredibly, by themselves, the black hole mathematical solutions also announce the limitations of General Relativity. They suffer from the same problem of Electromagnetism's point particle solution: A region with infinite charge density is a physical nonsense. In that case, a completely different theory, Quantum Electrodynamics, was needed to solve the problem and give an accurate description of how actually the fields behave at short scales, and, along the way, it changes the concepts of particles and interactions. In black holes, the presence of singularities in their mathematical description, a region with infinite matter density, is also a physical nonsense. It highlights the need to go beyond General Relativity and look for a new theory which provides a deeper understanding.

A more interesting and fruitful issue concerning black holes is the theoretical result that—already at the classical level—stationary black holes actually mimic the laws of thermodynamics. For instance, the mass of a stationary black hole can be thought as its energy content. The variation of the mass is naturally related to the variation of other intrinsic properties of the black hole such as the charge, the angular momentum, and the area of the black hole horizon. That relation is realized in a very particular way that is completely analogue to the *first law of thermodynamics*. Similarly, Hawking's area theorem states that in any classical process the area of the horizon can just grow, this is analogy with the *second law of thermodynamics* if the area of the horizon is identified with the entropy of the black hole system. The *zeroth law of thermodynamics* states that in thermal equilibrium the temperature of physical systems should be uniform. In black holes, the *surface gravity* is uniform over the horizon, this is again in analogy with the zeroth law, and further suggests that surface gravity should be identified with the black hole temperature. Finally, there is also an analogy with one of the formulations of the *third law of thermodynamics* that tells us that it is impossible for a thermodynamic system to reach an absolute zero temperature by any physical process made of finite steps. The so called *extreme black holes* are particular solutions characterized by having a zero surface gravity. For them, we have exactly the same situation that in the third law: It is impossible to reach an extreme black hole from a non-extreme black hole by a finite number of physical steps. All this thermodynamical analogue picture for stationary black holes was unveiled in the 70s and 80s, and today, it stands as a standard piece of knowledge summarized in the so called *laws of black hole mechanics.* It is called that way to highlight the fact that these laws are purely classical results, and, further, that they do not provide a complete understanding of black holes as thermodynamic systems.

That thermodynamical picture for black holes can be in fact fulfilled with the Hawking's semiclassical calculation [1]. By considering quantum matter fields around classical black holes, the thermal analogy between black holes and standard thermodynamic systems can be made exact. Large black holes produced by gravitational collapse behave like perfect black bodies at a *temperature* proportional to their surface gravity once they have reached their stationary equilibrium state. From the classical perspective, this predicted phenomenon is surprising, black holes are viewed as extremely simple objects which principal characteristic is that they swallow matter and never let it escape. But, from the Quantum Field Theory perspective, its radiation can be seen as quite natural. It is a generic property that localized potentials disturbing quantum massless fields in a flat spacetime would induce a thermal radiation on those fields (see 14.2 in [2]). In fact, it is by using this machinery that the temperature, or more precisely, the Hawking radiation effect, was found.

As it was the case for the singularity problem, black hole thermodynamics also claims for a deeper quantum understanding of General Relativity. Today, it is largely accepted that a quantum field interacting with a black hole produces a thermal spectrum of radiating particles. Furthermore, through the analogue first law, it is possible to associate with each black hole a particular *entropy* proportional to the area of the horizon. Therefore, temperature and notably entropy, appear as generic properties of real black holes. With these elements the understanding of black holes as thermodynamic systems is fully accomplished. However, immediately a natural question emerges: Can black hole thermodynamics be understood from the statistical mechanics perspective? The answer is unknown. But, as in most thermodynamic systems that answer should be provided by a quantum theory. The expectation is that *microscopic quantum gravitational degrees of freedom* should be able to explain the entropy found indirectly by the semiclassical method. In that sense, black hole temperature and entropy can be considered as strong insights that push the research to explore the microscopic quantum degrees of freedom associated with black holes. And, in a more general sense, as a motivation to study the quantum extensions of General Relativity.

To address the problem a quantum theory of gravity is needed. As there is not such a theory this is a difficult problem. Nevertheless, a lot of work has been done from quite different perspectives, mainly from String Theory and Loop Quantum Gravity. These are the more extensively studied quantum gravity candidates nowadays. However, the concrete progress in solving the quantum black hole dilemma are very subtle as those theories are not complete. The reason is that to compute physically interesting consequence several assumptions should be done. This is a standard way to proceed in a context where the theory—quantum gravity—is not complete.

This thesis addresses the problem of quantum black holes at a range of different levels: Starting from the very classical framework of the thermodynamical laws to the deeper microscopic quantum model that provides the degrees of freedom to explain their thermal properties.

To deal with the quantum gravity regime we adopt the **Loop Quantum Gravity approach**. This is a relatively new theory, not yet complete, but developed enough to the point that a physically sensible description of black holes can be given. We use this theory at the mathematical level through its specific mathematical structure. But, we also use it as a source of intuition to guide the assumptions when needed.

In addition, as a general guiding principle to deal with black holes, we use a **quasilocal perspective**. It means that we make special emphasis in studying the problem of black holes by looking for a classical and quantum description in the vicinity of the black hole horizon where the geometric nature of gravity and the thermodynamical analogy strongly reveals.

From the quasilocal perspective the concept of black hole itself is reviewed. At the very beginning, we introduce a quasilocal definition for black hole that is based in the use of Isolated Horizons. This definition will be the starting point for some of the computation presented in this dissertation.

In the same quasilocal spirit, we explore the first law of black hole thermodynamics from a new perspective based on observers close to the horizon.

It turns out that the first law can be reformulated, quasilocally, simply in terms of variations of the area of the black hole horizon. The preponderant role of the area is very convenient in the context of Loop Quantum Gravity as will be pointed out later.

On the same grounds, the quasilocal perspective based on near horizon observers is used to review the Euclidean quantum gravity approach. This approach allows for the construction of a partition function for black holes in a semiclassical approximation. We show that the quasilocal framework is consistent both with the standard Bekenstein-Hawking entropy result and with the quasilocal first law mentioned before.

The quasilocal approach to quantum black holes has been already adopted in previous works by using Isolated Horizons. In this thesis we also explore, starting from a *symplectic structure analysis*, the extension of the quantization of the spherically symmetric Isolated Horizon model to the more general axially symmetric Isolated Horizon case.

Finally, as a part of this thesis, we also explore, through a statistical analysis, the black hole entropy from the microscopic quantum model based on the Loop Quantum Gravity discrete approach. Special emphasis is put on the rotating quantum black hole model. The results are not conclusive as several assumptions must be made on the way. However, the perspective is still promising as some of the semiclassical results regarding, for instance, the entropy, can be reproduced.

Now, let us take a step back and comment on the general motivations of this work.

The Problem of Quantum Gravity. From the theoretical point of view the formulation of a coherent theory that naturally reproduces Quantum Field Theory and General Relativity as particular limits, is a main open problem in physics today. This is also called the problem of unification as such a theory should by itself explain the four forces present in nature. A less ambitious step in the direction of unification is the reformulation of General Relativity as a quantum theory. Today, this separate issue is also a cornerstone open problem. The research domain that looks for that theoretical reformulation is called Quantum Gravity.

In spite of the strong theoretical arguments for the existence of a quantum gravity theory, and the conceptual revolution of the world view it would imply, the a priori identification of the application domain of such a theory is a very subtle and difficult problem. Furthermore, the experimental data that we can access to guide the research is scarce. These problems put the theoretical physics research in a serious crisis. Most of the research progress, within the unification or quantum gravity problems, currently resides in the sphere of mathematical physics, out of the experimental range, and, in particular, all the surprising consequences those theories propose today—extra dimensions, new particles, discretized spacetime, etc.—cannot be directly corroborated.

However, the theoretical challenge of formulating such a theory is by itself a mayor task and, as we now argue, a guiding principle. One could think that because of the lack of experiments (in the critical regime for quantum gravity) one has a lot of competing theories which agree in the experimental regime we can access today. The situation is quite the opposite. Today, there is not even one mathematical consistent theory able to satisfy the basic criteria of being a quantum theory and at the same time reproduce General Relativity as its classical limit. In this sense, at the current stage, the simple consistent formulation of such a theory can be thought of as a first challenge.

From that perspective a historically fruitful strategy to learn more about the theories themselves, staying in the theoretical domain, has been to formulate *gedankenexperiments*. That is, to imagine and try to solve thought physical experiments only with theoretical tools. Here is where black holes could play a mayor role in the road to quantum gravity and a further unification of forces in physics.

Why Black Holes? The Hawking radiation is one of the concrete phenomena where quantum gravity should play a role. Surprisingly, black holes have an entropy which cannot be simply explained with the classical fluctuating degrees of freedom around a stationary solution—the same happens with the black hole temperature. That suggests that this entropy, up to now computed through an indirect method, is an intrinsically quantum phenomenon that should in fact be related to the quantum degrees of freedom that the black hole can access. This simple picture is in complete analogy with the grey body radiation whose spectrum is ultimately explained by the quantum structure of the body itself. In this case, the relevant entropy for the thermodynamical processes of the system is the full entropy computed out of its atomic structure description. And obviously, this is not the entropy computed out of the simple perturbation of the shape of the macroscopic system—e.g. vibration modes of a solid object.

The general problem here is to identify the degrees of freedom that are excited in a thermodynamical process. As expressed in [3] "The thermal properties of a macroscopic state are captured by the expression of its entropy as a function of macroscopic variables. The physical interpretation of the entropy is that is a measure of the volume of the microscopic phase space of the set of states sharing the same macroscopic variables. That is, it is a measure of the information lost in the macroscopic characterization of the state.” In this sense, it seems that in black hole thermodynamics the quantized gravitational degrees of freedom are the essential ones to explain the information of the system that its entropy accounts for.

Thus, we observe that the study of black holes is a path to get more insight about what a theory of quantum gravity should look like, and at the same time a place where any theory of quantum gravity should be applied. It is because of this interplay between black holes and quantum gravity that we choose to study black holes within the framework provided by one of the candidates for a theory of quantum gravity: Loop Quantum Gravity.

Why Loop Quantum Gravity? General Relativity tells us that the structure of space is dynamical and that it cannot be understood unless it is embedded in a more complex structure called spacetime. In contrast with the rest of physical theories, General Relativity is not built on top of any underlying spacetime structure, it is the theory of the spacetime itself. As a consequence, its formulation cannot rely on any a priori spacetime. This is a simple but deep property, usually referred as *background independence*, we think about it here as a fundamental principle to formulate truly physical theories.

Quantum Field Theories, and in particular the Standard Model, completely ignores this fact by using a fixed background spacetime, normally Minkowski spacetime, where all the fields are defined. But, in spite of the full experimental success of such theories, precisely because of its background dependence, we know that the Standard Model cannot be the end of the story, and a new true background independent theory is needed.

Loop Quantum Gravity is an attempt to construct a background independent quantized version of *just* General Relativity. The hope is that if it is ultimately successful, a similar formalism and ideas can be used to address the full problem of quantizing gravity and understand the unification of all the interactions. The way to proceed is by reformulating General Relativity in terms of new connection variables, called Ashtekar-Barbero variables, and to construct a representation of the Hilbert space out of them: The loop representation. The introduction of this variables can be only accomplished if we consider a spacetime time-foliation, and perform a partial gauge choice: The time-gauge. One could think that this choice breakdown the background independence. However the time foliation is completely arbitrary and in fact is nothing but the standard starting point of the canonical quantization procedure.

The main open challenge for the Loop Quantum Gravity program remains to reproduce General Relativity. In this sense it should be taken as a theory in progress. In spite of this, particular developments of the theory concerning the *discretized nature of geometrical quantities* are straightforward and deserves already to be tested on concrete physical situations. Black holes is one of the natural phenomena where such tests can be done. The ambitious hope is that, by doing so, we can learn things in both directions, i.e., more about the quantum nature of black holes but also about Loop Quantum Gravity as an incomplete theory.

Now, let us comment further on the structure of the present manuscript.

The logical structure of this thesis follows a standard physical analysis that goes from the simpler macroscopic description to the more complex microscopic description. As stated before, in this thesis we study black holes with the underlying motivation of quantum gravity—this is the big open problem we are ultimately interested in. In direct connection with the motivation is the guiding question about the origin of the black hole entropy. Therefore, in each chapter we deal with a subject concerning black hole entropy but at different levels. In spite of the fact that each chapter can be thought as independent to some extent, there is a clear logical path connecting them. Roughly, the four following chapters deal with:

Ch.2. Black holes thermodynamics laws
Ch.3. Semiclassical derivation of the entropy
Ch.4. Quantum model of black holes from quantum gravity
Ch.5. Statistical mechanics analysis to compute the entropy

Thus, we are reproducing a standard framework of a globally physical analysis that goes from the more simple to the more complex description.

In Fig. 1.1 we present our understanding of the different approaches to the study of black hole entropy. As represented there, we have at our disposal a few derivations for the entropy with a value $S = A/4$ (in natural units). In particular, in this thesis we stress the relevance of the quasilocal approach. We exploit that, already without any fundamental quantum gravity theory, the standard Bekenstein-Hawking entropy can be indirectly derived from a quasilocal Euclidean appraoch. However, all those derivations are indirect in the sense that none of them really clarify the degrees of freedom responsible for that entropy. The hope is that a quantum gravity theory can do the work from first principles. Our specific hope in doing this work is that Loop Quantum Gravity will shed some light on the problem.

In the following we summarize the content of each chapter.

Already, as a part of this introduction, in Section 1.1, we set up a quasilocal definition for black hole: **Isolated Horizon**. In simple terms, Isolated Horizon stands for the minimum quasilocal properties we would ask to any

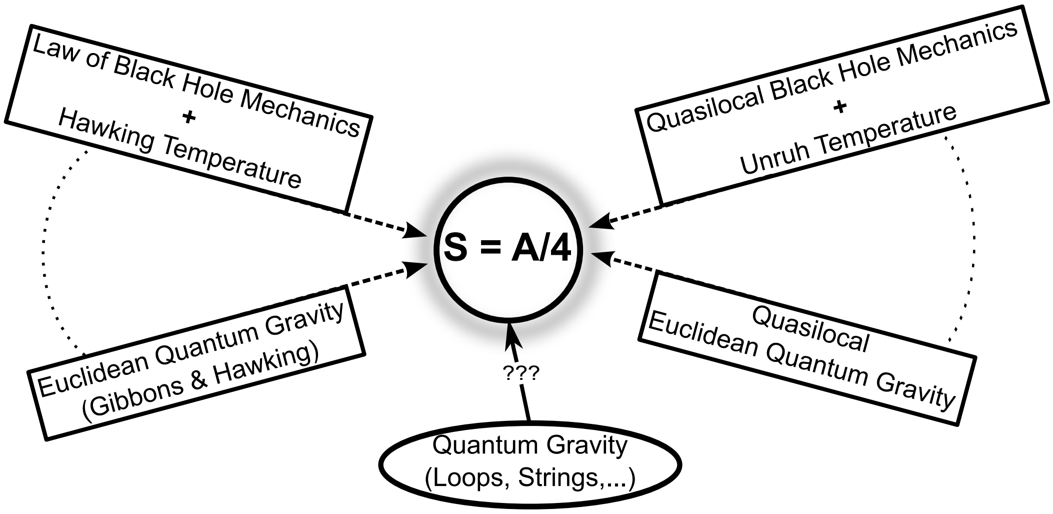

Fig. 1.1: The r.h.s. of the diagram shows the standard approaches to compute the black hole entropy. The l.h.s. displays the derivations based on the quasilocal notions that we develop in this work. Dashed arrows are used to remind the reader of the fact that the derivations are indirect. The dotted lines are used to express the fact that those derivations on the top are intimately linked with its Euclidean counterpart. At the bottom other approaches to explain black hole entropy are symbolized.

spacetime in order to ensure that it contains a black hole which is actually in equilibrium.

Then, the **Chapter 2** is devoted to discuss a new **quasilocal** version of the **first law of black hole mechanics**. The simple result is that by considering a particular near horizon family of observers the only relevant energy exchange of the black hole horizon with its surrounding is measured by the area changes of the horizon itself. The idea is fully developed in Section 2.2. As a preamble, three derivations of the standard first law are presented. A reader familiar with the laws of black hole mechanics could skip them. The new result of this chapter can be found in [arXiv:1110.4055/gr-qc] and has been published in [4].

In **Chapter 3** is presented the first incursion of this work into the quantum gravity problem by discussing the **Euclidean quantum gravity** semi-classical approach. It is found that the Euclidean semiclassical approach can be adapted to a quasilocal framework in such a way that the local temperature, the quasilocal energy, and the Bekenstein-Hawking entropy law are recovered. Again, a reader fully familiar with Euclidean quantum gravity could go directly to the end of the chapter, Section 3.3, where the novel ideas are presented. However, in this case, to understand the logical path it is actually important to read the full chapter.

Chapter 4 presents in detail the **black hole quantization** program. It is divided in two main sections: In Section 4.1 we will review the spherically symmetric Isolated Horizon quantization procedure. In Section 4.2 we carry out the generalization of the quantization program to the axially symmetric Isolated Horizon case. We identify the main obstructions to the naive treatment, new ideas to avoid them are proposed, and a new quantization model for the axially symmetric case is sketched. Because of its complexity some technical material has been relegated to appendices D and E to facilitate the reading. The main results of this chapter can be found in [arXiv:1212.5166/gr-qc].

In **Chapter 5** we carry out several computations for the **entropy** of the black hole. They differ in the way that the angular momentum is incorporated at the microscopic level. Also, different mathematical tools are used to compute the asymptotics of the number of states. The results for the entropy are summarized in a table—in page 115—which allows to rule out some of the microscopic models. The work described in this chapter is in progress and will be published in due time [5].

Note: in page 143 there is a list of the symbols used in each chapter.

1.1 Isolated Horizons

The starting point to study black holes is to specify what they are. The usual definition of black holes, given for instance in [2], says that black holes are the spacetime regions that are left after subtracting from the whole spacetime manifold all points connected to the null future infinity $\mathscr{I}^+$ through null geodesics. Intuitively. It means we are defining black holes as spacetime regions from where any emitted light ray does not reach *the far away spacetime region*, that is, regions where all rays remains trapped.

In particular, the boundary of such regions is called the black hole *event horizons*. All the events happening inside this boundary are causally disconnected from the rest of the spacetime.

The preceding one, is a very intuitive definition of a black hole but it has a very important problem that makes it useless for most applications: It is *teleological*. To decide if a given point belongs or not to a black hole interior you need to know the whole spacetime, the future, and in particular the asymptotic structure of your spacetime, i.e., to know how your spacetime geometry and fields behave as you move away from the black hole.

Moreover, in dealing with black holes in quantum gravity further difficulties appear. As they evaporate, the previous definition based on global structure of spacetime is ill-posed. This has been illustrated in the context of two-dimensional models [6, 7]. Nevertheless, one would expect that the physical notion of a large black hole radiating very little and, thus, remaining close to equilibrium for a long time could be characterized in a suitable way and that such a characterization should help in studying the appropriate semiclassical regime of the underlying quantum theory.

To study black holes, and to avoid dealing with asymptotic structures we need a local notion. However, because of the *equivalence principle*, a strict local characterization in spacetime is meaningless. The most we can aspire to have is an extended characterization of spacetime over a finite region, that is, to have a *quasilocal* characterization.

Therefore, it is useful to have a definition of black hole that uses only the quasilocal properties of spacetime and such that it coincides with the previous teleological one in most cases. There are different approaches in this respect running from weak to strong conditions defined on surfaces that can be related to the event horizon. The variability responds to the specific application the definition is intended for (see [8] for a recent short review on various proposals). In this respect, the Isolated Horizon definition is the one we use because it is constructed to provide a good description of a black hole in equilibrium—such that neither radiation nor matter falls in—while at the same time allows for a dynamical evolution on exterior spacetime. The following definition can be found discussed in [9, 12, 13, 14] or [15].

Definition

To define Isolated Horizons we start with a more general quasilocal definition of black hole horizons called *non-expanding horizons*. From there, the Isolated Horizon definition can be obtained by simply imposing a couple of extra requirements.

A spacetime with a non-expanding horizon satisfies the following properties:

- The spacetime $\mathcal{M}$ has a null submanifold Δ with the topology $S^2 \times \mathbb{R}$.
- The expansion of any null normal $\ell^a \in T(\Delta)$, the tangent space to Δ, vanishes:
$$\theta_{(\ell)} \equiv q^{ab} \nabla_a \ell_b = 0, \tag{1.1}$$
with q_{ab} the degenerated metric induced on Δ, and ∇_a the covariant derivative.

- The Einstein field equations holds at Δ. The matter stress-energy tensor T_{ab} satisfy that $-T^a{}_b \ell^b$ is future causal for any future directed null normal ℓ^a.

These conditions give rise to a number of interesting properties, in particular we just mention the existence of a 1-form ω_a such that for any tangent vector X^a on Δ the connection operator in spacetime, ∇_a, satisfies

$$X^a \nabla_a \ell^b = X^a \omega_a \ell^b, \tag{1.2}$$

where ℓ^a is a null vector on Δ.

To complete the definition of Isolated Horizon two more properties are imposed:

- Δ is equipped with a class of future directed null normals $[\ell]$ related by a positive constant factor: $\ell^a = c\ell'^a \in [\ell]$ such that

  $$\mathscr{L}_\ell \omega_a = 0, \tag{1.3}$$

  where $\mathscr{L}_\ell$ is the Lie derivative with respect to the vector field ℓ^a.

- The connection ∇_a on $\mathcal{M}$ induces a connection D_a on Δ according to

  $$X^a D_a Y^b \equiv X^a \nabla_a Y^b, \tag{1.4}$$

  with X^a and Y^a two tangent vectors on Δ. The induced connection is preserved by ℓ^a

  $$[\mathscr{L}_\ell, D] = 0. \tag{1.5}$$

The Equation (1.3) allows for a definition of surface gravity out of ℓ^a given by $\kappa_{(\ell)} = \ell^a \omega_a$ such that it is constant on Δ: This is the so called *zeroth law of black hole mechanics*—in the context of Isolated Horizons. Note that the surface gravity is not uniquely defined as ℓ^a is not uniquely selected on $[\ell]$. However, we stress here that the uniqueness can be accomplished if we find a natural normalization for the null generators, this point will be exploited in Section 2.2.

Finally, it can be shown that ℓ^a is a symmetry of the degenerate intrinsic geometry: $\mathscr{L}_\ell q_{ab} = 0$, this together with the last condition (1.5) guarantees that the pair (q_{ab}, D_a), that fully determines the geometric properties of Δ, is preserved by any of the null generators ℓ^a. Hence, the picture of Δ as a horizon in equilibrium is accomplished, see Figure 1.2. In particular, the horizons of all stationary black hole solutions are Isolated Horizons.

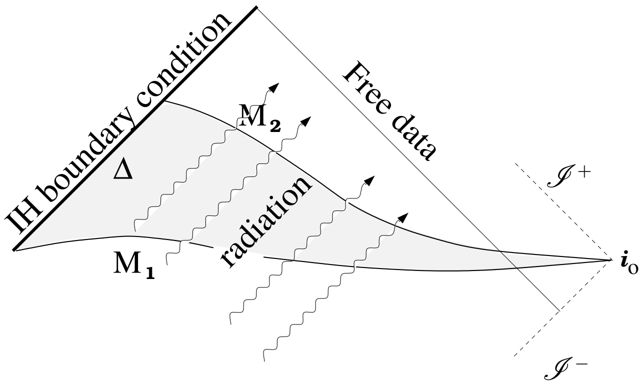

Fig. 1.2: In this picture the Isolated Horizon conditions are imposed just at Δ. M_1 and M_2 are Cauchy surfaces. For instance, one expect that in the characteristic formulation free radiation data can be imposed in the transversal null surface without affecting the Isolated Horizon. In addition, as no reference is made to the null infinity, $\mathscr{I}^+$, Isolated Horizons are not equivalent to event horizons. And, in a particular spacetime they may not coincide.

Consequences of these properties have been widely explored mostly in the context of black hole mechanics and black hole quantization [9, 12, 13, 14]. For our specific applications, in Chapter 2, we will consider the quasilocal surface for stationary observers that can be naturally constructed at the neighbourhood of the Isolated Horizon, and in Chapter 4 we will use the fact that Isolated Horizon conditions fix the geometry in such a way that an action principle—which incorporates the Isolated Horizon as an internal boundary—can be formulated.

2. BLACK HOLE MECHANICS

In this chapter we show that stationary black holes satisfy a simple quasilocal first law of black hole mechanics given by

$$\delta E = \frac{\overline{\kappa}}{8\pi}\delta A, \tag{2.1}$$

where we use the thermodynamical energy $E = A/(8\pi\ell)$ and the local surface gravity $\overline{\kappa} \approx 1/\ell$. Here, A, is the horizon area and ℓ is a proper length characterizing the distance to the horizon of a preferred family of near horizon observers. These observers are stationary with respect to the near horizon geometry and would see a locally isotropic thermal distribution [16], therefore, they are suitable for thermodynamical considerations. One of the main properties of the quasilocal first law is that it does not require any asymptotic structure to be defined, thus, we stress its true quasilocal nature in contrast with the standard first law of black hole mechanics.

As a preamble, in this chapter we briefly review three derivations for the first law of black hole mechanics: 1) by using the no-hair theorem and the Kerr-Newman solution 2) by studying perturbations and the Einstein equation on stationary black holes , and 3) by studying the phase space of the Isolated Horizons. Thereafter, the main new result is presented in detail through different physical arguments and also extended to the more general framework of Isolated Horizons. This new approach, based on physically motivated local measurements, is part of the program for a quasilocal description of black hole mechanics, where the idea is to translate some of the standard results to define new concepts in order to have a complete quasilocal framework useful, in particular, to rephrase the quantum analysis.

In this direction, the proposed quasilocal first law already motivates interesting applications in semiclassical computations of black hole physics, for instance, as a part of this thesis, we develop the euclidean partition function approach to compute black hole entropy based on the quasilocal first law, see Chapter 3. Also, the new proposal has raised interest among the people working on Loop Quantum Gravity. The interest is in part explained because in Loop Quantum Gravity the area is a rather simple geometric quantum operator and one of the central outputs of the theory. More precisely, if the area can be interpreted as a notion of energy—as we claim—it appeals for a direct statistical mechanics analysis by using the eigenvalues of the area quantum operator as the energy eigenvalues of the system. This idea has been already employed in [17, 18] and will be further used in Section 5.3 to build a partition function.

2.1 First Law of Black Hole Mechanics (preamble)

Two standard derivations of the first law of black hole mechanics are presented. First, considering a simple derivation based on the *no-hair theorem* and using a specific charged and rotating black hole solution (Kerr-Newman). Second, by a perturbation of stationary black holes with Killing Horizons [19, 20, 21]. Furthermore, a third—less standard—derivation in the more general context of Isolated Horizons is reviewed at the end.

2.1.1 First Law from the No-Hair Theorem

The no-hair theorem states that stationary black holes are simply described by three quantities: the mass M, the angular momentum J, and the electric charge Q. It implies that any dynamical perturbation of a black hole should settle down to another black hole characterized again by these three quantities. We use this argument for the Kerr-Newman black hole solution.

The Kerr-Newman solution to the Einstein equation represents precisely a black hole with a mass M, angular momentum J, and charge Q with asymptotically flat conditions, see the explicit metric in (A.1). The area of the black hole horizon is

$$A = 4\pi(r_+^2 + a^2), \quad a \equiv \frac{J}{M}, \quad r_+ = M + \sqrt{M^2 - a^2 - Q^2}. \tag{2.2}$$

The area is a function of only M, J, and Q. Thus, with $A(M, J, Q)$ it is possible to compute how small variations of the area are related to small variations in the mass, the angular momentum and the charge

$$\delta A = \frac{\partial A}{\partial M}\delta M + \frac{\partial A}{\partial J}\delta J + \frac{\partial A}{\partial Q}\delta Q, \tag{2.3}$$

a straightforward calculation shows

$$\frac{\partial A}{\partial M} = \frac{8\pi}{\kappa}, \quad \frac{\partial A}{\partial J} = -\frac{8\pi\Omega_H}{\kappa}, \quad \frac{\partial A}{\partial Q} = -\frac{8\pi\Phi_H}{\kappa}, \tag{2.4}$$

where we have used the following definitions of *surface gravity*, *horizon angular velocity*, and *horizon electric potential*

$$\kappa = \frac{r_+ - M}{2Mr_+ - Q^2} \tag{2.5}$$

$$\Omega_H = \frac{a}{r_+^2 + a^2} = \left.\frac{d\phi}{dt}\right|_H \tag{2.6}$$

$$\Phi_H = \frac{Qr_+}{r_+^2 + a^2} = -\left. A^b \chi_b \right|_H , \tag{2.7}$$

where in each line the first equality is the result of the straightforward calculation. In the second equality we use the fact that there is actually a natural physical interpretation such that the same quantity can be computed from an alternative expression. For the horizon angular velocity, it is the rapidity of the azimuth angle coordinate ϕ—describing a point on the horizon—with respect to the coordinate time t, see (A.1). For the electric potential, it is the projection of the electromagnetic potential vector A^b on a Killing field normal to the horizon χ_b. In the case of surface gravity κ the physical interpretation if explained after Equation (2.15). Hence, we have the relation

$$\delta M = \frac{\kappa}{8\pi}\delta A + \Omega_H \delta J + \Phi_H \delta Q. \tag{2.8}$$

As explained before, the no-hair theorem provides us with a physical process interpretation of the previous variations. It ensures us that actually the variations relate different stationary black hole solutions. In other words, the highly dynamical spacetime of a black hole that is being perturbed by a small amount of matter (characterized by certain mass, charge, and angular momentum), and that after some evolution time stabilizes, can be simply described by two stationary black hole solutions (one corresponding to the initial state and one corresponding to the final state) in such a way that their intrinsic parameters are related exactly by (2.8).

The simple equation in variations, (2.8), has a further and radical physical interpretation. To understand it we first should note that the black hole mass is in fact an energy notion for black holes. The ADM approach defines the energy content of the spacetime as the one naturally seen by static observers at infinity. For Kerr-Newman black hole solutions this energy turns out to be simply the mass M. Similarly, the angular momentum and the charge content of the spacetime can be defined by those static observers, respectively, as the generator of rigid rotational symmetry and the integral of the electric flux at infinity (charges in the ADM approach). Therefore, previous equation controls how this energy notion changes as we change the charge, the angular momentum or the area of the black hole. In analogy with the laws of thermodynamics, Equation (2.8) is naturally called *first law of black hole mechanics*. From $A(M,J,Q)$ it is trivial to check the *integrability condition* of (2.8), i.e., that cross derivatives of "intensive" quantities are satisfied, in fact they are simply the ordering interchange of second derivatives of $M(A,J,Q)$.

This rich physical interpretation of the first law suggests a more general derivation based on the black hole horizon, that is the so called "physical process version". We discuss it in the following.

2.1.2 First Law from Perturbation of Stationary Black Holes

Suppose we have a black hole in a stationary and axially symmetric spacetime. Let ξ^a and ϕ^a be the Killing fields associated with these symmetries (here we follow [20]), with ξ^a normalized to -1 at infinity and ϕ^a normalized such that closed orbits have a period 2π. The mass and angular momentum of the black hole can be computed as Komar integrals [2]

$$\begin{aligned} M &= -\frac{1}{8\pi}\oint_H \nabla^a\xi^b \, dS_{ab} \\ J &= \frac{1}{16\pi}\oint_H \nabla^a\phi^b \, dS_{ab}, \end{aligned} \tag{2.9}$$

where H is the two-surface defined by the black hole horizon. These quantities are exactly the ADM charges defined as integrals at infinity, but because we assume empty spacetime outside the black hole and we have the Killing equations $\nabla_{(a}\xi_{b)}=0$, $\nabla_{(a}\phi_{b)}=0$; Stokes' theorem allows us to express them as integrals at H. Let us stress here that if we do not consider the connection with the ADM charges and, consequently, with the notion of asymptotic observers previous definition would be meaningless, i.e., the real physical interpretation of the previous integrals as mass and angular momentum resides in the asymptotic definitions.

Now, let us consider a physical process in which some *small* quantity of matter falls through the horizon. After some complicated dynamical evolution the black hole will stabilize again. That falling matter can be described by a small energy-momentum tensor δT_{ab} which is small enough in such a way that changes of black hole geometry are negligible in the sense that we can still use ξ^a and ϕ^a as approximate Killing field. The total mass and angular momentum that crosses the horizon are

$$\delta M = -\int_\Delta \delta T^a{}_b \xi^b \, d\Sigma_a \tag{2.10}$$

$$\delta J = \int_\Delta \delta T^a{}_b \phi^b \, d\Sigma_a, \tag{2.11}$$

where Δ is the horizon worldsheet. Again, this δM and δJ has the meaning of mass and angular momentum for the observers at infinity, but now to use Stokes' theorem we should use, in addition to the Killing equation, the energy-momentum conservation $\nabla^a \delta T_{ab} = 0$. The change in the area of the horizon can be computed by looking at the transversal section of a congruence of geodesics normal to the horizon. The rate of infinitesimal change of area of this section is called *expansion* and denoted by θ. We compute the expansion for the affine parametrized geodesic normal null vectors $k^a = \partial_V{}^a$. In general, the expansion satisfies the *null*[1] Raychaudhuri equation

$$\frac{d\theta}{dV} = -\frac{1}{2}\theta^2 - \sigma^{ab}\sigma_{ab} + \omega^{ab}\omega_{ab} - R_{ab}k^a k^b, \tag{2.12}$$

where V is the affine parameter of k^a, σ_{ab} and ω_{ab} are, respectively, the shear and twist of the transversal section of the geodesic congruence, and R_{ab} is the Ricci tensor [2]. In the background—unperturbed—geometry the expansion, the shear, and the twist vanish at the horizon. As the amount of matter is assumed to be small θ^2, σ^2, and ω^2 are second order terms and hence negligible. Furthermore, we use the Einstein equation $R_{ab} - \frac{1}{2}g_{ab}R = 8\pi\delta T_{ab}$, which projected on null vectors is simply $R_{ab}k^a k^b = 8\pi\delta T_{ab}k^a k^b$. Hence, the Raychaudhuri equation simplifies to

$$\frac{d\theta}{dV} = -8\pi\delta T_{ab}k^a k^b. \tag{2.13}$$

In static black holes the Killing field ξ^a is also normal to the horizon. Instead, in the rotating case there is a particular linear combination of the Killing fields ξ^a and ϕ^a that is normal to the horizon, and therefore, proportional to k^a. It is given by

$$\chi^a = \xi^a + \Omega_H \phi^a, \tag{2.14}$$

and defines the horizon angular velocity Ω_H (it coincides with (2.6) for the Kerr-Newman solution) as well as the surface gravity κ through

$$\chi^a \nabla_a \chi^b|_\Delta = \kappa \chi^b|_\Delta. \tag{2.15}$$

At the horizon the proportionality between the two null normals[2] is given by $\chi^a|_\Delta = \partial_v{}^a = \kappa V \partial_V{}^a = \kappa V k^a$.

[1] Note that if we consider null geodesic curves the coefficient of the first term in the r.h.s. is $-1/2$ instead of the $-1/3$ for the spacelike or timelike geodesics Raychaudhuri.

[2] Proof: Let us use V as the affine parameter such that $k^a\nabla_a V = 1$, and v the parameter associated with χ^a, $\chi^a \nabla_a v = 1$. The surface gravity κ can be defined by the equation $\chi^a \nabla_a \chi^b = \kappa \chi^b$ at the horizon. Then, it is easy to show that $V = e^{\kappa v}$, where it has been used that κ is constant over the horizon. With $V = e^{\kappa v}$ it is trivial to show that $\chi^a|_\Delta = \kappa V k^a$.

Integration of the Raychaudhuri equation on the horizon worldsheet Δ of the Raychaudhuri equation gives

$$\begin{aligned}
\delta M - \Omega_H \delta J &= \int_\Delta \delta T_{ab}(\xi^a + \Omega_H \phi^a) k^b \; dV \; dS \\
&= -\oint_H \int_0^\infty \delta T_{ab} \; \chi^a k^b \; dV \; dS \\
&= -\kappa \oint_H \int_0^\infty dV \; V \delta T_{ab} k^a k^b \; dS \\
&= -\frac{\kappa}{8\pi} \oint_H \int_0^\infty dV \; V \frac{d\theta}{dV} dS \\
&= -\frac{\kappa}{8\pi} \oint_H dS \left(V\theta|_0^\infty - \int_0^\infty dV\theta \right) \\
&= \frac{\kappa}{8\pi} \int_0^\infty dV \oint_H \theta \; dS \\
&= \frac{\kappa}{8\pi} \delta A,
\end{aligned} \tag{2.16}$$

where the fact that surface gravity κ is constant over H has been used in the third line. The Raychaudhuri equation has been used in the fourth line.

Let us explain in more detail the step in the fifth line. The surface integral of $V\theta$ is neglected because of two reasons: First, the expansion θ is zero at the infinite future—black hole reaches the stationary state. Second, as presented in [20], the term is evaluated at $V = 0$. However, this last argument is not exactly true as we now explain. V is the affine parameter of the geodesic null generators of the horizon which is being perturbed. In fact, when evaluating at the lower limit $V = 0$ what we aim to is to the integration of the term $V\theta$ on the surface obtained by the intersection of the tracing back of the future perturbed horizon to the past and the past (unperturbed) horizon.[3] However, as far as there is a perturbation such a surface is not a bifurcated horizon,[4] that is, V does not vanishes as it would be in the unperturbed case. However, the differences with zero of the affine parameter V which truly parametrize that surface are of the order of the perturbation, and, as the expansion is already first order, then, the term $V\theta$ evaluated on that intersection is second order and therefore negligible. This subtlety has been explained in [22] and we made it explicit here as the argument will be the same in the more general case of Isolated Horizons.

[3] Note that, precisely because of the parametrization, the are not the same surface.

[4] Two-surfaces where the affine parameter of the horizon generator defined through the Killing fields vanishes.

Finally, as mentioned before, the expansion is the rate of an infinitesimal area change $\theta = \frac{1}{dS}\frac{d}{dV}dS$, and the last integral in (2.16) is precisely the sum of all these small changes over the whole horizon and through its whole evolution. Therefore, it is the change in the total area due to the proces, that is, the difference in the area of the past unperturbed horizon and the future perturbed horizon. Written in a more standardized form

$$\delta M = \frac{\kappa}{8\pi}\delta A + \Omega_H \delta J, \tag{2.17}$$

we have obtained the first law of black hole mechanics for uncharged black holes.

If charged black holes are considered the generalization is straightforward. However, in the literature the derivations are usually done with different and less intuitive techniques,[5] so, we sketch a simpler calculation here.

Consider a *charged* and rotating black hole spacetime, as before, ξ^a and ϕ^a are Killing fields. Again, we consider some small matter distribution that falls through the horizon, but now, possibly charged. Due to the charge the whole energy-momentum tensor is not small any more but can be decomposed as

$$T_{ab} = T^{(e)}_{ab} + \delta T_{ab} + \delta T^{(e)}_{ab}, \tag{2.18}$$

where, as before, δT_{ab} is a small perturbation of *neutral* matter fields that falls through the horizon. $T^{(e)}_{ab}$ is the electromagnetic energy-momentum tensor

$$T^{(e)}_{ab} = \frac{1}{4\pi}\left(F_{ac}F_b{}^c - \frac{1}{4}g_{ab}F^{cd}F_{cd}\right), \tag{2.19}$$

where $F_{ab} = \nabla_a A_b - \nabla_b A_a$ stands for the electric field produced by the charged black hole as well as for the charged matter that falls in, then:[6] $\nabla^a F_{ab} = -4\pi\delta j_b$, with δj_b the *small* current source of the field. Finally

$$\delta T^{(e)}_{ab} = -2A_{(a}\delta j_{b)} + g_{ab}A^c\delta j_c, \tag{2.21}$$

is the energy-momentum tensor associated with the current source.

[5] For instance, with the Hamiltonian formalism see [10], or with a generalization of the Smarr formula see [11].

[6] We recall that the action is given by

$$S = \frac{1}{2\kappa}\int_{\mathcal{M}}\left(R - \frac{1}{4}F^{ab}F_{ab} + 4\pi A_b\delta j^b + \mathscr{L}_{neutral}\right)\sqrt{-g}\; d^4x, \tag{2.20}$$

and $T_{ab} \equiv -\frac{1}{2\pi}\frac{1}{\sqrt{-g}}\frac{\delta(S_M\sqrt{-g})}{\delta g^{ab}}$, with S_M the action of all matter fields.

Now, we compute the flux integrals on the horizon

$$\begin{aligned}
\int_\Delta T^{(e)}_{ab}\chi^a d\Sigma^b &= \frac{1}{4\pi}\int_\Delta F_{ac}F_b{}^c\chi^a d\Sigma^b = \frac{1}{4\pi}\int_\Delta (\nabla_a A_c - \nabla_c A_a)F_b{}^c\chi^a d\Sigma^b \\
&= \frac{1}{4\pi}\int_\Delta \left(F_b{}^c(\chi^a\nabla_a A_c + A^a\nabla_c\chi_a) + A_a\nabla_c F_b{}^c\chi^a\right) d\Sigma^b \\
&= \frac{1}{4\pi}\int_\Delta \left(F_b{}^c\mathscr{L}_\chi A_c + 4\pi A_a\chi^a\delta j_b\right) d\Sigma^b \\
&= \int_\Delta A_a\chi^a\delta j_b d\Sigma^b = -\Phi_H\int_\Delta \delta j_b d\Sigma^b \\
&= -\Phi_H\delta Q, \qquad (2.22)
\end{aligned}$$

where integration by parts, the Killing equation, $\mathscr{L}_\chi A^b = 0$ (as χ^a is a symmetry on the background), and the Maxwell equation have been used. Furthermore, we have used that the electric potential $\Phi_H = -A^b\chi_b\big|_\Delta$ is constant on the null surface, and that the integral of the electric flux over Δ corresponds to the total charge that enters the black hole

$$\delta Q \equiv \int_\Delta \delta j_b d\Sigma^b. \qquad (2.23)$$

On the other hand, the integral of the current energy-momentum tensor at the horizon becomes

$$\int_\Delta \delta T^{(e)}_{ab}\chi^b d\Sigma^a = -2\int_\Delta \delta A_b\chi^b\delta j^a d\Sigma_a = 2\Phi_H\delta Q. \qquad (2.24)$$

Finally, to show the first law in this more general case it is enough to reproduce the proof in Equation (2.16) with the sole difference that, now, the whole energy-momentum tensor should be considered

$$\frac{\kappa}{8\pi}\delta A = -\int_\Delta T_{ab}\chi^a d\Sigma^b = \delta M - \Omega_H\delta J - \Phi_H\delta Q, \qquad (2.25)$$

where the integrals (2.22) and (2.24), as well as the same expressions for δM and δJ, (2.10) and (2.11), have been used.

Thus, the physical process argument applied to the generic rotating and charged black hole spacetime confirms the validity of the first law

$$\delta M = \frac{\kappa}{8\pi}\delta A + \Omega_H\delta J + \Phi_H\delta Q. \qquad (2.26)$$

Let us remark that, even if the present derivation looks quasilocal as the integrals involved in the first law are computed over the horizon worldsheet, it strongly rests on the asymptotic notions of charges M, J, and Q. All the quasilocal integrals over H or over Δ can be rewritten as integrals at infinity as Killing fields are globally defined, and it is just there, in the asymptotically flat spacetime region, that they have a natural interpretation in terms of static asymptotic observers. Otherwise, the quasilocal definitions of δM, δJ and δQ would be meaningless. Thus, to solve what would be an ambiguity of the law (2.26), we are implicitly using an extra physically motivated argument: The measurements—of the energy, angular momentum, and charge flux—by observers represented by the Killing fields at the asymptotic region. Those measurements are interpreted along all the way up to the horizon through the globally defined Killing fields. This remark would be relevant in the next section, where we deal with a different derivation of the first law, but the same underlying asymptotic assumptions will be present.

2.1.3 First Law for Isolated Horizons in the Hamiltonian Formalism

Now, we review a rather different derivation of the first law. The strategy is to study the covariant symplectic structure,[7] $\Omega(\delta_1, \delta_2)$, on the phase space of General Relativity under the Isolated Horizon restrictions—as defined in Section 1.1. In particular, we consider the construction of a time evolution field t^a and the related Hamiltonian flow. The result is that the analogous first law for Isolated Horizon mechanics is a necessary and sufficient condition to have a Hamiltonian flow. However, we will see that in this derivation ambiguities are present unless we make use of extra input from the asymptotic spacetime structure. We sketch the derivation in the simpler case of a charged Isolated Horizon that does not have a rotational symmetry [12]. The analysis for the rotating case is more involved and has been done in [14].

A first law for Isolated Horizons needs a notion of energy on the Isolated Horizon. To achieve this a time evolution vector field t^a is introduced on the spacetime $\mathcal{M}$ from which it is possible to define a vector field δ_t on the phase space Γ. In order to guarantee that δ_t is a Hamiltonian flow we should find a function H_t that satisfies

$$\delta H_t = \Omega(\delta, \delta_t), \tag{2.27}$$

therefore, one should compute $\Omega(\delta, \delta_t)$ and work out which condition must be imposed on t^a to ensure that (2.27) is satisfied. As we will see in a moment the symplectic structure $\Omega(\delta, \delta_t)$ defined on Cauchy surfaces M can

[7] In Appendix D we introduce the symplectic structure of General Relativity in the context we will use for Chapter 4 but it could be useful now if the reader is unfamiliar with the approach.

be decomposed into boundary terms, one on the intersection with the Isolated Horizon $S_\Delta = \Delta \cap M$ and another at infinity—a time-like cylinder at infinity. At infinity, it is natural to require that t^a coincide with the time symmetry generator of flat spacetime—flatness is imposed on the asymptotic boundary—in this way the usual ADM energy, E_{ADM}, is recovered. On S_Δ, as proposed in [12], the vector field t^a is identified with the class of generators that defines the Isolated Horizon: $t^a|_\Delta \in [\ell]$, the specific relation can change at each point of the phase space.[8] A vector field t^a with this property is called a *live* evolution field.

The symplectic structure is computed through the boundary term that appears in the arbitrary variation of the action, see for example Appendix D. The action of a phase space vector field—constructed out of a spacetime vector field—on a generic field variable ψ is given by the Lie derivative: $\delta_t\psi = \mathscr{L}_t\psi$. After standard computations, we just show the final result in the notation of [12], where $\tilde{\varepsilon}$ denotes the two-volume form on S_Δ and F is the electromagnetic field 2-form

$$\begin{aligned}\Omega(\delta,\delta_t) &= -\frac{1}{8\pi}\int_{S_\Delta} t^a\omega_a\delta(\tilde{\varepsilon}) - \frac{1}{4\pi}\int_{S_\Delta} t^a A_a \delta(\star F) + \delta E_{ADM} \quad (2.28)\\ &= -\frac{1}{8\pi}\kappa_t\delta A - \phi_t\delta Q_\Delta + \delta E_{ADM}, \quad (2.29)\end{aligned}$$

we have used that the horizon charge is given by $Q_\Delta = \frac{1}{4\pi}\int_{S_\Delta}\star F$, and the area horizon by $A = \int_{S_\Delta}\tilde{\varepsilon}$. In the second line we have used that $\kappa_t = t^a\omega_a$ and $\phi_t = -t^a A_a$ are constant on the horizon. Therefore, in order to guarantee that δ_t is Hamiltonian we have to require the existence of a function E^t_Δ such that

$$\delta E^t_\Delta = \frac{1}{8\pi}\kappa_t\delta A + \phi_t\delta Q_\Delta. \quad (2.30)$$

The last condition is the first law for Isolated Horizons in this context, and the Hamiltonian is

$$H_t = E_{ADM} - E^t_\Delta. \quad (2.31)$$

Summarizing, the horizon first law appears as a condition for the live vector field t^a to be Hamiltonian.

Finally, let us stress that the notion of horizon energy E^t_Δ is not unique as it depends on the chosen t^a. This ambiguity is explicitly present in κ_t and ϕ_t. The reason is that asymptotic flatness condition fixes the metric and defines a natural time translation vector field at infinity. On the other hand, on the Isolated Horizon the metric is not fixed at all, thus there is not a

[8] The class $[\ell]$ is one for all the phase space but the constant in the relation $t^a = c\ell^a$ is a function of the phase space.

local preferred time evolution vector field. If we consider static spacetimes, it is possible to use the normalization of the Killing field at infinity to fix the ambiguity at the horizon, see last comment on the previous section. However, for general time evolution fields the behaviour near the horizon is not related to the behaviour near the infinity.

2.2 Quasilocal First Law of Black Hole Mechanics

Now, we develop a truly quasilocal approach where the asymptotic structure and ADM definitions are avoided by using a new physically motivated structure: *quasilocal observers*. As in the previous section we explain the quasilocal first law at three levels: 1) by a simple analysis of free falling particle process and using the previous derived first law 2) by using perturbations and Einstein equations (analogous with 2.1.2), and 3) in the Isolated Horizon framework.

2.2.1 Near Horizon Observers

We study the thermodynamical properties of a Kerr-Newman black hole as seen by a family of stationary observers $\mathcal{O}$, surrounding the horizon at a small proper distance $\ell^2 \ll A$, where A is the area of the horizon. We define them in such a way that they follow integral curves of the Killing vector field

$$\chi^a = \xi^a + \Omega_H \phi^a = \partial_t^a + \Omega_H \partial_\phi^a, \tag{2.32}$$

where ξ^a and ϕ^a are the Killing fields associated with the stationarity and axisymmetry of Kerr-Newman spacetime respectively, while Ω_H is the horizon angular velocity introduced in (2.6).

The observers $\mathcal{O}$ four-velocity is given by

$$u^a = \frac{\chi^a}{\|\chi\|}. \tag{2.33}$$

These observers are the unique stationary ones that coincide with the *locally non-rotating observers* of [2] or *zero angular momentum observers,* ZAMOs of [23] as $\ell \to 0$. Therefore, the angular momentum of these observers is not exactly zero, but $o(\ell)$. Thus, in the approximation $\ell^2 \ll A$ we can consider them at rest with respect to the horizon: This makes them the preferred observers for studying thermodynamics issues from a quasilocal perspective.

The family of observers that we have introduced here defines a two-surface of stationary observers around the horizon. In spacetime their history is represented by a three-dimensional worldsheet $\mathcal{W}(\mathcal{O})$. For dynamical process, as the ones involved in the first law derivation, this two-dimensional hypersurface evolves in complicated ways depending on the details of the process—as the horizon itself—but it is valuable to stress that in the asymptotic stationary situation, when the black hole stabilizes, we recover the simple description of a surface at proper distance ℓ to the horizon.

In the standard first law we have derived before, (2.26), some of the quantities are defined for an asymptotic observer or have a global meaning, as we have already discussed, this is clear for M. Furthermore, Φ_H can be interpreted as the difference in electrostatic potential between the horizon and infinity, Ω_H is the angular velocity of the horizon as seen from infinity, and κ—if extrapolated from the non-rotating case—is the acceleration of the stationary observers as they approach the horizon as seen from infinity.

To construct a quasilocal form of the first law of black hole mechanics, it is crucial to describe physics from the viewpoint of our family of observers $\mathcal{O}$.

2.2.2 Thought Experiment: Throwing a Test Particle

The first situation that we will consider involves the process of absorption of a test particle by the black hole. Let us imagine we throw a test particle of unit mass and charge q from infinity to the horizon. The geometry as well as the electromagnetic field are stationary and axisymmetric, namely

$$\mathscr{L}_\xi g_{ab} = \mathscr{L}_\phi g_{ab} = \mathscr{L}_\xi A_a = \mathscr{L}_\phi A_a = 0.$$

The particle, described by the four-velocity vector w^a, moves according to the Lorentz force equation

$$w^a \nabla_a w_b = q\, F_{bc} w^c, \tag{2.34}$$

with four-velocity w^a. In the asymptotic region where the electromagnetic and the gravitational fields can be neglected, the energy of the particle is $\mathcal{E} = -w^a \xi_a|_\infty$, but along the whole particle trajectory the conserved energy is

$$\mathcal{E} \equiv -w^a \xi_a - qA^a \xi_a. \tag{2.35}$$

Similarly, the conserved angular momentum is

$$L \equiv w^a \phi_a + qA^a \phi_a. \tag{2.36}$$

As the particle gets absorbed, the black hole settles down to a new state with $\delta M = \mathcal{E}$, $\delta J = L$ and $\delta Q = q$. The standard first law of black hole mechanics, (2.26), implies

$$\frac{\kappa}{8\pi} \delta A = \mathcal{E} - \Omega_H L - \Phi_H q. \tag{2.37}$$

For our observers having four-velocity u^a the local energy of the particle is

$$\mathcal{E}_{loc} = -w^a u_a. \tag{2.38}$$

Using (2.33), the definitions of $\mathcal{E}$, L and $\Phi_H \equiv -\chi^a A_a$ we find

$$\mathcal{E}_{loc} = -\frac{w^a \xi_a + \Omega_H w^a \phi_a}{\|\chi\|} = \frac{\mathcal{E} - \Omega_H L - q\Phi_H}{\|\chi\|}. \tag{2.39}$$

Finally from (2.37)

$$\mathcal{E}_{loc} \quad = \quad \frac{\overline{\kappa}}{8\pi}\delta A, \quad \text{where} \quad \overline{\kappa} \equiv \frac{\kappa}{\|\chi\|}. \tag{2.40}$$

From the point of view of our quasilocal observers, the horizon has absorbed a particle of energy $\mathcal{E}_{loc}$. The change in energy of the system E as seen by $\mathcal{O}$ must be $\delta E = \mathcal{E}_{loc}$. All this implies a quasilocal version of the first law

$$\delta E = \frac{\overline{\kappa}}{8\pi}\delta A. \tag{2.41}$$

A direct calculation, Appendix A, shows that

$$\overline{\kappa} \equiv \frac{\kappa}{||\chi||} = \frac{1}{\ell} + o(\ell). \tag{2.42}$$

The local surface gravity $\bar{\kappa}$, up to $o(\ell)$ terms, is nothing else but the minimal acceleration these observers need just to keep their position without falling into the black hole.[9] In contrast with the standard surface gravity (2.5), the previous equation shows that $\bar{\kappa}$ for the locally non-rotating stationary observers is universal, i.e., it is independent of the mass M, angular momentum J and charge Q of the Kerr-Newman black hole. From (2.41) we get the quasilocal notion of energy

$$E = \frac{A}{8\pi\ell} + o(\ell), \tag{2.43}$$

as the above quantity leads to the local first law when varied. This provides a natural quasilocal notion of horizon energy that can be useful for thermodynamics considerations. Its physical interpretation is restricted to the realm of small changes close to equilibrium.

[9] The acceleration $a^b = u^a \nabla_a u^b = \frac{\chi^a \nabla_a \chi^b}{\|\chi\|^2}$ and $\|a\| = \frac{\kappa}{\|\chi\|} + o(\ell)$ which is equivalent to one of the definitions of surface gravity $\kappa \equiv \lim(\|a\|\|\chi\|)$, where the limit is taken approaching to the horizon.

The idea is to associate the above energy and first law to the horizon itself by taking our ℓ as small as possible without being zero. An effective quantum gravity formulation suggests that ℓ should be of the order of the Planck scale [17], $\ell \sim \ell_p$, but this is not essential for the analysis presented here.

One can wonder about the constant of integration in (2.43), however, in thermodynamical systems the absolute value of the energy is defined always up to a constant, therefore what is relevant for thermodynamical processes is the dependence that the energy has in terms of the physical variables of the system. Here, we learn that black holes are thermodynamics systems that are described by special hovering observers who have an energy which is primarily dependent on the area horizon.

2.2.3 Refined Thought Experiment: The Field Theoretical Version

A stronger field theoretical version of the previous arguments can be formulated as an analogy with the so called "physical process version" that proves the standard first law discussed in 2.1. In our case it has the virtue of being even simpler than the standard one.

Let the matter falling into the stationary black hole be described by a small perturbation of the energy-momentum tensor δT_{ab} whose back-reaction to the geometry will be accounted for in the linearized approximation of Einstein's equations around the stationary black hole background. Let us consider an uncharged system for the moment, the generalization will be addressed at the end. Because of energy-momentum tensor conservation and the Killing equation for χ^b, the current defined as

$$J^a = \delta T^a{}_b \chi^b, \tag{2.44}$$

is also conserved $\nabla_a J^a = 0$.

Applying Gauss's law to the spacetime region bounded by the black hole horizon Δ and the timelike worldsheet of the observers $\mathcal{W}(\mathcal{O})$ we get

$$\int_\Delta dV dS \; \delta T_{ab} \chi^a k^b = \int_{\mathcal{W}(\mathcal{O})} J_b N^b, \tag{2.45}$$

where N^a is the inward normal of $\mathcal{W}(\mathcal{O})$ and $k^a = \partial^a_V$ a null geodesic normal to Δ, with V an affine parameter along the generators of the horizon. The origin $V = 0$ is chosen to coincide with point 1 in Figure 2.1 (*bifurcate horizon*). We have also assumed that δT_{ab} vanishes in the far past and far future of the considered region. Using the fact that $\chi^a = \kappa V k^a$ on H, the previous identity takes the form

$$\kappa \int_{\Delta} dV dS \ V \delta T_{ab} k^a k^b = \int_{\mathcal{W}(\mathcal{O})} \| \chi \| \, \delta T_{ab} u^a N^b, \tag{2.46}$$

Notice that the integral on the right is closely related to the energy-flux associated with the observers, which is equal to δE. Now, the Raychaudhuri equation in the linear approximation is

$$\frac{d\theta}{dV} = -8\pi \delta T_{ab} k^a k^b, \tag{2.47}$$

where θ is the expansion of the null generators k^a. Finally, a direct calculation shows that $\|\chi\|$, evaluated on $\mathcal{W}(\mathcal{O})$ in terms of the proper distance to the horizon ℓ, is constant up to first order in a stationary solution. Then, we can define $\mathcal{W}(\mathcal{O})$ such that, this is true for the whole evolution. Therefore, we can simply take out $\|\chi\|$ from the integral on the r.h.s. of (2.46) and obtain

$$\int_{\Delta} dV dS \ V \frac{d\theta}{dV} = -\frac{8\pi \| \chi \|}{\kappa} \delta E + o(\ell^2), \tag{2.48}$$

By an integration by parts the integral on the left is equal to $-\delta A$. Explicitly,

$$-\int_{V_1}^{\infty} dV \oint dS \ V \frac{d\theta}{dV} = \int_{V_1}^{\infty} dV \oint dS \, \theta(V) + \oint dS \ V_1 \theta(V_1) = \delta A, \tag{2.49}$$

where in the last term we dropped the boundary contribution at $V = \infty$ using that $\theta(\infty) = 0$. If V_1 is the value of the affine parameter of the horizon generators before enough the falling matter, we can apply the same argument detailed after the demonstration (2.16), and therefore, that term is second order in the perturbation .

Finally, using $\overline{\kappa} \equiv \kappa / \| \chi \|$ we get the desired result

$$\delta E = \frac{\overline{\kappa}}{8\pi} \delta A + o(\ell). \tag{2.50}$$

Note that, in the expression for the energy as seen by the quasilocal observers it is clear that neither the angular momentum or the charge entering into the black hole appear explicitly, i.e., we do not split up neither χ^a nor δT_{ab} as in the standard case.

However, to incorporate charge into the analysis we should note that if we consider a small perturbation of the Killing fields $\chi^a = \chi^a_{(0)} + \lambda \chi^a_{(1)}$, and that—because of the electromagnetic field—the energy momentum tensor of the background is not zero any more. This implies

$$\nabla^a J_a = T_{ab} \nabla^a \chi^b \neq 0, \tag{2.51}$$

in fact this terms has the order of the perturbation, and therefore it could be relevant. Thus, when using Gauss' law the following volume term should be considered in the r.h.s. of (2.48)

$$\int_{Vol} T_{ab} \nabla^a \chi^b d^4 x. \tag{2.52}$$

Nevertheless, because the four-volume Vol is the thin region enclosed between the horizon Δ and the observers surface $\mathcal{W}(\mathcal{O})$, this term is order $o(\ell)$ and therefore negligible. The rest of the steps follow in the same manner such that (2.50) is still true.

The previous field theoretical argument can be further generalized to include Isolated Horizons as we will see in the following.

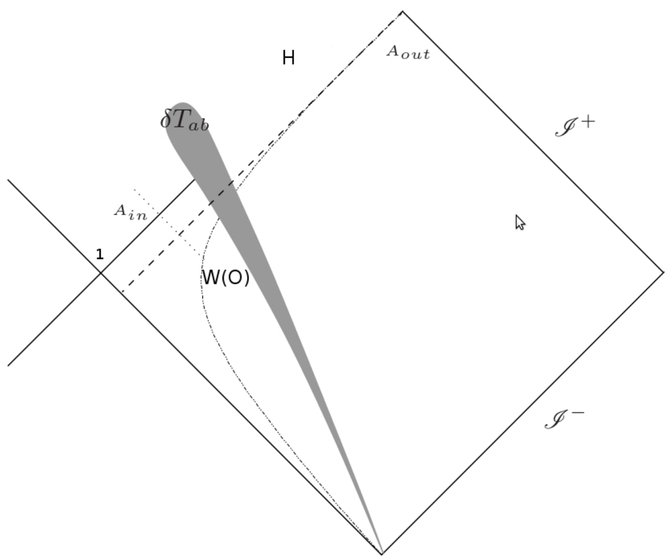

Fig. 2.1: Conformal diagram representing the perturbation of an initially stationary black hole with a bifurcate horizon. The dashed line represents the true black hole horizon, the stationary observers worldsheet is denoted by $\mathcal{W}(\mathcal{O})$. The quantity A_{in} is the area of the initially stationary background while A_{out} is the final area of the black hole horizon. The grey region represents the matter δT_{ab} falling into the black hole.

2.2.4 Isolated Horizons

Here we prove the validity of the quasilocal form of the first law (2.41) in the more general framework of Isolated Horizons introduced in Section 1.1. In contrast with the previous derivation of the first law for Isolated Horizon—shown is Subsection 2.1.3—that was found as a formal mathematical consistency, the first law derived here is more physical in character, i.e., changes in the area and energy of the system can be seen as the consequence of the absorption of matter fields by the horizon along its history.

Isolated Horizons are equipped with an equivalence class of null normal $[\chi]$ where equivalence is defined up to constant scaling[10]. The generators χ^a define a notion of Isolated Horizon surface gravity κ_{IH} through the equation $\nabla_\chi \chi_a = \kappa_{IH}\chi_a$. It is clear that κ_{IH} is not completely defined within $[\chi]$ because it gets rescaled when χ^a is rescaled. The near Isolated Horizon geometry can be described in terms of Bondi-like coordinates that we now explain in detail [24, 9].

Near Isolated Horizon Coordinates

From the class $[\chi]$ we can choose a particular generator and define an affine parameter v along Δ: $\chi^a\nabla_a v = 1$. Then, v induces a foliation of Δ of topological two spheres S_v. Let us take a particular one S_v and define the null surface $\mathcal{N}_v$ generated by the *past light cone at each point* of S_v, of course, this can be repeated for each S_v (see figure 2.2). This surface is the geodesic extension of $-n^a$, with n^a a future directed null vector orthogonal to S_v and normalized such that $\chi^a n_a = -1$. Let r be an affine parameter along $-n^a$, such that $r = r_0$ on Δ, this defines a second coordinate. Finally choosing x^1, x^2 two coordinates constant along χ^a orbits: $\chi^a\nabla_a x^i = 0$, and set also (v, x^i) constant along integral curves of n^a. Thus, (v, r, x^i) is a coordinate system in the neighbourhood of Δ. In particular χ^a is naturally extended in the neighbourhood outside Δ. In these coordinates, the near Isolated Horizon metric can we written [9]

$$g_{ab} = q_{ab} + 2dv_{(a}dr_{b)} - 2(r-r_0)[2dv_{(a}\omega_{b)} - \kappa_{IH}dv_a dv_b] + o[(r-r_0)^2], \quad (2.53)$$

where q_{ab} is the induced metric on S_v. Then, $q_{ab}n^a = q_{ab}\chi^a = 0$, and ω_a is the 1-form intrinsic to Isolated Horizons with the property $\chi^a\omega_a|_\Delta = \kappa_{IH}$. The *form* notation is used: $dr_a = \nabla_a r$, such that $r^a = \partial_r^a$ and $r^a\nabla_a r = 1$. Also, one has from the Isolated Horzion definition [25]

$$\mathscr{L}_\chi g_{ab}|_\Delta = 0. \quad (2.54)$$

[10] here we adopt $\chi^a \in [\chi]$ notation to denote a vector in the class of null generator of the horizon in order to avoid confusion with the constant proper distance to the horizon ℓ.

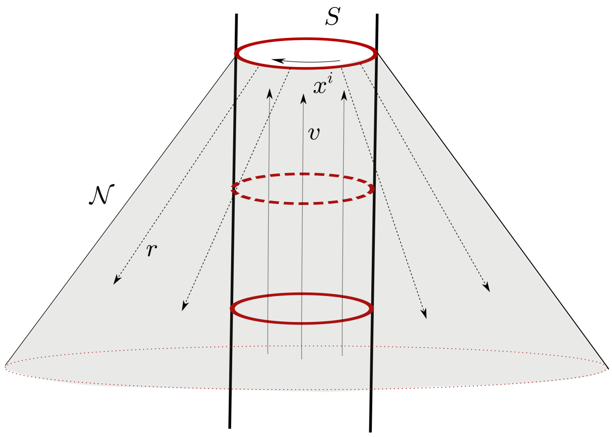

Fig. 2.2: Bondi-like coordinates

Thus, χ^a can be used to define the near horizon observers $\mathcal{O}$. The proper distance ℓ to the horizon from a point with coordinate r along a curve normal to both χ^a and q_{ab} can be computed by using a spacelike vector orthogonal to the observers worldsheet $\mathcal{W}(\mathcal{O})$, N^a such that

$$N^a \chi_a|_{\mathcal{W}(\mathcal{O})} = 0, \quad \text{and} \quad N^a q_{ab}|_{\mathcal{W}(\mathcal{O})} = 0. \tag{2.55}$$

The surface $\mathcal{W}(\mathcal{O})$ is in the neighbourhood of Δ, therefore, the metric (2.53) is a good approximation to compute these conditions. The vector turns out to be

$$N^a = \partial_r^a + \frac{1}{2\kappa_{IH}(r - r_0)}\partial_v^a, \tag{2.56}$$

remember that $\chi^a = \partial_v^a$. In terms of these coordinates the proper distance is

$$\ell = \int \sqrt{g_{ab}N^a N^b} dn = \sqrt{\frac{2(r-r_0)}{\kappa_{IH}}}, \tag{2.57}$$

on the other hand

$$\|\chi\| = 2\kappa_{IH}(r - r_0). \tag{2.58}$$

Therefore, in this context, we define the local surface gravity by

$$\overline{\kappa} = \frac{\kappa_{IH}}{\|\chi\|} = \frac{1}{\ell}. \tag{2.59}$$

Notice that in contrast with κ_{IH}, $\overline{\kappa}$ is uniquely defined for the class $[\chi]$: $\bar{\kappa}$ is invariant under rescaling of χ^a. The form of the Raychaudhuri Equation

(2.47) is the same for the generators of Isolated Horizons, but now this is due to the fact that their expansion, shear, and twist vanish by definition. Then, the argument following the Equation (2.43) holds. Similarly, the quasilocal first law is

$$\delta E = \frac{\overline{\kappa}}{8\pi}\delta A, \tag{2.60}$$

where the energy $E = \frac{A}{8\pi\ell}$, and we have used that $\ell^2 \ll A$.

Summarising, even though we have first justified the quasilocal first law (2.60) starting from the analysis of the standard first law for stationary space-times and its translation in terms of the quasilocal observers $\mathcal{O}$, the final analysis in the context of Isolated Horizons implies that the result can be recovered entirely from quasilocal considerations that know nothing about the global structure. In this thesis we are proposing to use this remarkable fact in order to reverse the perspective, and take the quasilocal definition of Isolated Horizons with their null normals $[\chi]$, the quasilocal first law (2.60), the energy (2.43), and the intrinsic notion of surface gravity (2.59)—defined in terms of quasilocal observers (2.33)—as an alternative structure behind black hole thermodynamics. Furthermore, because the notion of energy proposed is simply proportional to the area, this framework is specially suitable for the statistical mechanical procedures which are applied in Loop Quantum Gravity, where the area, as a quantum operator, has been studied in great detail.

As an extra comment, notice also that the quasilocal first law and the universality of $\overline{\kappa}$ implies the Gibbs relation $E = TS$ where $T = \ell_p^2\bar{\kappa}/(2\pi)$, and $S = A/4\ell_p^2$. That simple property of usual thermodynamical systems is not realized by the quantities taking part in the standard first law (2.26). Therefore, it is an extra bonus of this quasilocal description.

On the Physical Interpretation of the Quasilocal Energy

In previous section we derived a quasilocal first law Isolated Horizons. To do so, we have studied the response of black holes to matter infall from the point of view of near horizon observers, fundamental ingredients were the measure of local matter flow entering through the horizon and local surface gravity $\bar{\kappa} \approx 1/\ell$ that keeps observers fixed in their trajectories. The result (2.60) immediately suggests the following notion of local energy

$$E = \frac{\bar{\kappa}}{8\pi}A, \tag{2.61}$$

now, we will see how this energy can be written as a Komar-like integral.

First we establish some basic results. The acceleration of observers following χ^a trajectories is

$$a^b = u^a \nabla_a u^b = \frac{\chi^a}{\|\chi\|} \nabla_a \left(\frac{\chi^b}{\|\chi\|} \right) = \frac{\chi^a \nabla_a \chi^b}{\|\chi\|^2} = \frac{1}{2} \nabla^b \log \|\chi\|^2. \tag{2.62}$$

This is the local force which should be exerted to hold a unit test mass in place [2]. Then, for an arbitrary topological two-sphere S embedded in a spacelike hypersurface, we can compute the total local force needed to keep in place a unit surface mass density distributed on S.

$$F = \int_S N^b a_b dS, \tag{2.63}$$

where N^a is a spacelike unit normal orthogonal to u^a and also orthogonal to S. Since $u^b a_b = 0$ clearly, $N^b = a^b/\|a\|$. Now, this total force can be written as a Komar-like integral

$$F = \int_S N^b a_b \, dS = - \int_S \epsilon_{abcd} \nabla^c u^d, \tag{2.64}$$

which is derived as follows

$$\begin{aligned} F &= \int_S N^b u^a \nabla_a u_b \, dS = \int_S (N^b u^a - N^a u^b) \nabla_a u_b \, dS \\ &= \int_S \tilde{\epsilon}_{ab} N_{cd} \nabla^c u^d = \int_S 6 \tilde{\epsilon}_{[ab} N_{cd]} \nabla^c u^d \\ &= - \int_S \epsilon_{abcd} \nabla^c u^d, \end{aligned} \tag{2.65}$$

in the first line we have used $N^a u^b \nabla_a u_b = \frac{1}{2} N^a \nabla_a (u^b u_b) = 0$, then $N^{ab} = 2u^{[a} N^{b]}$ is the normal bi-vector to S, and $\tilde{\epsilon}_{ab}$ is the area element of S. The surface 2-form has been rewritten as a volume 4-form, by using the normal bi-vector $\epsilon_{abcd} = -6 N_{[ab} \tilde{\epsilon}_{cd]}$.

For our preferred family of near horizon observers we have

$$N^b a_b = \|a\| = \frac{\kappa}{\|\chi\|} + o(\ell) = \frac{1}{\ell} + o(\ell). \tag{2.66}$$

Therefore, provided we use $S = S_{\mathcal{O}}$ as the spacelike surface where near horizon observers lay, it is possible to compute the integral trivially

$$\int_{S_{\mathcal{O}}} N^b a_b \, dS = \frac{1}{\ell} A + o(\ell). \tag{2.67}$$

Thus, neglecting $o(\ell)$ terms, a Komar-like energy formula is obtained for the energy of the quasilocal observers

$$E = \frac{1}{8\pi} \int_{S_{\mathcal{O}}} N^b a_b \, dS = -\frac{1}{8\pi} \int_{S_{\mathcal{O}}} \epsilon_{abcd} \nabla^c u^d, \tag{2.68}$$

the coefficient is fixed by using the differential first law. Note that, as u^a is not a Killing field, this formula depends on the surface we choose: $\mathcal{W}(\mathcal{O})$. However, this surface can be constructed for every black hole. Therefore, this construction can be used as a notion of energy.

3. EUCLIDEAN PARTITION FUNCTION FOR BLACK HOLES

Thermodynamics deals with the study of *macroscopic* properties of systems which interact through *quasi-static* processes with the surrounding environment. Macroscopic quantities such as the energy, volume, temperature, entropy, as well as the processes that change them, are studied through the standard *four laws of thermodynamics*.

While the macroscopic quality of a system can be understood as a certain limit once we have a microscopic model of the same, a quasi-static process is difficult to define. It can be defined tautologically by saying that it is a process such that thermodynamics laws work. The mathematical framework of thermodynamics was originally developed to explain the physics of gases, and they are still today the paradigmatic example. However, this framework can be also successfully applied to a wide range of systems even outside physics. The key is that all them share a general feature: They are described by a huge number of degrees of freedom. From that description, the thermodynamics of the system can be understood, and emerges, from the probabilistic study of its microscopic components. The study of systems with a large number of microscopic degrees of freedom is the branch of physics known as *statistical mechanics* [27].

The main tool in statistical mechanics is the so called *partition function*. It is defined as the sum of all distributional probabilities where each one is assigned to a microstate configuration that the system can access. Different general computing procedures can be established by considering some of the macroscopic quantities fixed, these are called *ensembles*. For instance, in the *canonical ensemble* the sum is over all probability distribution of microstates such that volume and the number of particles are fixed. This sum is sometimes called the *canonical* partition function. Analogously, in the *grand canonical ensemble*, which describe a system where particles are not conserved, we can define the *grand canonical* paritition function.

From the partition function, macroscopic averages of the thermodynamics quantities can be explicitly computed and, consequently, the laws of thermodynamics can be reproduced. In this respect, the role of the entropy is paradigmatic as, even if it is a macroscopic quantity, it cannot be measured directly on systems. Whereas, in most cases, statistical mechanics concretely offers a way to compute it. This is exactly our perspective on the problem of the black hole entropy: One must have a statistical mechanics description to explain its entropy.

In the previous chapter we developed a new version of the first law of black hole mechanics that tells us how black holes exchange energy with their surroundings from a quasilocal setup. This new law—as well as the usual law—is in analogy with the thermodynamical first law, the only missing piece is the concept of temperature that is completely absent for a classical description of black holes. In the standard approach the temperature is provided by the Hawking radiation [1], which is a purely quantum effect. In our approach it is enough to consider the Unruh temperature [28] for local accelerated observers—consistent with the Hawking temperature—which characterize the radiation near the horizon. This is the temperature that a thermal bath should have in order to keep the black hole in equilibrium.

Once the laws of thermodynamics for black holes are established one should ask about the statistical mechanics description which could explain it. That is, the microscopic description that allows us to build a partition function.

When Hawking predicted that black holes radiate, he analyzed quantum fields on a curved spacetime as a background. On one side its prediction is strong because it is not necessary to know the quantum details of the black hole, but, by the same reason his approach is indirect as it did not identify the microscopic black hole states associated with this thermal properties. Nevertheless, two years after this prediction, Gibbons and Hawking [29] explored in this direction and found a simpler, formal, and fruitful way to study the black hole thermodynamics. They borrowed a different strategy to construct a statistical partition function through path integrals from Quantum Field Theory: The *Euclidean partition function* [30].

The Euclidean partition function allows us to put classical black holes directly in a proper quantum context and gives us a way to understand the thermodynamics of black holes. However, as we will see, it is a *semiclassical* approach as quantum degrees of freedom are not singled out. Therefore, though Euclidean black hole partition function is a mayor step towards a black hole quantization, nevertheless, at the end of the analysis, black hole still deserve a fundamental quantum treatment. Such a fundamental treatment should provide a standard partition function constructed by summing probability distributions associated with black hole microstates. This is the approach we follow in Chapter 5.

Hence, in this chapter we follow Hawking's way, to make a step towards quantization, and use the Euclidean partition function approach. We do it by using the quasilocal perspective from where our new first local law is built.

The main result is that the Euclidean partition function method using quasilocal observers is also a framework suitable to study the thermodynamical properties of black holes. In particular, we will be able to compute its entropy and show that its leading behaviour, as a function of the horizon area, is given by $S = A/4$, i.e., the usual Bekenstein-Hawking entropy law. As a by-product, we show that the role of the energy is played exactly by the quasilocal notion of energy defined in Chapter 2, and, that the temperature is the Unruh local temperature of accelerated observers. Thus, the approach is consistent with the quasilocal framework.

This chapter is organized in three sections as follows: First, we review the construction of the Euclidean partition function. Second, the Gibbons and Hawking application to gravity is reviewed for Schwarzschild geometry. Finally, we apply the quasilocal perspective developed in the previous chapter to define a quasilocal Euclidean partition function.

3.1 Preliminaries: Partition Function as a Path Integral

Let us start by considering a quantum system characterized by a Hamiltonian operator $\hat{H}$ and a complete basis of normalized eigenstates $\{|\Psi_n\rangle\}_n$, i.e. $\hat{H}|\Psi_n\rangle = E_n|\Psi_n\rangle$. The statistical partition function for the canonical ensemble is defined by

$$Z_s(\beta) = \sum_n e^{-\beta E_n} = \sum_n \langle\Psi_n|e^{-\beta\hat{H}}|\Psi_n\rangle, \tag{3.1}$$

where $\beta = \frac{1}{kT}$ is the inverse of the temperature of the system, and k is the Boltzmann constant. From now on, we choose units such that $k = \hbar = 1$. The sum in (3.1) is over each individual state, but, another way to express the partition function is by summing over energies. In this case, we should introduce a degeneration due to states with the same energy. If the energy spectrum is "dense enough" we can approximate the sum by an integral. Let us denote the density of states for a given energy by $N(E) = e^{S(E)}$, then

$$Z_s(\beta) = \int_0^\infty dE\; e^{-\beta E} e^{S(E)}, \tag{3.2}$$

thus, the partition function is simply the Laplace transform of the density of states. We will see that the function $S(E)$, when evaluated at the equilibrium energy, gives in fact the canonical entropy of the system.

On the other hand, in standard Quantum Mechanics it is possible to compute the transition amplitude between two states by simply using the evolution operator, $e^{-i\hat{H}t}$, which controls the deterministic dynamics in quantum theories. In Quantum Field Theory, the transition amplitude can be formally written as a *path integral* of the phase $e^{iS[\phi]}$, with $S[\phi]$ the classical action of the theory, and ϕ representing all fields involved. Thus, the transition amplitude formally satisfies

$$\langle\Psi_f|e^{-i\hat{H}t}|\Psi_i\rangle = \int D[\phi]e^{iS[\phi]}, \tag{3.3}$$

the similarity between the r.h.s. of (3.1) and the l.h.s. of (3.3) suggests a different way to compute the partition function. Let us consider time as a pure imaginary number $t = -i\tau$ with τ real. Then, take both, the initial and the final states in the transition amplitude, as the same eigenstate $\Psi_f = \Psi_i = \Psi_n$. Finally, by summing over all possible eigenstates of the system we obtain the partition function exactly as defined by (3.1)

$$Z_e(\beta) = \int D\,[\phi]\,e^{-S_E[\phi]}, \tag{3.4}$$

where now $\beta = \tau$. Thus, the "time" τ, which by virtue of the state identification is periodic, is interpreted as the inverse of the temperature of the system (for a pedagogical example see [31]). The complexification of time is usually called Wick rotation as it can be represented by a $-\pi/2$ rotation on a time complex plane. Because of the periodicity in τ, the classical field ϕ in the path integral should satisfy time periodic boundary conditions. For Lorentz invariant field theories the Wick rotation changes the signature of spacetime from Lorentzian $(-,+,+,+)$ to Euclidean $(+,+,+,+)$, thus, the previous expression is called the Euclidean path integral. In the same vein

$$S_E[\phi] \equiv -\left. iS[\phi]\right|_{t\to -i\tau}, \tag{3.5}$$

is called the Euclidean action.

A path integral formula for the partition function is interesting because it expresses the statistical mechanics fundamental tool in terms of the classical fields directly. This provides a framework suitable for semiclassical approximations as we will see in a moment. It is also a completely different and covariant method to compute the partition function for quantum systems.

If the system is not in a strong quantum regime the path integral can be estimated around classical solutions by the method known as the *steepest descend approximation*, also called the saddle point approximation. In our case, it consists basically in the estimation of the path integral by evaluating the integrand at the saddle point of $S_E[\phi]$. The leading order is

$$Z_e(\beta) \sim e^{-S_E[\phi_0]+\cdots}, \tag{3.6}$$

where $S_E[\phi_0]$ is the Euclidean action evaluated at the saddle point ϕ_0, i.e., a stationary point of the action or equivalently a particular solution of the equations of motion that satisfies the boundary conditions. The dots in (3.6) contain corrections due to quantum fluctuations that will be ignored at this level. We use $\sim$ to relate quantities that have a similar asymptotic behaviour.

Now, let us use the partition function in its more standard form (3.1) to compute physical macroscopic quantities. We are assuming that the system under consideration satisfies the Boltzmann probability distribution, thus, we have that

$$p_n = \frac{e^{-\beta E_n}}{Z}, \tag{3.7}$$

is the probability for the system to be in a quantum state with an energy E_n. The von Neumann entropy is defined by

$$S = -\sum_n p_n \log p_n = \beta \bar{E} + \log Z, \tag{3.8}$$

and the average energy of the system is given by

$$\bar{E} = -\partial_\beta \log Z(\beta) = \sum_n E_n p_n. \tag{3.9}$$

The equation for the entropy can also be formally obtained by identifying the statistical and the Euclidean partition functions

$$Z_s \sim Z_e \tag{3.10}$$

$$\int_0^\infty dE e^{-\beta E} e^{S(E)} \sim e^{-S_E[\phi_0]+\cdots} \tag{3.11}$$

$$S = \beta \bar{E} + \log Z_{cl} + \cdots, \tag{3.12}$$

where in the last line we have used the saddle point approximation, such that, the energy that defines this point is $E = \bar{E}$, then, we took the logarithm and used the notation $Z_{cl} = e^{-S_E[\phi_0]}$ for what would be the "classical contribution" to the partition function.

Path integrals are not rigorously defined for general physical theories thus, there is a possible mismatch between the two different expressions for the partition function in (3.10). This is frequently manifest by the appearance of divergences on the path integral side, so, the Euclidean path integral partition function should be treated carefully for each specific example. In particular, it should also be kept in mind that by using the path integral we have not control of the fundamental quantum degrees of freedom of the system. Of course, these are problems concerning the semiclassical approximation methods and must be cured by an ultimate underlying quantum theory.

In [29] the use of this semiclassical method in General Relativity is proposed as a glimpse of a quantum treatment for gravity. The starting point is to compute

$$Z_e(\beta) = \int D\,[\phi]\, e^{-S_E[\phi]}, \tag{3.13}$$

where an Euclidean action for General Relativity should be defined. We use the Einstein-Hilbert action plus the Gibbons-Hawking-York boundary term which makes the variational principle well-defined [29, 32]. Written in terms of the metric tensor $\phi = g_{\mu\nu}$ the Euclidean action is

$$S_E\,[g] = -\frac{1}{16\pi}\int_{\mathcal{M}} R\; dV + \frac{1}{8\pi}\int_{\partial\mathcal{M}} (K - K_0)\; d\Sigma, \tag{3.14}$$

where R is the scalar curvature, dV the four-volume element on $\mathcal{M}$, K the trace of the extrinsic curvature of the three-surface $\partial\mathcal{M}$, which is the boundary of $\mathcal{M}$ and has $d\Sigma$ as three-volume element. K_0 is an arbitrary constant that does not depend on the spacetime metric and, hence, its variation is automatically zero. This constant is fixed by using a physical argument and will play an important role in what follows.

Before going further, a word of caution is in order. The previous expression (3.13) does not have a precise mathematical definition. Gravity is not a renormalizable theory in the standard quantum field theoretical approach and the perturbative expansion of its path integral is not well-defined. Furthermore, when trying to define the Euclidean continuation for the Einstein-Hilbert action, $S_E\,[g]$ in (3.14), there is not a general prescription to analytically continue general spacetimes with Lorentzian signature to spacetimes with Riemannian signature [2] (for example in the Kerr metric, in the usual coordinates, the transformation $t \to -i\tau$ makes the metric and the action complex). In addition to this, the partition function defined this way is not bounded, as shown in [33] for a certain family of conformal Euclidean metrics it diverges. Nevertheless, having said that, for specific examples in gravity the method explained before gives interesting results, see for instance its use in the simpler case of 2+1 black holes in [34]. In four dimensions there are metric solutions, notably static metrics, where the Wick rotation can be performed consistently, allowing for the use of the method.

Summarizing, in this section we have introduced the Euclidean partition function and described a method to estimate it. The goal of this chapter is to explore the possibility of giving a *quasilocal* meaning for the Euclidean partition function in the context of black holes. Before that, in order to contextualize the result, we briefly review the Euclidean partition function approach with the spherically symmetric black hole.

3.2 Partition Function for Schwarzschild Solution

Here we will partially follow the classic work of Gibbons and Hawking [29]. Let us start by writing the Schwarzschild solution

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2. \tag{3.15}$$

The ADM formalism provides a notion of global energy for gravitating systems, for the Schwarzschild solution—a spherically symmetric black hole at rest with respect to the asymptotic structure—it is simply given by the mass, $E = M$. Thus, the statistical partition function (3.2) is

$$Z_s(\beta) = \int_0^\infty dM e^{-\beta M} e^{S(M)}. \tag{3.16}$$

In the last expression the quantum corrections are disregarded. This partition function can be seen as a mere toy model, however, it will be useful for us when we compare it with the Euclidean version.

On the other side, we can use the Euclidean approach. In the Euclidean section[1] the real "time" parameter $\tau = it$ should be β-periodic to avoid a conical singularity,[2] and the period can be identified with the temperature

$$\beta = \frac{1}{T_H} = 8\pi M, \tag{3.19}$$

which is the temperature associated with the Hawking radiation for the spherically symmetric black hole. The scalar curvature vanishes for Schwarzschild, $R = 0$. Thus, the Euclidean action (3.14) is just the boundary term used to define correctly the action principle [29]

$$\log Z_{cl}(\beta) = -S_E(\Psi_0) = -\frac{1}{8\pi} \int_{\partial \mathcal{M}} (K - K_0) d\Sigma, \tag{3.20}$$

[1] If the coordinates are thought of complex variables the *Euclidean section* is the sector where time is purely imaginary.

[2] The Euclidean Schwarzschild metric reads

$$ds_e^2 = \left(1 - \frac{2M}{r}\right) d\tau^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2, \tag{3.17}$$

there is possibly a singularity at $r = 2M$. By using the coordinate $\rho = 4M\sqrt{1 - 2M/r}$, the metric is

$$ds_e^2 = \rho^2 d\left(\tau/(4M)\right)^2 + \left(r/(2M)\right)^4 dr^2 + r^2 d\Omega^2, \tag{3.18}$$

and we see that at the origin, $\rho = 0$, there is a conical singularity unless we restrict the variable τ to be β-periodic with $\beta = 8\pi M$.

where $\partial\mathcal{M}$ has the topology $\mathcal{S}^1 \times \mathcal{S}^2$ due to the periodicity in τ. K_0 is a constant under variations of the action. The variational principle is defined in such a way that the metric is keep fixed at the boundary surface, thus, K_0 can almost depends on the intrinsic metric at that boundary surface and it is still constant. To study the asymptotic behaviour of the boundary we can take a sphere of coordinate radius r and study the far away regime $r \gg 2M$. The boundary integral of K blows up with r, but, we can use the K_0 term to regularize it. A natural way is to choose K_0 as the trace of extrinsic curvature of a sphere with radius r but computed as embedded in a flat spacetime,[3] see Fig. 3.1.

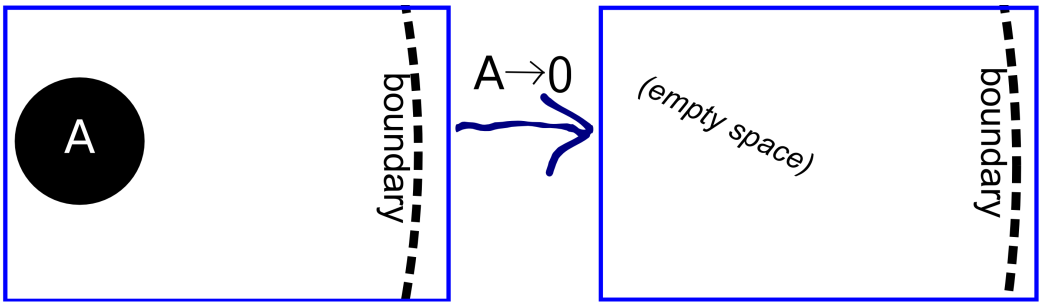

Fig. 3.1: The extrinsic curvatures are computed on a large sphere which is the boundary of a spacetime: With a black hole for K (left), and in an empty spacetime for K_0 (right). As sketched, the empty spacetime can be thought as the area zero limit of the black hole spacetime.

The result is

$$\log Z_{cl}(\beta) = \frac{\beta}{2}\left[(2r - 3M) - \sqrt{1 - \frac{2M}{r}}\, 2r\right], \tag{3.21}$$

the second term naturally diverges in the same way for $r \gg 2M$ and the whole quantity is regular for $r \to \infty$. This choice for K_0 has the additional property that it produces $S_E = 0$ for a flat spacetime. The boundary, characterized by the asymptotically flat conditions, is reached when the radius of the sphere goes to infinity. Expanding in the regime $r \gg 2M$ we get a finite contribution

$$\log Z_{cl}(\beta) \approx -\beta\frac{M}{2}. \tag{3.22}$$

Now, we can try to estimate the entropy of the system by identifying both definitions of partition functions $Z_s \sim Z_e$. The direct computation cannot

[3] This definition is ad-hoc for the Schwarzschild spacetime, in a general case there could be boundaries that cannot be simply embedded in flat spacetime.

be performed directly,[4] but we can avoid this problem by simply computing the entropy directly from the formal expression (3.8), where Z is replaced by the estimation of the Euclidean partition function, and $\beta = 8\pi M$. If we use the notion of energy, $\bar{E} = M$, provided by the ADM framework as an *extra ingredient*, the result is the standard Bekenstein-Hawking entropy

$$S = 4\pi M^2 = \frac{A}{4}, \tag{3.24}$$

where we have have used the fact that the area of the horizon is $A = 16\pi M^2$.

Now, if we look closely at (3.22), it is clear that

$$\bar{E} \equiv -\partial_\beta \log Z_{cl} \neq M, \tag{3.25}$$

which is in fact inconsistent with the use of $\bar{E} = M$. To obtain consistency we should study the topological issues concerning the Euclidean black hole.[5]

The main remark is that for the Euclidean metric the appearance of a *conical singularity* implies a *curvature singularity* that has been overlooked up to here. We pay attention to it in the following.

The partition function defined as the Euclidean path integral, with the boundary conditions discussed above is a function of β, and this curvature contribution modifies the dependence. For an Euclidean manifold with a conical singularity and a deficit angle of $2\pi(1-\alpha)$ the curvature acquires a non trivial term [35, 36]. Near the region with the conical singularity the curvature scalar is

$$R = \bar{R} + 4\pi(1-\alpha)\delta_{\Sigma_2} \tag{3.26}$$

where Σ_2 is the two-horizon surface, δ_{Σ_2} is the Dirac delta on the four-manifold such that: $\int_{\mathcal{M}} f\delta_{\Sigma_2} = \int_{\Sigma_2} f$, and $\bar{R}$ is the scalar curvature computed in the smooth manifold. Note that in the four manifold the conical singularity is a two-dimensional extended object, in consequence it can also be understood as a two-dimensional internal boundary term, also called *corner term* [37]. It can be shown, by studying the *canonical action* [38], that for black hole boundary conditions the equations of motion imply that this extra

[4] We might use the inverse Laplace transform to compute the entropy, formally

$$e^{S(M)} = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} d\beta\; e^{\beta M - \beta\frac{M}{2} + \cdots}, \tag{3.23}$$

this integral does not converge, to make it convergent we can *rotate the contour* to the real axis as done in [30], however, this procedure does not seem very satisfactory.

[5] Note that in the Hawking approach [30] *before* the relation $\beta = 8\pi M$ is used before computing the energy, such that $\log Z_{cl} = -\frac{\beta^2}{16\pi}$, with this $\bar{E} \equiv -\partial_\beta \log Z_{cl} = M$ is trivially obtained, but, this procedure seems arbitrary to us.

term vanishes for any exact solution (see (19) in [38]). Thus, it does not contribute to the leading saddle point approximation. However, the Euclidean partition function as a quantum object has contributions from metrics that are not solutions of the Einstein equation. Some of those can, in principle, have conical singularities, then, there is a *hidden* dependence on β in the Euclidean partition function than can be relevant to compute derivatives as in (3.25). We should make this dependence manifest in the Euclidean action. For the Euclidean black hole we introduced the imaginary time $t = -i\tau$, and the periodicity should be imposed on the real coordinate $\tau \to \tau + \beta$. If we leave β free for a moment, the deficit angle, which can be easily read from (3.18), is

$$2\pi(1-\alpha) = 2\pi - \kappa\beta, \tag{3.27}$$

where $\kappa = 1/(4M)$ is the surface gravity for Schwarzschild already introduced in the previous chapter. Thus, the Euclidean action evaluated on a metric which can have a conical singularity is the boundary term plus the deficit angle

$$\begin{aligned} S_E[\Psi_0] &= 4\pi \int (1-\alpha)\delta_{\Sigma_2} dV + \frac{1}{8\pi}\int_{\partial\mathcal{M}} (K - K_0) d\Sigma. \\ &= -\frac{1}{4}(1-\alpha)A - \frac{\beta}{2}\left[(2r-3M) - \sqrt{1-\frac{2M}{r}}\,2r\right] \\ &\approx -(2\pi - \kappa\beta)\frac{A}{8\pi} + \beta\frac{M}{2}, \end{aligned} \tag{3.28}$$

where in the second line we have used that $\int \delta_{\Sigma_2} dV = A$ is the area of the horizon, and in the third line we have used that the boundary is at $r \gg 2M$. Now, with this refinement the Euclidean partition function allows us to compute the energy of the system. For Schwarzschild, we have

$$\begin{aligned} \log Z_{cl} &= -\beta\frac{M}{2} + (2\pi - \kappa\beta)\frac{A}{8\pi} \\ \bar{E} &= -\partial_\beta \log Z_{cl} = \frac{M}{2} + \frac{\kappa A}{8\pi} = M, \end{aligned} \tag{3.29}$$

where the exact value of $\beta = 8\pi M$ has not been used in the derivation. By doing this the partition function is simply self-consistent with the ADM energy: We do not need it as an extra input.

In summary, by fixing correctly the arbitrariness K_0 in the boundary term and by considering the contribution of the conical singularity, the Euclidean approach—in the leading saddle point approximation—provides us with a temperature $T_H = \frac{1}{8\pi M}$, an entropy $S = \frac{A}{4}$ and a notion of energy $\bar{E} = M$

which are consistent with the known results of Hawking temperature, its use on the standard first law which gives the entropy, and the ADM energy, respectively. The analysis of the conical singularity is not included in the original work [29].

3.3 Partition Function in the Quasilocal Approach

To construct a quasilocal description of the black hole mechanics in terms of the Euclidean partition function, we basically need two ingredients. First we need to define a three-dimensional timelike surface $\mathcal{W}(\mathcal{O})$ at a proper distance $\ell^2 \ll A$ to the horizon. The proper distance ℓ is measured by hypothetical observers laying on $\mathcal{W}(\mathcal{O})$ and stationary with respect to the horizon (see Appendix A for a description in terms of coordinates). The surface $\mathcal{W}(\mathcal{O})$ will be thought of the boundary of the spacetime. This allows us, as discussed in the previous chapter, to define a physically motivated notion of energy in terms of local measurements carried out by the observers on this surface. However, in this context the quasilocal energy found in (2.61) is not part of the assumptions. Instead, in the same way that the ADM energy emerges in the procedure of the previous section, the quasilocal energy emerges naturally from the following quasilocal approach.

For the Schwarzschild solution, the quasilocal Euclidean partition function in the saddle point approximation is

$$\log Z_e \approx \log Z_{cl} = -S_E = -\frac{1}{8\pi}\int_{\mathcal{W}(\mathcal{O})} (K - K_0)\, d\Sigma + \frac{1}{8\pi}(2\pi - \frac{\alpha}{2\pi})A, \quad (3.30)$$

where, up this point, the only difference with the previous treatment is the introduction of $\mathcal{W}(\mathcal{O})$ as the spacetime boundary.

The second ingredient is the fixation of the arbitrariness of the boundary term, K_0. In the standard approach the required boundary condition is asymptotic flatness of spacetime. In that case it is natural to fix the arbitrary quantity, K_0, in order to make $S_E = 0$ for flat spacetime, i.e., when there is not black hole. But, this choice is not useful in the quasilocal approach. In the quasilocal case we need to have the horizon to make sense of $\mathcal{W}(\mathcal{O})$, the stationary observers, and their quasilocal energy measurements. However, in the limit where there is no black hole, $A \to 0$, the surface over which we define the quasilocal energy does not exist. To address this issue we consider the opposite limit, $A \to \infty$, which can be thought of a huge black hole and where, therefore, we still have a horizon. In this limit, if we take any small patch of the surface $\mathcal{W}(\mathcal{O})$ where the near horizon observers lay, as $A \to \infty$ the surrounding spacetime of those observers becomes exactly

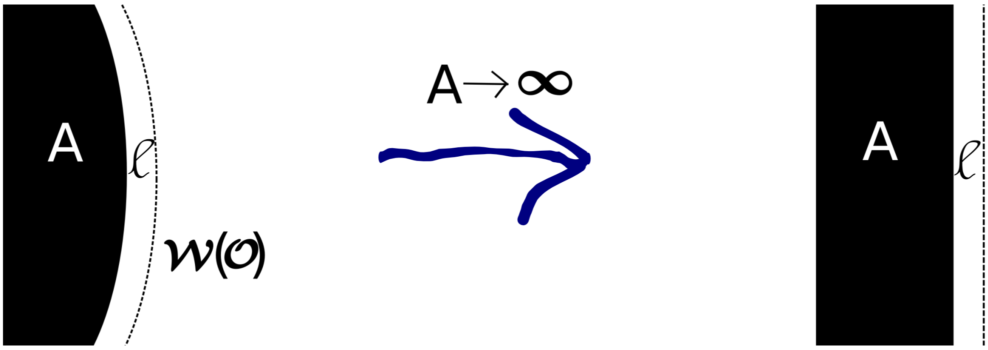

Fig. 3.2: For a large black hole any small patch of the horizon can be seen as a Rindler spacetime.

Rindler, see Fig. 3.2. This means that all the possible corrections due to the curvature of the spacetime near the horizon, disappear in this limit. In Rindler spacetime we can still define a surface $\mathcal{W}(\mathcal{O})$ a distance ℓ from the horizon. The extrinsic curvature on this surface is exactly $K_0 = 1/\ell$. As the Rindler spacetime is flat, this choice has a similar interpretation than that of the asymptotic boundary in the empty spacetime.

To avoid the conical singularity we need again to ask for a "time" periodicity. In the quasilocal frame the natural time variable is the proper time of the observers on $\mathcal{W}(\mathcal{O})$. In the near horizon region the metric reads

$$ds_e^2 = \ell^2 d(\kappa\tau)^2 + [r/(2M)]^4 d\rho^2 + r^2 d\Omega^2, \tag{3.31}$$

thus, the proper time $\tilde{\tau} = \ell\kappa\tau$ should have the following periodicity $\tilde{\tau} \to \tilde{\tau} + 2\pi\ell$, then, the period is simply

$$\tilde{\beta} = 2\pi\ell, \tag{3.32}$$

which is naturally the inverse of the Unruh temperature that can be obtained using quantum field theoretic consideration about accelerated observers in Rindler spacetime. Taking all this into account, the boundary term can be straightforwardly computed

$$-\frac{1}{8\pi}\int_{\mathcal{W}(\mathcal{O})}(K-K_0)d\Sigma = \tilde{\beta}\frac{5\ell}{16M} + o(\ell^4), \tag{3.33}$$

where the extrinsic curvature K has been computed explicitly in Appendix B (for Schwarzschild see directly (B.13)). Hence, we can express our partition function as a function of $\tilde{\beta}$

$$\log Z_{cl}(\tilde{\beta}) = \tilde{\beta}\frac{5\ell}{16} + (2\pi\ell - \tilde{\beta})\frac{A}{8\pi\ell}, \tag{3.34}$$

in particular we can compute the energy

$$\bar{E} = -\partial_{\tilde{\beta}} \log Z_{cl} = \frac{A}{8\pi\ell} + o(\ell) = E_{loc}, \tag{3.35}$$

which is consistent with the expected quasilocal energy defined in the previous chapter (2.43). Note that in comparison with the asymptotically far away view, here, the energy comes entirely from the conical defect. The entropy can be computed again using the general expression (3.8), but now we replace β by $\tilde{\beta}$ and $\bar{E}$ by E_{loc}, from (3.32) and (3.35), respectively,

$$S = 2\pi\ell\frac{A}{8\pi\ell} + \frac{\pi\ell^2}{4} = \frac{A}{4} + o(\ell^2). \tag{3.36}$$

Therefore, the result is that the quasilocal energy of the observers plus the local temperature—obtained as the period used to regularize the Euclidean metric—combined in the general entropy formula produce, again, the so called Bekenstein-Hawking entropy.

In summary, the Euclidean partition function within the quasilocal approach, at first order in ℓ, naturally reproduces the local (Unruh) temperature $\widetilde{T} = \frac{1}{2\pi\ell}$, the Bekenstein-Hawking entropy $S = \frac{A}{4}$, and the quasilocal energy notion $E_{loc} \approx \frac{A}{8\pi\ell}$.

As a last remark it should be emphasized that, as in the previous chapter, the use of the quasilocal approach in the Schwarzschild example is easy to generalize, as far as the quasilocal structure around the black hole horizons is similar. For instance, in the Kerr solution, from the far away perspective the Wick rotation turns the metric complex due to the crossed terms with the from $dtdx$. But, as shown in Appendix C, for the quasilocal approach those crossed imaginary terms are $o(\ell)$ and can be neglected in the first approximation.

4. BLACK HOLE QUANTIZATION

The program of black hole quantization, that we develop here, is the product of the combination of three main subjects: Isolated Horizons, Loop Quantum Gravity, and Chern-Simons theory. The core of our strategy to get a quantum description of black holes is to impose the Isolated Horizons as a condition already at the classical level, i.e., we reduce the General Relativity phase space by imposing an internal boundary such that it has the properties defining an Isolated Horizon. In this sense, we have a theory of gravity with a generic spacetime that contains one black hole. Afterwards, quantization is carried on by using the techniques of Loop Quantum Gravity. The Chern-Simons theory appears at the quantum level as a natural way to describe the degrees of freedom of the Loop Quantum Gravity over the black hole horizon.

This general strategy has been developed as an application of Loop Quantum Gravity to study black hole quantum physics originally in [39] and restudied recently in [15, 40] by considering a less restricting gauge condition.

Our approach puts special emphasis on the symplectic structure defining the phase space. In physics, the symplectic formulation of a theory state that over the phase space Γ there is a symplectic structure $\Omega(\delta_1, \delta_2)$ defined as a 2-form on the cotangent space of Γ. In fact, the symplectic structure is a reformulation of the Poisson bracket and therefore, it is a fundamental ingredient to have a well-defined Hamiltonian evolution in any theory. In standard mechanics, this structure has the important property that does not depend on time. On diffeomorphism covariant field theories this property is translated to an independence of the Cauchy surfaces, if this is true, the symplectic structure is said to be *preserved*.

Another interesting property of the symplectic formulation is that canonical transformation of variables are simply equivalent to diffeomorphism on the Γ space—where the fields are the *coordinates* on the phase space Γ—such that $\Omega(\delta_1, \delta_2)$ is preserved. A fundamental result in this respect is the Darboux's theorem: It states that *on each small patch* of a given phase the symplectic structures is essentially unique as all possible symplectic structures on Γ are locally related by a diffeomorphism transformation. Since we deal with different sets of variables, it is exactly the former property what makes the approach useful for us. The progress achieved in the quantum gravity proposal known as Loop Quantum Gravity is deeply based in the use of an alternative set of variables called Ashtekar-Barbero connection variables. In the standard Hamiltonian formalism they can be obtained, precisely, by using a canonical transformation of the standard Palatini conjugate variables.

On the other hand, as we will see, to show that $\Omega(\delta_1, \delta_2)$ is preserved in covariant theories, we need to study the behaviour of fields at the boundaries of the spacetime [41]. In fact, the preservation of the syplectic structure can be rephrased by saying that $\Omega(\delta_1, \delta_2)$ is preserved if and only if boundary conditions guarantee that the symplectic flux does not escape through spacetime boundaries, i.e., it vanishes on spacetime boundaries.

Our characterization of black holes involves the particular type of spacetime boundaries given by the Isolated Horizon conditions. Hence, we should pay special attention when performing the diffeomorphism transformation of $\Omega(\delta_1, \delta_2)$ to obtain the Ashtekar-Barbero connection variables such that Isolated Horizon conditions still allow for a preserved symplectic structure.

The quantization procedure does not differ much from the Loop Quantum Gravity one. Actually, what we aim at is to adapt the classical formulation such that Loop Quantum Gravity procedure can be applied and the quantum degrees of freedom of the horizon singled out. So, basically the strategy consists in incorporating the Isolated Horizon on the symplectic structure formulation of the theory. Then, we chose a particular time foliation of the spacetime such that we can impose the corresponding time gauge, and, within that framework, the Ashtekar-Barbero variables can be introduced. Afterwards, the partial quantization of Loop Quantum Gravity can be done in such a way that it is possible to directly use the coarse grained spacetime degrees of freedom that the theory provides.

However, because of the existence of an internal boundary, there are differences with respect to the usual Loop Quantum Gravity quantization recipe. Already at the classical level the symplectic structure acquires a boundary term. Interestingly, this boundary term can be thought as a symplectic structure of a purely connection theory on a three-dimensional manifold, which in fact has not classical degrees of freedom. This is how the Chern-Simons theory comes into the game.

As a general remark, it should be noted that the procedure developed in this chapter does not deal with the quantization of black holes in General Relativity from first principles, in the sense that we borrow results from a proposal to quantize General Relativity—Loop Quantum Gravity—and we make an effort to adapt them in a classical situation where a black hole is present. Our approach does not obtain black holes as a solution of a full quantum gravity theory. Nevertheless, the hope is that the Loop Quantum Gravity theory reduces to General Relativity as a classical limit and therefore, it should recover black holes as solutions. As this is still an open problem here we make a shortcut and simply impose the black hole solution from the very beginning in the general sense of Isolated Horizons. By doing it, we do not fix the geometry in the whole spacetime but just some specific geometric structure that defines the black hole horizon.

To address the quantization problem further simplifications are used. Instead of dealing with a general Isolated Horizon we will assume some symmetries. The simplest Isolated Horizon is the *spherically symmetric* one. In this family the horizon of the Schwarzschild solution is the main representative. Thus, we start by presenting the complete program sketched above for the spherically symmetric case, Section 4.1. Then, we deal with the next natural generalization: The *axially symmetric* Isolated Horizon. In this family the main representative is the horizon of the Kerr solution. This is the subject of Section 4.2 and the main new contribution contained in the present chapter.

As stated above, a big portion of the program is concerned with the definition of the symplectic structure. In spite of the slight generalization—from spherically symmetric to axially symmetric—we will see that many deep technical differences emerge such that they almost rule out the applicability of the program. However, one of our results is a method to construct a new set of connection variables, such that, with them it is possible to mimic the approach of the spherically symmetric description. At the same time, we show that the symplectic structure for the axially symmetric Isolated Horizon built out of these variables is preserved. This is a cornerstone proof needed for the rest of the program which we present at the end of Section 4.2.

4.1 Spherically Symmetric Isolated Horizon

In this section, we briefly review the results about the quantization procedures developed in [15, 40], for the spherically symmetric Isolated Horizons (also called Type I Isolated Horizons in the classification proposed in [15]) in the context of Loop Quantum Gravity. We present in some detail the quantization program of the spherically symmetric case as a natural introduction of the perspective and the notation we will use in the next section. For a reader unfamiliar with symplectic geometry formalism we recommend reading Appendix D first.

4.1.1 Symplectic Framework with Ashtekar-Barbero Variables

The main goal of this section is to prove that the introduction of Ashtekar-Barbero connection variables when considering Isolated Horizon as an internal boundary of our spacetime, and, with the further requirement of spherical symmetry of the horizon, gives rise to the following symplectic structure in the phase space

$$\kappa\gamma\Omega(\delta_1,\delta_2) = 2\int_M \delta_{[1}\Sigma^i \wedge \delta_{2]}A_i - \frac{a_H}{\pi(1-\bar{\gamma}^2)}\int_H \delta_1\bar{A}_i \wedge \delta_2\bar{A}^i. \tag{4.1}$$

The interesting feature of $\Omega(\delta_1,\delta_2)$ is that—in addition to the usual bulk term defined on the Cauchy surface M—there is a boundary term at the horizon $H = \Delta \cap M$, where Δ is the null surface in the definition of Isolated Horizon. Notice that the horizon term—integral on H—correspond to a Chern-Simons symplectic structure.

To obtain (4.1), let us start by writing the symplectic structure of General Relativity in terms of the Palatini variables, for a detailed derivation see Appendix D,

$$\underleftarrow{J}(\delta_1,\delta_2) = \frac{1}{\kappa}\delta_{[1}\underleftarrow{\Sigma_i} \wedge \delta_{2]}\underleftarrow{K}^i, \tag{4.2}$$

where $\Sigma_i = \varepsilon^i{}_{jk}e^j \wedge e^k$ is called the *desitized triad*—or *gravitational electric field* [43]—while K^i is the extrinsic curvature[1]. The left-pointing arrow denotes pullback to the Cauchy surface M. The partial time gauge fixing condition $\underleftarrow{e}^0 = 0$ is imposed on the whole spacetime and in particular can be used on the Cauchy slide M. These are the three-dimensional hypersurfaces where the symplectic density is integrated in order to define a proper symplectic structure.[2]

Now consider the Isolated Horizon. The region H is assumed to have a fixed spherically symmetric geometry. The only allowed variations of the fields are non-physical, i.e., the diffeomorphisms on H—those which preserve the Isolated Horizon conditions—and the internal gauge symmetry. By using it, it is possible to show [15] that the symplectic structure constructed out of $J(\delta_1,\delta_2)$ is preserved in the presence of a spherically symmetric Isolated Horizon.

[1] Without using forms, a more common way to express these variables is [42]

$$E^i{}_a = \frac{1}{2}\varepsilon^{abc}\varepsilon_{ijk}e^j{}_b e^k{}_c, \qquad K^i{}_a = \frac{1}{\sqrt{E}}K_{ab}E^{ib}, \tag{4.3}$$

with $E = det(E^i{}_a)$, $e^i{}_a$ the triad obtained by a ADM decomposition and $K_{ab} = \frac{1}{2}\mathscr{L}_n q_{ab}$ the extrinsic curvature, n^a is the unit normal and $q_{ab} = g_{ab} + n_a n_b$ the induced metric of any layer of the time foliation.

[2] Technically, as far as we are not factoring out the degeneracy subspace of Γ produced by precisely the symmetries in the allowed variations, we are dealing with a presymplectic structure, but certainly the focus is on this one as we will study the interplay between gauge symmetries and boundary conditions, then, for symplicity let us just call it symplectic structure [41].

Let us make a technical observation about the previous statement. As we are already considering Isolated Horizons, the preservation of the symplectic structure is a non-trivial assertion. In the proof, total derivatives appear, so, we should get rid of boundary terms. In fact, precisely because we are in the *spherically symmetric* Isolated Horizon case, the following expression vanishes on the horizon

$$v \lrcorner K^i \wedge \Sigma_i = 0, \tag{4.4}$$

and, as it will be clear below, even if it appears fleetingly in the proof of (4.1) it is essential for it. We stress it here because it is precisely this expression which will become a problem when we try the same recipe for the axially symmetric case: In that case it does not longer vanish.

Thus, for the spherically symmetric Isolated Horizon, using the equations of motion, and in particular (4.4), it can be proved (see Equation (73) in [15]) that

$$\Omega(\delta_1, \delta_2) = \frac{1}{\kappa} \int_M \delta_{[1} \Sigma^i \wedge \delta_{2]} K_i, \tag{4.5}$$

is independent of the chosen Cauchy surface, M.

Let us go back to the main goal: To introduce the Ashtekar-Barbero variables. The symplectic structure is defined on the space of solutions of the theory, thus, when manipulating $\Omega(\delta_1, \delta_2)$ the equations of motion can be explicitly used. In order to introduce the Ashtekar-Barbero variables it is necessary to make explicit use of the First Cartan equation,

$$de^I + \omega^I{}_J \wedge e^J = 0, \tag{4.6}$$

let us write its pullback to the Cauchy surface, M, and use time gauge $\underleftarrow{e}^0 = 0$

$$\begin{aligned} \underleftarrow{\omega}^0{}_j \wedge \underleftarrow{e}^j &= 0 \\ \underleftarrow{de}^i + \underleftarrow{\omega}^i{}_j \wedge \underleftarrow{e}^j &= 0, \end{aligned} \tag{4.7}$$

the second equation can be rewritten using the $SO(3)$ connection defined as $\Gamma^i = -\frac{1}{2}\varepsilon^i{}_{jk}\omega^{jk}$

$$\underleftarrow{de}^i + \varepsilon^i{}_{jk} \underleftarrow{\Gamma}^j \wedge \underleftarrow{e}^k = 0, \tag{4.8}$$

from this it is easy to prove

$$\underleftarrow{\Sigma}_i \wedge \delta \underleftarrow{\Gamma}^i = -d(\underleftarrow{e}^i \wedge \delta \underleftarrow{e}_i). \tag{4.9}$$

Given a connection we can simply define another connection by adding a vector as this does not change its transformation properties. With the time gauge condition the full SO(3,1) connection ω^{IJ} naturally decomposes in two pieces: The $SO(3)$ connection Γ^i and the vector—under the $SO(3)$ action—given by $K^i = \omega^{0i}$. In this way, we introduce the Ashtekar-Barbero connection variable

$$A^i \equiv \Gamma^i + \gamma K^i = -\frac{1}{2}\varepsilon^i{}_{jk}\omega^{jk} + \gamma\omega^{0i}, \tag{4.10}$$

where γ is an arbitrary constant known as the Immirzi parameter.

Equation (4.9) allows us to write the symplectic structure density in terms of the Ashtekar-Barbero variables plus an exact 3-form

$$\begin{aligned} \gamma\kappa \underleftarrow{J}(\delta_1,\delta_2) &= \delta_{[1}\underleftarrow{\Sigma_i} \wedge \delta_{2]}(\gamma\underleftarrow{K^i} + \underleftarrow{\Gamma^i}) + d(\delta_1\underleftarrow{e^i} \wedge \delta_2\underleftarrow{e_i}). \\ &= \delta_{[1}\underleftarrow{\Sigma_i} \wedge \delta_{2]}\underleftarrow{A^i} + d(\delta_1\underleftarrow{e^i} \wedge \delta_2\underleftarrow{e_i}). \end{aligned} \tag{4.11}$$

To obtain the symplectic structure we integrate the density over the Cauchy surface M, namely

$$\gamma\kappa\Omega(\delta_1,\delta_2) = \int_M \delta_{[1}\Sigma_i \wedge \delta_{2]}A^i + \int_{\partial M}\delta_1 e^i \wedge \delta_2 e_i, \tag{4.12}$$

notice that a boundary term appears explicitly. If the border of our spacetime is taken to be asymptotically flat, this boundary term is automatically zero and we can simply keep the bulk term being certain that there is not symplectic flow escaping to infinity [44]. This result is what makes the Ashtekar-Barbero variables suitable to describe the gravitational degrees of freedom consistently at the classical level. When further boundaries and/or different boundary conditions are imposed we should be particularly careful with this term. Here, besides the asymptotically flat infinity, we also impose Isolated Horizon conditions as an internal boundary of our spacetime and this term is no longer zero, so, we should keep it and study it.

To prove the main result of this section (4.1) we need to study in detail the allowed phase space variations of the boundary Isolated Horizon term, i.e., take $\partial M = H$ in (4.12). By doing this, we will be able to prove

$$\int_H \delta_{[1}e_i \wedge \delta_{2]}e^i = -\frac{a_H}{2\pi(1-\bar{\gamma}^2)}\int_H \delta_{[1}\bar{A}_i \wedge \delta_{2]}\bar{A}^i, \tag{4.13}$$

from which (4.1) follows immediately. This is a key equation of the approach as the r.h.s. can be interpreted as a Chern-Simons symplectic structure on the boundary for the connection $\bar{A}^i$.

We must start by establishing some preliminary equations satisfied by the variables at the Isolated Horizon. The first crucial equation is

$$\underset{\Leftarrow}{F^i}(A) = -\frac{\pi(1-\gamma^2)}{a_H}\underset{\Leftarrow}{\Sigma^i}, \tag{4.14}$$

where $F^i(A) = dA^i + \frac{1}{2}\varepsilon^i{}_{jk}A^j \wedge A^k$ is the curvature of the Ashtekar-Barbero connection, a_H is the horizon area, and the double left-pointing arrows $\underset{\Leftarrow}{\Sigma^i}$ denotes the pullback onto the horizon H. There is a general way to prove it by using the Isolated Horizon condition applied to the Second Cartan equation, it can be found in [15]. For completeness, we give a simpler proof in the concrete example of Schwarzschild horizon geometry in Appendix E. In the proof of (4.14), the following equation is also needed

$$\epsilon^i{}_{jk}\underset{\Leftarrow}{K^j} \wedge \underset{\Leftarrow}{K^k} = \frac{2\pi}{a_H}\underset{\Leftarrow}{\Sigma^i}, \tag{4.15}$$

we also refer the reader to [15] for a general version in the context of spherically symmetric Isolated Horizons, or to Appendix E, where it is checked in the simplified case of Schwarzschild horizon geometry. Combining (4.14), with (4.15) we obtain that for a connection $\bar{A}^i = \Gamma^i + \bar{\gamma}K^i$—which is reduced to the Ashtekar-Barbero connection just if we choose $\bar{\gamma} = \gamma$—the curvature at the horizon is

$$\underset{\Leftarrow}{F^i}(\bar{A}) = -\frac{\pi(1-\bar{\gamma}^2)}{a_H}\underset{\Leftarrow}{\Sigma^i}, \tag{4.16}$$

thus, at the horizon the connection is not unique, there is a $\bar{\gamma}$-dependent family of connections $\bar{A}^i$ which satisfy the same curvature equation. The parameter $\bar{\gamma}$ is a second ambiguity of the description that appears at the classical level when we introduce connection variables at the horizon.

With the previous elements established we devote the rest of this subsection to a detailed proof of (4.13).

As we noticed before, the region H is assumed to have a fixed spherically symmetric geometry. The only allowed variations of the fields are non-physical: The diffeomorphisms on H—those who preserve the Isolated Horizon conditions—and the $SU(2)$ gauge internal symmetry, i.e., the symmetry remaining after the partial time gauge fixing ($\underset{\leftarrow}{e^0} = 0$).

At H, the infinitesimal field variations decompose in two pieces

$$\delta = \delta_v + \delta_\alpha, \tag{4.17}$$

where δ_v are the infinitesimal diffeomorphism transformations while δ_α are the infinitesimal gauge transformations. Both are vectors in the tangent space of the phase space Γ and are "parametrized" by v and α, respectively. The parameter v is a vector field tangent to the horizon, generator of diffeomorphisms, $v : H \to T(H)$. The space $T(H)$, is the tangent space of the two-manifold H. Analogously α is an element of the $su(2)$ algebra and therefore an $SU(2)$ group generator, $\alpha : H \to su(2)$. As infinitesimal variations of fields are linear objects we can study infinitesimal diffeomorphisms

δ_v and infinitesimal internal gauge transformations δ_α independently. In the following the pullback to the manifold H will be assumed for all fields to simplify the notation. The general transformations of our fields restricted to H are simply

$$\begin{aligned} \delta \bar{A}^i &= \delta_v \bar{A}^i + \delta_\alpha \bar{A}^i \\ \delta e^i &= \delta_v e^i + \delta_\alpha e^i. \end{aligned} \tag{4.18}$$

The infinitesimal diffeomorphisms are given by the Lie derivative, $\delta_v = \mathscr{L}_v$, explicitly

$$\begin{aligned} \delta_v \bar{A}^i &= \mathscr{L}_v \bar{A}^i = d(v \lrcorner \bar{A}^i) + v \lrcorner d\bar{A}^i \\ \delta_v e^i &= \mathscr{L}_v e^i = d(v \lrcorner e^i) + v \lrcorner de^i, \end{aligned} \tag{4.19}$$

we use the notation $v \lrcorner e^i = v^a e^i{}_a$ for the so called *interior product*,[3] and *Cartan's formula* for the Lie derivative (see Equation (5.80) in [31]).

The infinitesimal group actions on the variables differ because e^i is a vector while $\bar{A}^i$ is a connection, explicitly

$$\begin{aligned} \delta_\alpha \bar{A}^i &= -d_{\bar{A}} \alpha^i = -d\alpha^i - \varepsilon^i{}_{jk} \bar{A}^j \alpha^k \\ \delta_\alpha e^i &= [\alpha, e]^i = \varepsilon^i{}_{jk} \alpha^j e^k. \end{aligned} \tag{4.20}$$

Now, let us rewrite the Equation (4.16) simply as $F^i(\bar{A}) = c\ \Sigma^i$.

We will prove the following

$$\delta_v \bar{A}^i \wedge \delta \bar{A}_i = 2c\ \delta_v e^i \wedge \delta e_i + t.d. \tag{4.21}$$

$$\delta_\alpha \bar{A}^i \wedge \delta \bar{A}_i = 2c\ \delta_\alpha e^i \wedge \delta e_i + t.d. \tag{4.22}$$

where *t.d.* stand for *total derivative*, a term that will disappear when we integrate on H as it is a manifold without boundary. For infinitesimal diffeomorphisms we have[4]

[3] In our notation $\lrcorner$ has *priority*, for example $v \lrcorner e^i A^j = v^a e^i{}_a A^j{}_b dx^b$ while $v \lrcorner (e^i A^j) = v^a (e^i{}_a A^j{}_b - e^i{}_b A^j{}_a) dx^b$.

[4] The form products defined on a 2-manifold satisfy the following rules with regard to ordering interchange of $v \lrcorner$

$$\begin{aligned} A^{(2)} v \lrcorner B^{(1)} &= -v \lrcorner A^{(2)} \wedge B^{(1)} \\ A^{(1)} v \lrcorner B^{(2)} &= v \lrcorner A^{(1)} \wedge B^{(2)}, \end{aligned}$$

where the superscript keeps tracks of the degree of the form, for example $A^{(2)}$ is a two-form.

$$
\begin{aligned}
\delta_v \bar{A}^i \wedge \delta \bar{A}_i &= -\delta(v \lrcorner \bar{A}^i d\bar{A}_i) + d\left(v \lrcorner \bar{A}^i \delta \bar{A}_i\right) \\
&= -\delta\left(v \lrcorner \bar{A}^i F_i(\bar{A})\right) + t.\ d. \\
&= -c\ \delta\left(v \lrcorner \bar{A}^i \Sigma_i\right) + t.\ d. \\
&= -c\ \delta\left(\varepsilon_{ijk} v \lrcorner \Gamma^i e^j \wedge e^k\right) + t.\ d. \\
&= -2c\ \delta\left(\varepsilon_{ijk} \Gamma^i (v \lrcorner e^j) \wedge e^k\right) + t.\ d.
\end{aligned} \tag{4.23}
$$

where we have used that[5] $v \lrcorner \bar{A}^i \varepsilon_{ijk} \bar{A}^j \wedge \bar{A}^k = 0$ to reconstruct $F^i(\bar{A})$, the Equation (4.16), that in spherically symmetric case $K^i \sim e^i$ (see (E.35)) such that $v \lrcorner K^i \Sigma_i \sim v \lrcorner e^i \varepsilon_{ijk} e^j \wedge e^k = 0$, and the definition of $\Sigma^i = \varepsilon^i{}_{jk} e^j \wedge e^k$.

For the triad term, we have

$$
\begin{aligned}
\delta_v e^i \wedge \delta e_i &= -\delta\left(v \lrcorner e^i d e_i\right) + d(v \lrcorner e^i \delta e_i) \\
&= \delta\left(v \lrcorner e^i \varepsilon_{ijk} \Gamma^j \wedge e^k\right) + t.d. \\
&= -\delta\left(\varepsilon_{ijk} \Gamma^i (v \lrcorner e^j) \wedge e^k\right) + t.d.
\end{aligned} \tag{4.24}
$$

where we have used the pullback of the First Cartan equation in the time gauge: $de^i + \varepsilon^i{}_{jk} \Gamma^j \wedge e^k = 0$.

For the infinitesimal group transformation we have

$$
\begin{aligned}
\delta_\alpha \bar{A}^i \wedge \delta \bar{A}_i &= -(d\alpha^i + \varepsilon^i{}_{jk} \bar{A}^j \alpha^k) \wedge \delta \bar{A}_i \\
&= \alpha^i \left(\delta d\bar{A}_i + \varepsilon_{ijk} \bar{A}^j \wedge \delta \bar{A}^k\right) - d(\alpha^i \delta \bar{A}_i) \\
&= \alpha^i \delta F_i(\bar{A}) + t.d. \\
&= c\ \alpha^i \delta \Sigma_i,
\end{aligned} \tag{4.25}
$$

where we use again the Equation (4.16).

And, finally for the triad term

$$
\begin{aligned}
\delta_\alpha e^i \wedge \delta e_i &= \varepsilon_{ijk} \alpha^j e^k \wedge \delta e^i \\
&= \frac{1}{2} \alpha^i \delta\left(\varepsilon_{ijk} e^j \wedge e^k\right) \\
&= \frac{1}{2} \alpha^i \delta \Sigma_i.
\end{aligned} \tag{4.26}
$$

Now, notice that (4.23) and (4.24) imply (4.21), while (4.25) and (4.26) imply (4.22). Therefore, we have proven that for the allowed variations on H

$$
\delta \bar{A}^i \wedge \delta \bar{A}_i = 2c\ \delta e^i \wedge \delta e_i + t.d. \tag{4.27}
$$

[5] Note that we are using the pullback on a two-surface and v is precisely tangent to this surface. Then, using an arbitrary basis the proof is easy.

integrating in H and using the constant c from (4.16) we find

$$\int_H \delta_{[1} e_i \wedge \delta_{2]} e^i = -\frac{a_H}{2\pi(1-\bar{\gamma}^2)} \int_H \delta_{[1} \bar{A}_i \wedge \delta_{2]} \bar{A}^i. \tag{4.28}$$

which finishes the proof.

The appearance of $\bar{\gamma}$ as an *arbitrary parameter* in the boundary term is natural as the original boundary term does not contain such a parameter. This ambiguity is analogous with the Barbero-Immirzi ambiguity γ defining the Ashtekar-Barbero variables: Both parameters γ and $\bar{\gamma}$ are introduced to construct a *connection* description, on the bulk and the boundary, respectively.

In summary, the symplectic structure for General Relativity written in terms of Ashtekar-Barbero connection variables, with a spherical symmetric Isolated Horizon as an internal boundary, and asymptotic flatness condition as an external boundary, is

$$\kappa\gamma\Omega(\delta_1, \delta_2) = \int_M 2\delta_{[1}\Sigma^i \wedge \delta_{2]} A_i - \frac{a_H}{\pi(1-\bar{\gamma}^2)} \int_H \delta_1 \bar{A}_i \wedge \delta_2 \bar{A}^i. \tag{4.29}$$

The boundary term is a *purely connection term* living in the 2+1 manifold $\Delta = H \times \mathbb{R}$ defining the Isolated Horizon. Therefore, it is natural to consider a 2+1 Chern-Simons theory with the connection $\bar{A}^i$ living in Δ. This theory would have exactly the same symplectic term on H. Furthermore, it fits well in the framework because the Isolated Horizon condition requires that the variations $\delta\bar{A}^i$ at the internal boundary to be pure gauge/diffeomorphism. On the other hand, a 2+1 Chern-Simons theory (in empty spacetime) is precisely a pure gauge theory:[6] There are not local degrees of freedom. However, we have shown that the curvature of the connection, $F^i(\bar{A})$, actually does not vanish but satisfies

$$\underset{\Leftarrow}{F^i}(\bar{A}) = -\frac{\pi(1-\bar{\gamma}^2)}{a_H} \underset{\Leftarrow}{\Sigma^i}, \tag{4.30}$$

therefore, Σ^i would play the role of an external source for the Chern-Simons curvature at Δ. So, at this classical level the interpretation is not completely fulfilled. In fact, we have not given an action principle for General Relativity coupled to a Chern-Simons theory at the boundary Δ such that (4.30) appears as an equation of motion. However, here we can take a step further: *Classical theories are just approximations for more fundamental quantum theories.* Hence, we use these indications at the classical level to construct

[6] Actually if the spacetime is not simply connected or has a non-trivial homotopy group there are global degrees of freedom. In fact, this is the property we will exploit in the following, when we introduce topological defect.

a quantum model for the spherically symmetric Isolated Horizon based on a quantum Chern-Simons description of horizon degrees of freedom. We will see that, in this quantized model, Equation (4.30) is not a problem as Σ^i will be a distributional source for the Chern-Simons curvature, such that the source is not everywhere but is just concentrated on particular points: Topological defects on H.

4.1.2 Quantization

Now, we review the elements used to build a quantum description of the spherically symmetric Isolated Horizons developed in [40].

To construct a quantum model of the black hole we use the framework of Loop Quantum Gravity. We briefly sketch it to introduce the quantum area operator.

Loop Quantum Gravity is a theory that attempts to quantize gravity (see a review in [42], and the full proposal in [43]). It has been developed from the Hamiltonian formulation for vacuum General Relativity as a starting point. In this sense, it is a canonical quantization of the gravitational field. In particular, it means that time plays a special role because the Hamiltonian approach requires a time foliation of the spacetime. The central idea in the Loop Quantum Gravity strategy is to use the *holonomy* or *loopy* representation for quantum states. To build the holonomies it is essential to have a connection variable: In this case the Ashtekar-Barbero connection.

The standard procedure for a canonical quantization consists in identify the pair of classical canonical conjugate variables and promote them to operators. Such operators act on a certain Hibert space, $\mathscr{H}$, which is the quantized version of the off-shell phase space of the theory. Consequently, the equations of motion of the theory also become operators on $\mathscr{H}$, and should in principle be solved there. For example, in standard quantum mechanics the classical Hamiltonian evolution is transformed into a quantum evolution by promoting the Hamiltonian to an operator, then, in a particular basis, this is translated into the Schödinger equation, and by solving it, the quantum evolution of states defined on a Hilbert space is obtained.

In the Hamiltonian analysis of General Relativity the equations of motion are contained in the so called *constraints*. In the Cartan formulation we have the *scalar*, the *vector*, and the *Gauss* constraints [42].

The loop representation space in Loop Quantum Gravity partially solves the quantum constraints (classical constrains where conjugate variables are promoted to operators). For instance, there is a simple basis where this representation solves the Gauss constraint. It is the so called spin-network basis. Spin-networks are graphs (networks) $\mathscr{G} \in M$ composed of edges and nodes where each edge carries an $SU(2)$ spin irreducible representation (*a coloured graph*). The quantum states are functions defined on this graph. This basis does not solve the scalar constraint which controls the dynamical evolution of the quantum states. The vector constraint can be formally solved by studying the dual of the space of function over the spin-network space (see Section 4.3 in [42]). As far as the dynamical part is missing, the Hilbert space constructed out of the states defined on the spin-network basis is simply called *kinematical* Hilbert space.

Now, we focus directly on the quantization of the conjugate variables (Σ^i, A_i) used in this framework. Given a spin-network $\mathscr{G} \in M$ the action of the densitized triad 2-form (4.29) is naturally defined on two-surfaces of M, in such a way that it only acquires non-trivial values at the intersections of the graph with the surfaces. Each of these intersections between the two-dimensional surface H and the one-dimensional link—which belong to the graph—is called *puncture*, it is denoted by $p \in \mathscr{G} \cap H$, and it carries a particular $SU(2)$ irreducible representation. See Fig. 4.1.

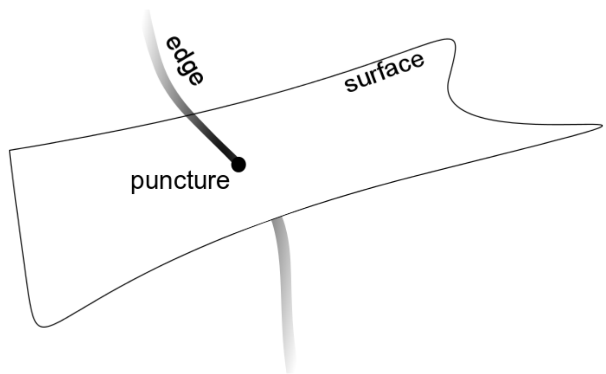

Fig. 4.1: Diagrammatic representation of an edge of the graph that pierce a surface. The intersection is called puncture.

Explicitly,

$$\varepsilon^{ab}\hat{\Sigma}^i_{ab}(x) = 2\kappa\gamma \sum_{p\in\mathscr{G}\cap H} \delta(x, x_p)\hat{J}^i(p), \tag{4.31}$$

with $\hat{J}^i(p) \in su(2)$ an algebra valued operator associated with each puncture p with coordinates x_p. It satisfies the $su(2)$ algebra commutation relations $[\hat{J}^i, \hat{J}^j] = \varepsilon^{ij}{}_k\hat{J}^k$. The action of $\hat{\Sigma}^i$ depicted in (4.31) is the same for an arbitrary two-surface $S \subset M$, here we are already using the horizon surface, $S = H$, because it will be the one used in this application.

Classically, the area of an arbitrary surface can be written in a coordinate invariant way as a surface integral of a simple function of the densitized triad.

Then, it is possible to construct a quantum description of the area as an operator acting on the spin-network basis of the Hilbert space, we find in this way

$$\hat{a}_H \left|\{j_p, m_p\}_1^n; \cdots\right\rangle = 8\pi\gamma\ell_p^2 \sum_{p=1}^{n} \sqrt{j_p(j_p+1)} \left|\{j_p, m_p\}_1^n; \cdots\right\rangle, \tag{4.32}$$

where $\left|\{j_p, m_p\}_1^n; \cdots\right\rangle$ denotes a particular spin-network state which pierces the surface n times and such that each edge piercing at p carries an $SU(2)$ irreducible representation labelled by j_p and m_p.

With these few ingredients we can already have an application to the previous classical analysis. The Equation (4.30) relates the curvature of the connection at the boundary with the densitized triad. Therefore, (4.76) suggests that the quantum version of (4.31) is simply

$$-\frac{a_H}{2\pi(1-\bar{\gamma}^2)}\varepsilon^{ab}\hat{F}^i_{ab} = \kappa\gamma \sum_{p\in\mathscr{G}\cap H} \delta(x, x_p)\hat{J}^i(p), \tag{4.33}$$

where in the r.h.s. a_H is assumed to be a constant fixed by the Isolated Horizon conditions.

Now, let us consider an $SU(2)$ Chern-Simons theory coupled to n particles defined on the 2+1 manifold Δ (for a general review of Chern-Simons theories at the classical level see [46, 47], for a short introduction closer to this context see [48]). Given an $SU(2)$ algebra valued connection $A = A^i\tau_i = A^i{}_a\tau_i dx^a$ with $\tau_i \in su(2)$,[7] we can define a Chern-Simons action over the Isolated Horizon as

$$S_0[A] = \frac{k}{4\pi}\int_\Delta Tr\left[A\wedge dA + \frac{2}{3}A\wedge A\wedge A\right], \tag{4.35}$$

where k is the Chern-Simons label, and $Tr[\cdot]$ is the trace over the group algebra (Killing form). The previous action can be generalized to describe particles on the manifold coupled to the connection field. Let us denote $\Lambda_p \in SU(2)$ the group valued particle degrees of freedom, it follows that

$$S_p[A, \Lambda_1, \cdots, \Lambda_n] = \sum_{p=1}^{n} \lambda_p \int_{c_p} Tr\left[\tau_3\left(\Lambda_p^{-1}d\Lambda_p + \Lambda_p^{-1}A\Lambda_p\right)\right], \tag{4.36}$$

[7] We have $\tau_i = -\frac{1}{2}\sigma_i$ where σ_i are the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4.34}$$

where $c_p \subset \Delta$ is the world-line of the p-particle, λ_p is an arbitrary coupling for each particle and $\tau_3 \in su(2)$ characterizes the particle at rest. The equation of motion for the connection is simply

$$\frac{k}{4\pi}\varepsilon^{ab}F^i{}_{ab}(A) = \sum_{p=1}^{n} \delta(x, x_p) S_p^i \ , \tag{4.37}$$

where we define

$$S_p^i = \lambda_p Tr\left[\tau_3 \Lambda_p \tau^i \Lambda_p^{-1}\right] . \tag{4.38}$$

Then, the curvature is different from zero just at the places where the defects p are situated. In the absence of particles the Chern-Simons connection is said to be trivial as it does not describe local degrees of freedom: Local gauge symmetry can be used to make the connection trivial. However, when particles are considered, the value of the curvature at the points where they are located adds local degrees of freedom to the connection. The particle source is distributional and can be thought of a collection of topological defects. The presence of the punctures modifies the topology of the manifold. In this sense, it is said that the theory is topological: It is characterized entirely by the topology of the manifold.

The similarities of equations (4.33) and (4.37) suggest an identification of both descriptions. This is the perspective we adopt here. Then, $\hat{J}^i(p)$, at the quantum level, can be interpreted as the source for the Chern-Simons curvature. As operators, the identification of the descriptions can be written formally as

$$\hat{\Sigma}^i \otimes \mathbb{I} = \mathbb{I} \otimes \frac{k}{4\pi}\hat{F}^i, \tag{4.39}$$

last expression means that the action of both operators is on the tensor product of two different Hilbert spaces (given below) but in such a way that their eigenvalues coincide. As a by product the Chern-Simons level is given by

$$k = \frac{a_H}{4\pi\ell_p^2\gamma(\bar{\gamma}^2 - 1)}. \tag{4.40}$$

In the identification (4.39) we are considering the quantized Chern-Simons theory, this is not a minor step, in fact, it is a technical one that goes beyond the purpose of this section. For an introduction to the canonical and covariant quantization of the Chern-Simons theory we refer the reader to [48]. Here we will just define the Hilbert space obtained and remember that from the path integral Chern-Simons quantization it is easy to prove that k is an integer.[8]

[8] If we perform a gauge transformation of the connection with group elements that are not connected to the identity the Chern-Simons action is not invariant: It produces a

For this identification (4.39) to make sense the Hilbert space for a fixed spin-network graph should be decomposed as a tensor product

$$\mathscr{H}_{\mathscr{G}} = \bigoplus_{\{j_p\}_{p\in\mathscr{G}\cap H}} \mathscr{H}_{\{j_p\}} \otimes \mathscr{H}^{CS}(j_1 \cdots j_n), \tag{4.41}$$

where $\mathscr{H}^{CS}$ is the Chern-Simons Hilbert space on $\Delta \cap M$ while $\mathscr{H}_{\{j_p\}}$ is the spin-network Hilbert space on M such that the endings of the edges at H are precisely the sources for the topological defects. More precisely, the representation labelled by j_p at each topological defect coincides with the one carried by the edge. Note that the structure of the graph beyond H is left completely free.

One interesting property of $\mathscr{H}^{CS}$ emerges when we study its observables. As Chern-Simons is a connection theory where we can define non-local gauge invariant quantities through holonomies along closed paths: Wilson loops, see [42] for their use in the context of Loop Quantum Gravity. This are the observables.

For instance, if a holonomy encloses a region of H without any defects/particles, its curve can be contracted to one point and therefore the holonomy is said to be *trivial*: Equal to the group identity, see Figure 4.2. On the other hand, one can analogously compute a holonomy that encloses all the defects, its value is given by the exponential of the sum of the curvature eigenvalues of all the punctures. As H has the topology of a sphere, both holonomies can be deformed one into the other, thus, they are simply *equivalent*. This means, that the second holonomy which encloses all the punctures is also the identity. Then, noticing that the value of the holonomy that encloses a puncture is given by the exponential of the curvature on this puncture modulo $k/2$, we can conclude that, the sum of all curvature eigenvalues at the deficits is zero modulo $k/2$ (for a formal approach see [49, 48]). This is the so called *closure constraint*. We will use its explicit expression in the next section.

The closure constraint strongly reduces the Chern-Simons Hilbert space in such a way that

$$\mathscr{H}^{CS}(j_1, j_2, \ldots, j_n) \subset \mathrm{Inv}(j_1 \otimes j_2 \otimes \cdots \otimes j_n), \tag{4.42}$$

where the r.h.s. is the *group invariant* part of the tensor product of all the representations associated with the punctures. The Hilbert space $\mathscr{H}^{CS}$ is

discretized winding number θ which depends in the topology of the group as well as the topology of the manifold. To make the path integral gauge invariant we must require $e^{i2\pi k\theta} = 1$. In last expression the winding number θ can take the particular value $\theta = 1$, therefore, to satisfy the equation in a general case k should be an integer.

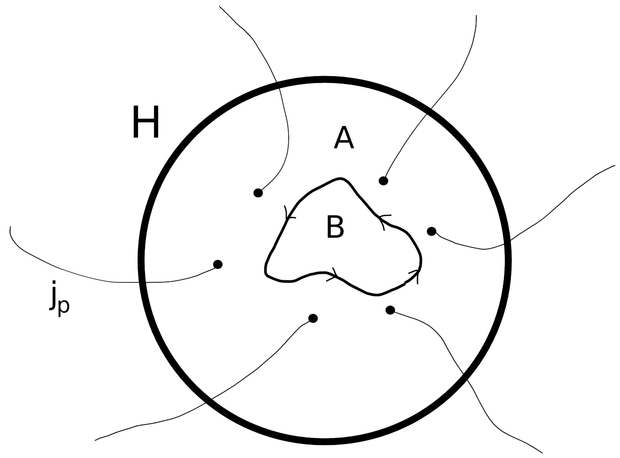

Fig. 4.2: The horizon H is a closed surface, therefore, the holonomy enclosing the empty region A is equivalent to the holonomy enclosing the region B which contains all the punctures (labelled by j_p).

contained in the l.h.s. of the Equation (4.42) because there is a subtlety on the group definition that we now introduce.

When we considered the Chern-Simons theory we started by looking at the classical theory, the group involved was the $SU(2)$, here we will consider a generalization of the group to a so called *quantum group*. Quantum groups appear as a useful tool to describe the quantization of the Chern-Simons theory [48].[9]

The symbol

$$j_1 \otimes j_2 \otimes \cdots \otimes j_n, \tag{4.43}$$

denotes the tensor product of the irreducible representations of $SU_q(2)$, and, $Inv(\otimes_\ell j_\ell)$ in (4.42) refers to its invariant part under the $SU_q(2)$ action. The main implication of the use of $SU_q(2)$ instead of $SU(2)$ is that the dimension of the Hilbert space changes and, in particular, it is finite. In principle, this allows us to count the states involved in the construction, and therefore, to compute the entropy. The explicit formula for the dimension of the Hilbert space will be further explained and used in Chapter 5. There, the statistical

[9] The emergence of quantum groups in the quantization of Chern-Simons theories is still controversial and not rigorously proven. However, they have been useful as a way to construct a regularized quantization of the phase space Poisson brackets, see Equation (3.28) in [48]. These objects are not really groups but are related to the original group, in this case $SU(2)$, as they are constructed by modifying the group algebra through the introduction of an extra parameter q. The symbol $U_q(su(2))$ is also used [50]. For our purpose it will be enough to remember that the Hilbert space for the punctures is constructed out of a generalization of the irreducible representations of $SU(2)$.

consequences of this quantization model are further explored.

This finishes our review of the quantum description of black holes in the reduced context of spherical symmetry. In the next section we explore the possibility of generalizing the same framework to the more general case of rotating Isolated Horizons.

4.2 Axially Symmetric Isolated Horizons

To deal with the rotating case we restrict the Isolated Horizons by asking an additional condition: The existence of a Killing vector field ϕ^a defined on the horizon Δ such that it generates a rotational symmetry around a preferred symmetry axis. We call this new structure the Kerr Isolated Horizon or rotating Isolated Horizon. The phase space of rotating Isolated Horizons has been characterized already in [25]. However, its quantization in the Loop Quantum Gravity framework has remained elusive due to what it seemed at first a technical issue. The presence of angular momentum means that there is a non-trivial charge generating rigid rotations around the symmetry axis. Therefore, as far as Isolated Horizons are internal boundaries, diffeomorphisms associated to vector fields tangent to the horizons are not simply gauge symmetries of the symplectic structure.[10]

Although the braking of some of the gauge symmetries by the boundary conditions has nothing pathological in itself and can be found in more familiar contexts,[11] it introduces serious problems for the quantum theory if one tries to approach the issue of quantization using Loop Quantum Gravity techniques. The reason is that diffeomorphism invariance is at the heart of the definition of the Loop Quantum Gravity framework. Consequently, it can only accommodate boundary conditions that respect this fundamental symmetry.

This is apparent from the central role played by diffeomorphism invariance in the models leading to the black hole entropy calculations for the spherically symmetric boundary condition considered in the previous section. More precisely, kinematical states of the spherically symmetric system are given by spin network states puncturing the horizon and endowing it with an area eigenvalue within the range $[a_H - \epsilon, a_H + \epsilon]$. The degeneracy of such kinematical states is infinite as it is labelled by the coordinates defining

[10] To prove that the symplectic structure is preserved the boundary terms should be zero even in the Palatini formulation, this is not true any more when rotating Isolated Horizons as an internal boundary are considered.

[11] Notice that this is in strict analogy to the fact that diffeomorphisms that do not fall off at infinity are not gauge symmetries of the phase space of asymptotically flat solutions of general relativity.

the embedding of the punctures on the horizon. Physical states are however finitely many. The reason is that they are obtained by modding out gauge symmetries which in this case include diffeomorphisms tangent to the horizons. This is crucial for the finiteness of the entropy. This central step is not justified in the naive treatments of the rotating case. The lack of diffeomorphism invariance in the phase space of the Kerr Isolated Horizon makes the usual program inapplicable.

An approach to deal with generic quantum Isolated Horizons (including rotation) has been proposed in [51]. However, the question of the fate of the diffeomorphism symmetry is unclear in such treatment. In particular, in such formulations both the leading order of the entropy calculation and the logarithmic corrections remain the same as the one of a non-rotating spherically symmetric model. In this work we emphasize the central role of diffeomorphism invariance in the construction of the model of quantum rotating horizons. This will change the nature of the admissible states to be counted in the entropy calculation.

One can recover a manifestly diffeomorphism invariant description of the phase space of a rotating Isolated Horizon by appropriately including new degrees of freedom that restore the broken symmetry. This has been shown explicitly in [52] using vector variables. We will adapt the same idea to the connection variable formulation presented here. In fact what we aim at is a generalization of the Chern-Simons formulation just presented.

However, the first naive attempt to follow this strategy fails due to the fact that—in contrast to the spherically symmetric case, and for instance, in the Kerr solution—the pullback to the horizon of the Ashtekar-Barbero connection does not satisfy the simple boundary condition found in (4.30), [53]

$$F^i(A) \neq c\Sigma^i. \tag{4.44}$$

for some constant c. As this boundary condition is a key equation for the Chern-Simons treatment in the non-rotating case this seems to rule out the possibility of describing the boundary degrees of freedom in terms of a Chern-Simons theory in the rotating model. Additional heuristics that seems to preclude the Chern-Simons treatment of the rotating case comes from the natural assumption, first put forward by Krasnov [54], that quantum states of rotating horizons with total angular momentum J should satisfy an additional constraint taking the form $J \sim \sum_p j_p$ (where j_p are the eigenvalues of the spin operators associated with punctures of the horizon). In other words one assumes that the total angular momentum of the black hole is made up from microscopic contributions from individual spins at the punctures. This suggestion is certainly appealing from an intuitive perspective. Nevertheless, the point is that if such a constraint would be true then this would further preclude the use of a Chern-Simons formulation as, as explained in the end of previous section, in such formulations one always obtains the closure constraint which naively tells us that the total angular momentum vanishes: $\sum_p j_p \sim 0$.

The two apparent difficulties described in the previous paragraph are nicely avoided as follows. We will show that one can introduce a new connection $\mathscr{A}^i$, defined on the horizon, such that by definition one has

$$F^i(\mathscr{A}) = c\Sigma^i, \tag{4.45}$$

for c a constant *almost* everywhere on the horizon (we get to this key subtlety in a moment). If one uses $\mathscr{A}^i$ as the connection dynamical field instead of A^i then the boundary symplectic structure takes the Chern-Simons form as far as the connection field is concerned. However, on the basis of our discussion in the previous paragraph, this would seem to contradict Krasnov's natural intuition that the total spin contributed by the bulk geometry Σ^i should be simply related to the spin of the black hole. Therefore, intuitively to have a Chern-Simons description Equation (4.45) should somehow be broken, at least in the deep quantum description. We will see that in fact there is a natural way to do it. The reason is that when we consider $\mathscr{A}^i$ there are *singularities* at the north and south poles of the horizon as defined by the symmetry axis. Classically, one expects these singularities to be related to some bad choice of coordinates as black hole horizons are smooth manifolds. The expectation is founded in the underlying standard property of smoothness of the manifolds. As we will deal with the quantization of the system where this assumption is not necessarily valid anymore, we sacrifice smoothness of the description on these two points and take the singularities as a part of the classical framework suitable to produce a quantum picture. We will show that the equation satisfied by the Cherns-Simons connection is

$$\frac{k\ell_p^2}{4\pi}F^i(\mathscr{A}) = \frac{\Sigma^i}{8\pi\gamma} + \frac{J}{2}\delta^i_1\delta_N + \frac{J}{2}\delta^i_1\delta_S, \tag{4.46}$$

where J is the macroscopic angular momentum and the delta symbols $\delta_{N/S}$ represent singularities of the curvature at the north and south poles of the horizon as defined by the singularities of the Killing field φ^a associated with the axial symmetry. The previous constraint implies, in the quantum theory, that the total spin contribution of spin network punctures must add up to J (actually modulo $k/2$ as we will see later). Note that when $J = 0$ we recover the k of the previous section. Admissible states can be depicted as in Figure 4.3.

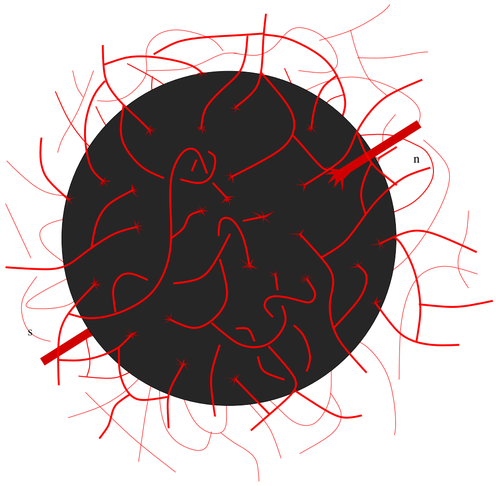

Fig. 4.3: The admissible states of the rotating black hole are in correspondence with invariant vectors in the Chern-Simons Hilbert space $\mathscr{H}^k_{CS} = \mathrm{Inv}[j_1 \otimes j_2 \otimes \cdots j_n \otimes j]_k$ where $\{j_i\}_{i=1}^n$ are the spins carried by spin network punctures (with arbitrary n) and there are two additional (macroscopic) punctures at the south and north poles carrying spins $J/2$ respectively. The subscript k is there to remind us that the notion of invariant space is that of the quantum group $SU(2)_q$ with q fixed by the level k.

The geometric picture associated with the admissible states is similar to the one presented in polymer models of the horizon geometry introduced in [55] and later in [56].

4.2.1 Rotating Horizons

In this section we present the variables used in order to describe the boundary degrees of freedom as a Chern-Simons theory. We will show explicitly a classical solution in these variables such that the Isolated Horizon conditions imply a consistent phase space description. The pullback of the Ashtekar-Barbero connection of Kerr geometry on the horizon has been computed in [53]. Here we follow a different approach: Instead of computing the pullback of a bulk connection in Kerr geometry we construct a connection field $\mathscr{A}^i$ from the Kerr horizon data. More precisely, the Chern-Simons connection $\mathscr{A}$ is required to satisfy the following set of conditions that will completely fix it, up to gauge transformations and diffeomorphisms tangent to the horizon H. First, we require that the following equation is satisfied

$$\frac{k}{4\pi} F^i(\mathscr{A}) = \frac{1}{8\pi\gamma\ell_p^2}\Sigma^i, \tag{4.47}$$

where k is the Chern-Simons level which is a function of the area a_H and the angular momentum J of the Isolated Horizon that will be determined in what follows. The 2-forms of the previous equation are pullback to the horizon two-surface H. The densitized triad field Σ^i (the pullback of $\epsilon^{ijk} e_i \wedge e_k$ to H, where e_i is the co-tretrad field) is part of the geometric data provided by the Kerr horizon geometry.

The above equation fixes the connection $\mathscr{A}^i$ up to an arbitrary rotation. That rotation is any rotation around the internal axis leaving Σ^i, seen as an internal vector, invariant. Explicitly, if $\mathscr{A}_1$ is also a solution of (4.47) then $\mathscr{A}_2 = g\mathscr{A}_1 g^{-1} - g dg^{-1}$ is a solution of (4.47) with the same Σ^i if $g \in U(1)_\Sigma \subset SU(2)$ is such that $g\Sigma g^{-1} = \Sigma$. We view this as an intrinsic ambiguity in the choice of the variable $\mathscr{A}^i$ and not as a gauge transformation. In particular the bulk connection is (by definition) unaffected by the transformation described above. Hence, we can and will exploit this freedom to fix our variable $\mathscr{A}^i$ so that an additional condition is satisfied, namely

$$v^\perp \lrcorner (A^i - \mathscr{A}^i)\Sigma_i = 0, \tag{4.48}$$

where $v^\perp$ is the unique normal direction to the axisymmetric Killing field $\phi^a = \partial^a_\phi$ on the horizon. In the usual spherical coordinates the previous condition can simply be written as $(\partial_\theta) \lrcorner (A^i - \mathscr{A}^i)\Sigma_i = 0$. We also require

$$\mathscr{L}_\phi(\mathscr{A}^i \Sigma_i) = 0, \tag{4.49}$$

where, again, ϕ^a is the axial Killing vector field on the Kerr horizon. With these conditions the connection $\mathscr{A}^i$ is almost completely fixed by the data provided by Σ^i and A^i of the Kerr Isolated Horizon. The remaining freedom is fixed by the condition

$$J = \frac{1}{8\pi\gamma} \int_H \phi \lrcorner (A^i - \mathscr{A}^i)\, \Sigma_i, \tag{4.50}$$

where J is the total angular momentum of the spacetime. These conditions are needed and will be explicitly used in the proof of the conservation of the symplectic structure in the next section. Given k, equations (4.47) to (4.50) uniquely determine the connection $\mathscr{A}^i$ up to gauge transformations and tangent diffeomorphisms (transforming A^i, Σ^i and $\mathscr{A}^i$ together).

In order to study the properties of $\mathscr{A}^i$ in more detail we will construct an explicit solution. The properties of this solution discussed below are all gauge and diffeomorphism invariant. We start with a spherically symmetric connection A_0^i (see Appendix E)

$$
\begin{aligned}
A_0^1 &= \cos(\theta)d\phi \\
A_0^2 &= \frac{1}{\sqrt{2}}(\sin(\theta)d\phi + \bar{\gamma}d\theta) \\
A_0^3 &= \frac{1}{\sqrt{2}}(\bar{\gamma}\sin(\theta)d\phi - d\theta).
\end{aligned}
$$

The parameter $\bar{\gamma}$ is not determined for the time being. The previous connection will be used as a *seed* to construct the Chern-Simons connection $\mathscr{A}^i$. The parameter $\bar{\gamma}$ labels a one-parameter family of suitable connections. In [15] the seemingly natural choice $\gamma = \bar{\gamma}$ was made. We will see here that the inclusion of rotation gives us the means to fix this ambiguity in a more physical way by requiring that the level of the Chern-Simons theory (computed below) vanishes for the extremal Kerr black hole $a_H = 8\pi J$. This is a suitable choice as the disappearance of the level in the extremal case will in turn imply that the entropy of an extremal black hole vanishes [57].

The curvature of the previous connection is

$$F^i(A_0) = \delta_1^i \frac{(\bar{\gamma}^2 - 1)}{2} \sin(\theta)d\theta \wedge d\phi. \tag{4.51}$$

The solution that we are looking for can be obtained via an active diffeomorphism ϕ_W acting on A_0 sending $d\phi \to \partial_\phi W(\phi)d\phi$. Such action should not be confused with a gauge transformation as the diffeomorphism acts only on A_0. The action on the connection is $A_0 \to \phi_W^* A_0$ and it follows that

$$F^i(\phi_W^* A_0) = \delta_1^i \frac{(\bar{\gamma}^2 - 1)}{2} \sin(\theta)\partial_\phi W(\phi)d\theta \wedge d\phi. \tag{4.52}$$

Now Equation (4.47) becomes the following Equation for $W(\phi)$

$$k\,\partial_\phi W(\phi) = \frac{a_H}{4\pi\gamma(\bar{\gamma}^2 - 1)\ell_p^2}. \tag{4.53}$$

Thus, $\phi_W^* A_0^i$ solves (4.47) if $W(\phi) = \frac{1}{k}\frac{a_H}{4\pi\gamma(\bar{\gamma}^2-1)\ell_p^2}\phi$. The area of the horizon, similarly to the spherically symmetric case, appears in the preceding formula because we have $\Sigma^i = \frac{a_H}{4\pi}\delta_1^i \sin(\theta)d\theta \wedge d\phi$.

As mentioned above our connection has to satisfy also the condition (4.48) which is accomplished by fixing the $U(1)_\Sigma$ ambiguity. Considering all this our solution is given by

$$\mathscr{A} = g[\phi_W^* A_0]g^{-1} + gdg^{-1}, \tag{4.54}$$

which is completely fixed (up to gauge transformations) by equations (4.47), (4.48), (4.49), and (4.50) and hence by the data contained in A^i and Σ^i

for a Kerr Isolated Horizon. Now, it is easy to show from (4.54) that in a circulation of an infinitesimal loop C around the poles our variables satisfy

$$\frac{k}{4\pi}\oint_C \mathscr{A}^1 = \frac{a_H}{8\pi\gamma(\bar{\gamma}^2-1)\ell_p^2}. \tag{4.55}$$

The previous equation will be used to fix the value of the Chern-Simons level k. We require that

$$\frac{k}{4\pi}\oint_C \mathscr{A}^1 = \frac{k}{2} + \frac{J}{2\ell_p^2}. \tag{4.56}$$

From equations (4.55) and (4.56) we obtain

$$k = \frac{a_H}{4\pi\gamma(\bar{\gamma}^2-1)\ell_p^2} - \frac{J}{\ell_p^2}. \tag{4.57}$$

The level of the Chern-Simons connection is given by the usual non-rotating level minus the Isolated Horizon angular momentum in Planck units. We choose to fix the ambiguity parameter $\bar{\gamma} = \sqrt{(2+\gamma)/\gamma}$ so that the Chern-Simons level takes the simpler form

$$k = \frac{a_H}{8\pi\ell_p^2} - \frac{J}{\ell_p^2} \tag{4.58}$$

which has the important property that it vanishes in the extremal case $a_H = 8\pi J$. We will comment further on the importance of this choice. Equation (4.56) implies the presence of conical singularities in the curvature $F^i(\mathscr{A})$ at the poles. We will see in the following section that these singularities are useful for the implementation of the Chern-Simons quantization of the rotating Isolated Horizon. One can remember the presence of the singularities at the poles if one writes the curvature equation over H in its entirety (including the poles) as

$$\frac{k}{4\pi}F(\mathscr{A})^i = \frac{\Sigma^i}{8\pi\ell_p^2\gamma} + p\delta_1^i\delta_N + p\delta_1^i\delta_S, \tag{4.59}$$

where δ_N and δ_S are Dirac delta functions centred on the north and south poles, respectively, and

$$p = \frac{k}{2} + \frac{J}{2\ell_p^2}. \tag{4.60}$$

4.2.2 Conservation of the Symplectic Structure

In this section we present the symplectic structure and prove that it is conserved provided that the boundary conditions hold. The symplectic structure is constructed in terms of the connection $\mathscr{A}^i$ introduced in the previous section. Additional variables are necessary to preserve diffeomorphism invariance in the rotating case (see [52] or the discussion above). These are a two-form $\mathcal{J}$ (that will acquire the physical meaning of the angular momentum density on shell) and its conjugate momentum, a scalar field Φ.

Now, we will study the allowed variation at the horizon in the same spirit that in the previous subsection. The only allowed variations on the horizon are tangent diffeomorphisms and $SU(2)$ gauge transformations [58]. We start with the $SU(2)$ gauge transformations denoted by δ_α for $\alpha^i(x) \in su(2)$, i.e, a Lie algebra valued scalar on M. For the bulk variables we have

$$\begin{aligned} \delta_\alpha \Sigma^i &= [\alpha, \Sigma]^i = \varepsilon^i{}_{jk}\alpha^j \Sigma^k \\ \delta_\alpha A^i &= -(d_A \alpha)^i = -d\alpha^i - \varepsilon^i{}_{jk}\alpha^j A^k, \end{aligned} \tag{4.61}$$

while for boundary variables the transformation is

$$\begin{aligned} \delta_\alpha \mathscr{A}^i &= -(d_{\mathscr{A}}\alpha)^i \\ \delta_\alpha \Phi &= (\alpha_1|_N + \alpha_1|_S)/2 \\ \delta_\alpha \mathcal{J} &= 0. \end{aligned} \tag{4.62}$$

Note that the angular momentum density $\mathcal{J}$ is gauge invariant by construction and the scalar field transforms in a distributional way: Only the values of α^i on H at the symmetry axis (the north and south poles) change Φ, (this particular transformation will justified in a better way in the proof of the symplectic structure preservation under gauge transformation).

We restrict diffeomorphisms to vector fields v that vanish at the north and south poles of H and, therefore, leave the north and south poles invariant. The transformation δ_v is

$$\begin{aligned} \delta_v \Sigma^i &= \mathscr{L}_v \Sigma^i = d(v \lrcorner \Sigma^i) \\ \delta_v A^i &= \mathscr{L}_v A^i = v \lrcorner dA^i + d(v \lrcorner A^i) \\ \delta_v \mathscr{A}^i &= \mathscr{L}_v \mathscr{A}^i = v \lrcorner d\mathscr{A}^i + d(v \lrcorner \mathscr{A}^i) \\ \delta_v \mathcal{J} &= \mathscr{L}_v \mathcal{J} = d(v \lrcorner \mathcal{J}) \\ \delta_v \Phi &= \mathscr{L}_v \Phi = v \lrcorner d\Phi. \end{aligned} \tag{4.63}$$

Proposition: In terms of the Ashtekar-Barbero variables the presymplectic structure of the rotating Kerr horizon takes the form

$$\begin{aligned}\Omega_M(\delta_1,\delta_2) &= \Omega_B(\delta_1,\delta_2)+\Omega_H(\delta_1,\delta_2)\\ &= \frac{1}{\kappa\gamma}\int_M 2\delta_{[1}\Sigma^i\wedge\delta_{2]}A_i+\frac{k}{4\pi}\int_H \delta_1\mathscr{A}_i\wedge\delta_2\mathscr{A}^i-\frac{16\pi}{\kappa}\int_H \delta_{[1}\Phi\,\delta_{2]}\mathcal{J},\end{aligned} \quad (4.64)$$

where k is the level of the Chern-Simons boundary term and $\kappa = 8\pi G$. $\Omega_B(\delta_1,\delta_2)$ denotes the first (bulk integral) term in the second line, while $\Omega_H(\delta_1,\delta_2)$ denotes the last two (surface integral) terms.

Proof: We prove the result by first looking at variations which are pure $SU(2)$ gauge transformations. Then, we show the invariance under pure diffeomorphisms. The idea is to prove that both kind of cotangent vectors are degenerate direction for the symplectic structure.

Invariance under Infinitesimal $SU(2)$ Transformations

We want to check that

$$\Omega_M(\delta_\alpha,\delta)=\Omega_B(\delta_\alpha,\delta)+\Omega_H(\delta_\alpha,\delta)=0, \quad (4.65)$$

for δ_α which is a local $SU(2)$ transformation as given in (4.61) and (4.62). The first contribution $\Omega_B(\delta_\alpha,\delta)$ yields

$$\Omega_B(\delta_\alpha,\delta) = \frac{1}{\kappa\gamma}\int_M\big([\alpha,\Sigma]_i\wedge\delta A^i+\delta\Sigma_i\wedge d_A\alpha^i\big) \quad (4.66)$$

$$= -\frac{1}{\kappa\gamma}\int_M\big[d(\alpha_i\delta\Sigma^i)-\alpha_i\delta(d_A\Sigma^i)\big] = -\frac{1}{\kappa\gamma}\int_H \alpha_i\delta\Sigma^i, \quad (4.67)$$

where we have used the Gauss law $\delta(d_A\Sigma)=0$ and that boundary terms at infinity vanish. At the boundary itself we have to take special care of the singular nature of our connection variables at the poles. Therefore, we split H in two small patches around the poles N and S, and an intermediate strip $H^* = H\backslash(N\cup S)$, see Fig. 4.4.

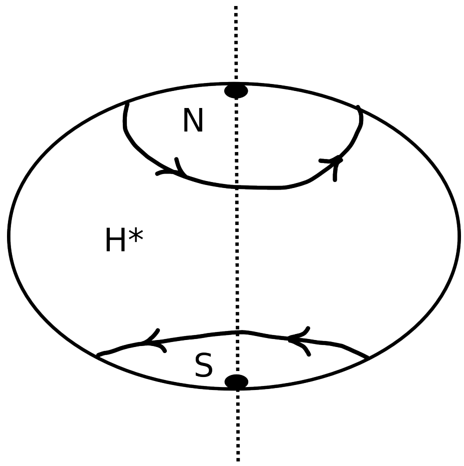

Fig. 4.4: The horizon H is divided into three regions $H = H^* \cup N \cup S$ such that the *north* and the *south* singularities lie inside N and S, respectively.

Thus, we obtain

$$\begin{aligned}
\frac{k}{4\pi}\int_H \delta_\alpha \mathscr{A}_i \wedge \delta \mathscr{A}^i &= -\frac{k}{4\pi}\int_H d_{\mathscr{A}}\alpha^i \wedge \delta\mathscr{A}_i \\
&= -\frac{k}{4\pi}\int_{H^*} d(\alpha^i \delta\mathscr{A}_i) + \frac{k}{4\pi}\int_{H^*} \alpha_i \delta F^i(\mathscr{A}) \\
&\quad -\frac{k}{4\pi}\int_{N\cup S} (d\alpha^i + \epsilon^i{}_{jk}\mathscr{A}^j\alpha^k) \wedge \delta\mathscr{A}_i \\
&= -\frac{k}{4\pi}\int_{\partial H^*} \alpha^i \delta\mathscr{A}_i + \frac{1}{\kappa\gamma}\int_{H^*} \alpha^i \delta\Sigma_i \\
&= \frac{k}{4\pi}\int_{\partial N} \alpha^i \delta\mathscr{A}_i + \frac{k}{4\pi}\int_{\partial S} \alpha^i \delta\mathscr{A}_i + \frac{1}{\kappa\gamma}\int_H \alpha^i \delta\Sigma_i \\
&= \int_H \alpha_i \delta\left(\frac{1}{\kappa\gamma}\Sigma^i + p\ \delta_N \delta^i_1 + p\ \delta_S \delta^i_1\right),
\end{aligned}$$

where on the second line we have integrated by parts, on the third line we have used (4.59) on H^*, on the fourth line we have used the oriented boundaries $\partial H^* = -(\partial N \cup \partial S)$, and finally on the fifth line we have used (4.56). Then

$$\begin{aligned}
\Omega_H(\delta_\alpha, \delta) &= \frac{k}{4\pi}\int_H \delta_\alpha \mathscr{A}_i \wedge \delta\mathscr{A}^i - \frac{8\pi}{\kappa}\int_H \delta_\alpha \Phi\, \delta\mathcal{J} \\
&= \frac{1}{\kappa\gamma}\int_H \alpha^i \delta\Sigma_i + (\alpha_1|_N + \alpha_1|_S)\delta p - \frac{4\pi}{\kappa}(\alpha_1|_N + \alpha_1|_N)\int_H \delta\mathcal{J}, \quad (4.68)
\end{aligned}$$

Hence, the symplectic structure has δ_α as a degenerate directions: $\Omega_M(\delta_\alpha, \delta) =$

$\Omega_{B}(\delta_\alpha,\delta)+\Omega_{H}(\delta_\alpha,\delta)=0$, if the following condition is satisfied

$$\frac{8\pi}{\kappa}\int_H \mathcal{J} \;\; = \;\; p, \tag{4.69}$$

then, it is because of the appearance of this equation in the symplectic conservation proof that we assume the distributional transformation rule for Φ, (4.62).

Invariance under Infinitesimal Diffeomorphisms

Now, we focus on the invariance under infinitesimal diffeomorphisms. We want to show that for a tangent vector field $v \in T(H)$ we have

$$\Omega_M(\delta_v,\delta)=\Omega_{B}(\delta_v,\delta)+\Omega_{H}(\delta_v,\delta)=0.$$

For the bulk term, using (4.63), we obtain

$$\begin{aligned}
\Omega_{B}(\delta_v,\delta) &= \frac{1}{\kappa\gamma}\int_M \left[\mathscr{L}_v\Sigma_i\wedge\delta A^i-\delta\Sigma_i\wedge\mathscr{L}_vA^i\right]\\
&= \frac{1}{\kappa\gamma}\int_M \left[d_A(v\lrcorner\Sigma)_i\wedge\delta A^i-\delta\Sigma_i\wedge v\lrcorner F^i+d(v\lrcorner A_i\,\delta\Sigma^i)\right]\\
&= \frac{1}{\kappa\gamma}\int_M \left[d(v\lrcorner\Sigma_i\wedge\delta A^i)+v\lrcorner\Sigma_i\wedge d_A(\delta A^i)-\delta\Sigma_i\wedge v\lrcorner F^i\right.\\
&\qquad\qquad\qquad\qquad\qquad \left.+d(v\lrcorner A_i\,\delta\Sigma^i)\right]\\
&= \frac{1}{\kappa\gamma}\int_M \left[d(v\lrcorner\Sigma_i\wedge\delta A^i)+v\lrcorner\Sigma_i\wedge\delta F^i-\delta\Sigma_i\wedge v\lrcorner F^i\right.\\
&\qquad\qquad\qquad\qquad\qquad \left.+d(v\lrcorner A_i\,\delta\Sigma^i)\right]\\
&= \frac{1}{\kappa\gamma}\int_M \left[d(v\lrcorner\Sigma_i\wedge\delta A^i)+\delta(\Sigma_i\wedge v\lrcorner F^i(A))+d(v\lrcorner A_i\,\delta\Sigma^i)\right]\\
&= \frac{1}{\kappa\gamma}\int_H \delta(v\lrcorner A_i\,\Sigma^i).
\end{aligned} \tag{4.70}$$

The horizon term yields

$$\begin{aligned}
\Omega_H(\delta_v,\delta) &= \frac{k}{4\pi}\int_H \mathscr{L}_v\mathscr{A}^i\wedge\delta\mathscr{A}_i - \frac{8\pi}{\kappa}\int_H[\mathscr{L}_v\Phi\,\delta\mathcal{J}-\delta\Phi\,\mathscr{L}_v\mathcal{J}] \\
&= -\frac{k}{4\pi}\int_H[\delta\mathscr{A}_i\wedge v\lrcorner F^i(\mathscr{A})+\delta\mathscr{A}_i\wedge d_{\mathscr{A}}(v\lrcorner\mathscr{A}^i)] \\
&\qquad\qquad -\frac{8\pi}{\kappa}\int_H[v\lrcorner d\Phi\,\delta\mathcal{J}-\delta\Phi\,d(v\lrcorner\mathcal{J})] \qquad (4.71)\\
&= -\frac{k}{4\pi}\int_H[\delta(v\lrcorner\mathscr{A}_i)\,F^i(\mathscr{A})+\delta F_i(\mathscr{A})\,v\lrcorner\mathscr{A}^i] \\
&\qquad\qquad -\frac{8\pi}{\kappa}\int_H[v\lrcorner d\Phi\,\delta\mathcal{J}+\delta(d\Phi)\wedge v\lrcorner\mathcal{J}] \\
&= -\frac{k}{4\pi}\int_H \delta(v\lrcorner\mathscr{A}_i\,F^i(\mathscr{A})) - \frac{8\pi}{\kappa}\int_H\delta(v\lrcorner d\Phi\,\mathcal{J}) \\
&= -\frac{1}{\kappa\gamma}\int_H\delta[v\lrcorner\mathscr{A}^i\,\Sigma_i+8\pi\gamma\;v\lrcorner d\Phi\,\mathcal{J}]. \qquad (4.72)
\end{aligned}$$

Now, equation $\Omega_M(\delta_v,\delta)=0$ is satisfied if the following equation holds

$$\frac{1}{\kappa\gamma}\int_H\delta[v\lrcorner(A^i-\mathscr{A}^i)\,\Sigma_i-8\pi\gamma\;v\lrcorner d\Phi\,\mathcal{J}]=0 \qquad (4.73)$$

for all $v\in T(H)$. Equation (4.73) is nothing else but the diffeomorphism constraint in these variables. The classical solution corresponding to Kerr is $\Phi=\phi$, where ϕ is the Killing parameter associated with axisymmetry. In this case $(A^i-\mathscr{A}^i)_\phi\Sigma_i/(8\pi\gamma)$ is the angular momentum density satisfying

$$J=\int_H\mathcal{J}=\frac{1}{8\pi\gamma}\int_H(A^i-\mathscr{A}^i)_\phi\,\Sigma_i, \qquad (4.74)$$

where J is the total angular momentum of the Kerr solution. This provides the physical interpretation of the l.h.s. of the Equation (4.69) found above. It tells us that the value of the pole singularities is given by the angular momentum of the black hole

$$p=\frac{8\pi J}{\kappa}. \qquad (4.75)$$

4.2.3 Quantization

Once the boundary description is captured by a Chern-Simons symplectic structure plus the Chern-Simons-like source Equation (4.59), the quantization is basically analogous to the one applied in the non-rotating case. There are, however, new aspects here that have to be treated carefully. The most obvious one is that in addition to the Chern-Simons connection $\mathscr{A}^i$ we have the field $\mathcal{J}$ and its conjugate Φ in the boundary symplectic structure and their quantization needs to be addressed too. The second issue is that the Chern-Simons constraint (4.59) contains two classical singularities at the north and south poles of the sphere and these are new features specific of the rotating system. Here we will start by ignoring the first problem and go directly to the second. In the last part of this section we will comment on the first.

As in the non-rotating case the form of the symplectic structure leads us to handle the quantization of the bulk and horizon separately. We first discuss the bulk quantization. As before we first consider the bulk Hilbert spaces $\mathscr{H}_{\mathscr{G}}$ provided by Loop Quantum Gravity and defined on a graph $\mathscr{G} \subset M$. The quantum operator associated with Σ^i in (4.59) reads again

$$\epsilon^{ab}\hat{\Sigma}^i_{ab}(x) = 2\kappa\gamma \sum_{p\in\mathscr{G}\cap H} \delta(x, x_p)\hat{J}^i(p). \tag{4.76}$$

Now, consider a basis of $\mathscr{H}_{\mathscr{G}}$ of eigenstates of both $\vec{J}(p)\cdot\vec{J}(p)$ and $J^3(p)$ for all $p \in \mathscr{G}\cap H$ with eigenvalues $\hbar^2 j_p(j_p+1)$ and $\hbar m_p$, respectively. These states are spin network states, here denoted by $|\{j_p, m_p\}_1^n; \cdots\rangle$, where j_p and m_p are the spins and magnetic numbers labelling the n edges puncturing the horizon at points x_p. They are eigenstates of the horizon area operator $\hat{a}_H$ as well

$$\hat{a}_H|\{j_p, m_p\}_1^n; \cdots\rangle = 8\pi\gamma\ell_p^2 \sum_{p=1}^{n} \sqrt{j_p(j_p+1)}|\{j_p, m_p\}_1^n; \cdots\rangle. \tag{4.77}$$

Now, along the same lines of the previous quantization treatment, we propose a quantum version of (4.59)

$$\frac{k}{8\pi}\epsilon^{ab}\hat{F}^i_{ab} = \sum_{p\in\gamma\cap H} \delta(x, x_p)\hat{J}^i(p) - \delta(x, x_N)\ \hat{J}^i_N - \delta(x, x_S)\ \hat{J}^i_S, \tag{4.78}$$

where, for all purposes, we define the operators associated with the singularities as

$$\hat{J}^i_N = \frac{J}{2\ell_p^2}\hat{z}^i \quad \text{and} \quad \hat{J}^i_S = \frac{J}{2\ell_p^2}\hat{z}^i, \tag{4.79}$$

where $\hat{z}^i$ is a normalized internal direction representing the symmetry axis. This means that we keep its classical value. To this choice makes sense we should select form the Hilbert space those states compatible with a classical angular momentum. Equation (4.79) tells us that the horizon Hilbert space $\mathscr{H}^H_{\mathscr{G}\cap H}$ that we are considering can be thought, as before, of the Hilbert space corresponding to the quantum Chern-Simons theory in the presence of particles but with a new feature: Two of the punctures are considered as having classical macroscopic values (4.79), *classical punctures*. By using the same argument based on holonomies enclosing all punctures, Fig. 4.2 (where now two special extra punctures should be added), we notice that the closure constraint is still present. However, instead of restricting the sum of all punctures coming from the spin-network to vanish, now, the sum is related to the value of the special punctures at the poles. The precise way this relation is realized is left for future work. However, at the operator level, a natural generalization of the closure constraint based on Equation (4.78) is

$$\begin{aligned}\sum_{p\in\gamma\cap H}\hat{J}^z(p) &= j\\ \sum_{p\in\gamma\cap H}\hat{J}^y(p) &= 0\\ \sum_{p\in\gamma\cap H}\hat{J}^x(p) &= 0,\end{aligned}\tag{4.80}$$

where we use $j = J/\ell_p^2$. These are *formal* expressions because they are actually inconsistent due to quantum uncertainties (they do not commute). Nevertheless, we can take a step further and propose a different quantum version of the previous conditions.

From the point of view of quantum geometry (bulk perspective), admissible states that solve the above constraint in the strongest possible way compatible with the uncertainty principle are *coherent states* of the collection of punctures satisfying the conditions

$$\sum_p m_p = [j]_k \tag{4.81}$$

$$\left[\sum_p \hat{J}^i(p)\right]\left[\sum_p \hat{J}_i(p)\right] = [j(j+1)]_{q(k)}, \tag{4.82}$$

where $-j_p \leq m_p \leq j_p$ denote the usual magnetic quantum numbers and in the last equality, the r.h.s. denotes the $SU(2)_{q(k)}$ Casimir operator. The spin-network state mimics the coherent state of the form $|j,j\rangle$. That is, in the Wigner notation $|j,m\rangle$, the state $|j,j\rangle$ is the coherent state that would have a particle with angular momentum j.[12] In other words, from the set of all possible spin-network states $|\{j_p, m_p\}_1^n; \cdots\rangle$ we select those which, by summing all their projected contribution of magnetic number m_p, contribute with a macroscopic angular momentum j. Furthermore, we incorporate in the condition (4.81) two quantum properties from the Chern-Simons formulation:

[12] These states are coherent in the sense that they minimize the uncertainty Δj.

First, the value of the defect (puncture) is defined modulo the Chern-Simons level, this is the meaning of $[\cdot]_k$. Second, if we work with the quantum group structure used in the quantization of Chern-Simons, the operators $\hat{J}^i(p)$ belong to the algebra deformation $SU(2)_{q(k)}$ and therefore we should consider the $SU(2)_{q(k)}$ Casimir operator. It is given simply by the standard Casimir operator but as a $q-$number

$$[x]_{q(k)} = \frac{q^x - q^{-x}}{q - q^{-1}}. \tag{4.83}$$

The states we are dealing with can be graphically represented as shown in Figure 4.3.

From the point of view of the boundary Chern-Simons theory the constraints are simpler. The two *classical punctures* (singularities) are aligned along the same axis. In the Chern-Simons description this amounts to a single puncture carrying the total macroscopic spin of the black hole. This one particular case explored in Chapter 5. Admissible states span the intertwiner space $j_1 \otimes j_2 \otimes \cdots \otimes j_n \to j$, if condition (4.81) is satisfied, and finally we also impose the area condition

$$a_H - \epsilon \leq 8\pi\gamma\ell_p^2 \sum_{p=1}^{n} \sqrt{j_p(j_p+1)} \leq a_H + \epsilon. \tag{4.84}$$

Finally we need to address the quantization of $\mathcal{J}$ and Φ and the imposition of the condition (4.73), namely

$$\int_H \delta[v \lrcorner (A^i - \mathscr{A}^i)\, \Sigma_i - 8\pi\ v \lrcorner d\Phi\, j] = 0,$$

for all vector fields v tangent to H. At the classical level the previous constraint completely reduces the $(\mathcal{J}, \Phi)$ degrees of freedom. This is due to the fact that it is an additional first class local constraint for two local degrees of freedom. Here we will simply assume that the constraint holds also at the quantum level without imposing any further constraint. This can be rephrased by saying that for each spin-network state satisfying the above restrictions there is only one solution of the previous equation for the quantum counterpart of $\mathcal{J}$ and Φ. In other words we expect the admissible states are indeed labelled by the spin quantum numbers satisfying the above constraints up to diffeomorphisms. This assumption is similar to the one made generically in the context of quantum states of Isolated Horizons as far as the bulk Hamiltonian (*scalar*) is concerned. It would certainly be useful to eliminate it; this is probably within the reach of present background independent quantization techniques.

With this comment we finish the description of our model of quantum rotating Isolated Horizon. This model has many heuristic ingredients coming from the classical black hole theory as well as from the quantization proposal of Loop Quantum Gravity or Chern-Simons Quantization. However, the model is detailed enough such that further analysis can be performed. In particular, it is possible to work out a statistical counting of the proposed quantum states based on (4.81), (4.82) and (4.84); and, therefore, to test further if the model predicts novel macroscopic physical properties for the black hole thermodynamics.

4.2.4 Discussion

In this section we have constructed a model of a rotating Isolated Horizon which is axisymmetric and has angular momentum J. The classical description of the system is based on a $SU(2)$ Chern-Simons connection plus additional auxiliary fields that restore diffeomorphism invariance. In the quantum theory the connection is constrained to be flat almost everywhere. As in spherically symmetric models, there are conical singularities with a strength that matches the quantum flux of the area encoded in the spin quantum numbers of spin network edges ending at the horizon. In addition to these, there are two conical singularities at the north and south poles (as defined by the singularities of the axisymmetric Killing field) with combined strength equal to $[J/\hbar]_{k/2}$.

The ambiguity condensed in the parameter $\bar{\gamma}$ appearing in the definition of the $SU(2)$ boundary Chern-Simons connection can be fixed in the rotating case by the requirement that the level of the Chern-Simons theory vanishes in the extremal case. This requirement implies that the number of states of an extremal horizon is unity and hence that their entropy vanishes as suggested in [57].

In [54] a tension was pointed out between the analogue of Equations (4.81), the area spectrum of Loop Quantum Gravity, and the fact that classically J can vary between 0 and $a_H/(8\pi)$. The tension disappears in the present model as we have incorporated the angular momentum as special *classical punctures* which allow the coexistence of the *closure constraint* and the intuitive idea of the angular momentum as composed by the microscopic structure.

In that reference the analogue of (4.81) was postulated with the important difference that the r.h.s. did not contain the $[\cdot]_k$ symbol. In that case one sees that there is a spin state configuration such that the maximum angular momentum of the horizon is $J_{max} \approx a_H/(8\pi\gamma)$. The fact that, classically, $J_{max} = a_H/(8\pi)$ would seem to imply $\gamma = 1$. Moreover, as the spectrum of the area is non-linear in the spins, it was conjectured in [54] that the extremal black holes would be represented by single puncture states with a large spin: In the large spin limit the spectrum becomes linear. None of these conclusions are valid in our model due to the appearance of the symbol $[\cdot]_{k/2}$ on the r.h.s. Indeed any classically allowed angular momentum value leads to a consistent set of constraints and there are no restrictions on the value of the Immirzi parameter γ. No matter how close we are from the extremal situation the black hole states that dominate the statistical mechanical treatment have many punctures (of the order of a_H/ℓ_p^2) which is compatible with the idea that these states approximate continuum geometries well.

5. ENTROPY COMPUTATIONS

The origin of the black hole entropy is still nowadays a subject of controversy.[1] For the physical black hole solution—in four dimensional spacetime—there are not crystal clear computations that reproduce the Bekenstein-Hawking semi-classical result from a fundamental approach, i.e., that identify the microscopic degrees of freedom. There are several attempts to solve the problem, for example, in the stringy community the calculation relies on a conformal symmetry. One idea is to recover the Bekenstein-Hawking entropy through the use of the Cardy formula which relates the entropy with the central charge that appears in the algebra of the generators of the conformal symmetry. Unfortunately, such a symmetry, which in the BTZ black hole emerges naturally in the asymptotic region [60, 61], is not generically present in the case of four-dimensional black holes. For example in the case of the axially symmetric (Kerr) black hole, there have been recent attempts to find such a symmetry in the near horizon region [62]—the Kerr/CFT correspondence—, however, the way to find such a symmetry is very particular, and it does not seem to be a generic property of black hole horizons. Furthermore, in this approach, it is not clear which are the local degrees of freedom that would reproduce the entropy.

In this chapter we compute the entropy of black holes from a completely different point of view given by the Loop Quantum Gravity approach presented in Chapter 4. In the seminal work [63], Rovelli proposed a very concrete way to compute the entropy by arguing that the degrees of freedom that are relevant in the computation of black hole entropy are those laying exclusively on the black hole horizon, and further, that they are in fact just those emerging by quantizing the gravitational field. In that proposal the computation is performed by simply counting the eigenvalues of the area quantum operator provided by Loop Quantum Gravity. In this respect, the quantization program—presented in the previous chapter—is the product of an effort to put these seminal ideas on better formal grounds. As it was shown, the program has progressed as a better control of the quantum grav-

[1] The stimulating dialogue transcript on [59] is a good example of the confrontation between different approaches.

itational degrees of freedom of the horizon proposed in [63] is accomplished: The punctures given by the piercing of the spin-network graph on the horizon surface can be described by a Chern-Simons theory with topological defects.

This chapter, in some sense, goes back to the historical origin of the Loop Quantum Gravity entropy computation and stars from there. That is because all the microscopic models presented here use the area spectrum for the horizon as it was the original idea. However, several new ingredients are implemented. For instance, we consider black holes with an angular momentum, the possibility of quantum group deviations, as well as new mathematical techniques to compute the asymptotics of the number of states in its leading and sub-leading approximation. In particular, dealing with rotating black holes—i.e., with angular momentum—is relevant because it is believed that the entropy computation for rotating black holes can be used to discern between different quantum gravity approaches [64]. The Bekenstein-Hawking entropy does not differ for rotating or static black holes, and therefore to recover this law is a first requirement for all quantum gravity proposals, however, the quantum correction to the dominant term can depend on the angular momentum—as indeed some of our models show—and thus they could be useful to compare theories which produce the same leading term.

The present chapter is organized as follows: In Section 5.1 we compute the dimension of the Hilbert space of the quantum states of the horizon. A further generalization by considering the representation of quantum group is used, the derivation of it is reviewed in Appendix F. Section 5.2 is the main body of this chapter. There, we introduce the mathematical tools used to compute the asymptotic behaviour of the number of states, the approximations and assumptions are discussed, and finally the computation is carried out for several models. The results are organized in a table at the end of the section. Section 5.3 is an extra bonus where we present a different computation of the entropy through the canonical partition function were ideas such as the indistinguishability of the punctures and holographic degeneration—explained there—are implemented.

5.1 Dimension of the Hilbert Space

In the previous chapter we introduced the Hilbert space were quantum states associated with the horizon live. The closure constraint strongly reduces the allowed states on the Hilbert space by selecting just those that are invariant under the action of $SU(2)_q$. In principle, there could be even more restrictions to select the physical states of the Hilbert space, for instance, extra constraints of dynamical originating in the Hamiltonian constraint. Here we ignore this possibility and assume that the reduction to physical states can be faithfully described by

$$\mathscr{H}^k_{CS}(j_1, \cdots, j_p) \equiv \mathrm{Inv}_k(j_1 \otimes \cdots \otimes j_p), \tag{5.1}$$

on the left hand side we have the $SU(2)_q$ invariant subspace of the tensor product of the vectors. Each vector transforms under the $SU(2)_q$ irreducible representations which are carried by each of the p punctures. Here we will explicitly work out the *dimension* of this space, which is in fact, the number of physical states in our quantum horizon model.

First, let us review in detail the derivation of the formula for the dimension of the Hilbert space associated with p punctures but assuming that each one carries simply an $SU(2)$ irreducible representation. Afterwards, the quantum group equivalent for $SU(2)_q$ will be worked as well.

Each $SU(2)$ irreducible representation can be labelled with half integers $j_\ell = \frac{1}{2},\ 1, \frac{3}{2}, \ldots$ and has a dimension $d_\ell = 2j_\ell + 1$. Irreducible representation are square $d_\ell \times d_\ell$ matrices $D^{(j_\ell)}_{mn}(g)$, with $g \in SU(2)$ and $m, n = -d_\ell,\ -(d_\ell - 1), \cdots,\ d_\ell - 1,\ d_\ell$.

Group integration is defined thought the Haar measure in such a way that arbitrary group dependent functions can be integrated. In the $SU(2)$ case, the Haar measure is left- and right-invariant. Let us use it to define the *intertwiner* operator as the integration of the product of p $SU(2)$ irreducible representations

$$I_{m_1 m_2 \cdots m_p n_1 n_2 \cdots n_p} \equiv \int_{SU(2)} D^{(j_1)}_{m_1 n_1}(g) D^{(j_2)}_{m_2 n_2}(g) \cdots D^{(j_p)}_{m_p n_p}(g)\ dg, \tag{5.2}$$

the intertwiner is by construction an $SU(2)$ invariant object (left and right), it can be proved by simply acting with the corresponding group representation and using the basic property ${D^{(j)}}_m{}^l(g') {D^{(j)}}_{ln}(g) = {D^{(j)}}_{mn}(g'g)$. Furthermore, by contracting the intertwiner with itself and using the properties of the Haar measure, in addition to the natural normalization $\int_{SU(2)} dg = 1$, it is easy to prove that intertwiners are projector operators $I^2 = I$

$$I_{m_1 m_2 \cdots m_p}{}^{l_1 l_2 \cdots l_p} I_{l_1 l_2 \cdots l_p n_1 n_2 \cdots n_p} = I_{m_1 m_2 \cdots m_p n_1 n_2 \cdots n_p}, \tag{5.3}$$

therefore, intertwiners are operators that projects tensor product of vectors of the corresponding representation space into its $SU(2)$ invariant component

$$I : j_1 \otimes j_2 \otimes \cdots \otimes j_p \longrightarrow \mathrm{Inv}\left(j_1 \otimes j_2 \otimes \cdots \otimes j_p\right). \tag{5.4}$$

It is always possible to choose a basis such that the projector operator is represented by a diagonal matrix with ones at its diagonal entries. In this basis it is apparent that the dimension of the invariant subspace is simply the trace of the intertwiner. This property is, of course, independent of the basis

$$\dim\left(\mathrm{Inv}(\otimes_\ell j_\ell)\right) \;=\; I^{m_1 m_2\cdots m_p}{}_{m_1 m_2\cdots m_p} = \int_{SU(2)} dg \prod_{\ell=1}^{p} \chi^{(j_\ell)}(g), \tag{5.5}$$

where $\chi^{(j)}(g)$ are the trace of the $SU(2)$ irreducible representations, also called *characters*. They are given by

$$\chi^{(j)}(g) \equiv D^{(j)}{}_{mm}(g) = \frac{\sin((2j+1)\theta/2)}{\sin(\theta/2)}, \tag{5.6}$$

in the last equality the standard Euler angles have been used to compute the $SU(2)$ characters. If the same coordinates are used to write the Haar measure the final formula for the dimension of the invariant $SU(2)$ intertwiner space is

$$D_\infty(\mathbf{d}) = \int_{SU(2)} dg \prod_{\ell=1}^{p} \chi^{(j_\ell)}(g) = \frac{1}{\pi}\int_0^{2\pi} d\theta\ \sin^2(\theta/2) \prod_{\ell=1}^{p} \frac{\sin(d_\ell\theta/2)}{\sin(\theta/2)}, \tag{5.7}$$

where $\mathbf{d} = (d_1, \ldots,\ d_\ell)$. The notation D_∞ will be clear in a moment when we consider the generalization of this formula for quantum groups.

Quantum groups are well-defined mathematical objects that can be constructed out of a deformation of any group Lie algebra. For an extended introduction on the subject see [65]. For our purpose, it is enough to consider a few properties which allow us to compute the equivalent of (5.7). They are presented in Appendix F. The algebra deformation for a quantum group is controlled by a parameter q. In particular, the quantum group $SU(2)_q$ is a more general structure defined in such a way that it reduces to the standard group $SU(2)$ when $q = 1$. In the general theory q is a complex number but when q is taken to be a root of unity the number of irreducible representations is finite. In Chern-Simons models of the quantum horizon the Chern-Simons level k is assumed to be proportional to the area of the classical black hole horizon. Then, it is natural to take $q(k)$ dependence such that $\lim_{k\to\infty} q(k) = 1$, and, at the same time, root of unity such that the dimension of the Hilbert space is finite. One simple option we take from the literature [50] is

$$q = \exp\left(\frac{i\pi}{k+2}\right). \tag{5.8}$$

Let us denote by $N_k(\mathbf{j}) \equiv \dim\left(\mathscr{H}^{CS}(j_1, \cdots, j_p)\right)$ the dimension of the Hilbert space constructed out of the quantum group structure. It is formally computed in Appendix F, the result is

$$N_k(\mathbf{j}) = \frac{2}{k+2}\sum_{d=1}^{k+1}\left(\sin\left(\frac{\pi d}{k+2}\right)\right)^{2-p}\prod_{\ell=1}^{p}\sin\left(\frac{\pi d(2j_\ell+1)}{k+2}\right), \tag{5.9}$$

where p is the number of punctures, and $\mathbf{j} = (j_1, \cdots, j_p)$ are the labels of the irreducible representation. There is a intuitive way to understand the previous formula, the use of the quantum group accounts for a sort of discretization of the angle in the formula (5.7) given by $\theta \to \frac{2\pi d_\ell}{k+2}$, consequently, the integral becomes a sum but in the index range $\ell \in [0, k+1]$. To slightly simplify the formula let us use $D_k(\mathbf{d}) = N_{k-2}(\mathbf{j})$, and for matter of convenience $\mathbf{d} = (d_1, \cdots, d_p)$ where $d_\ell = 2j_\ell + 1$.

$$D_k(\mathbf{d}) = \frac{2}{k}\sum_{d=1}^{k-1}\sin^2\left(\frac{\pi d}{k}\right)\prod_{\ell=1}^{p}\frac{\sin\left(\frac{\pi d d_\ell}{k}\right)}{\sin\left(\frac{\pi d}{k}\right)}, \tag{5.10}$$

this formula has an equivalent combinatorial expression which can be nicely interpreted as functions naturally appearing in discrete *random walks* [50]. From combinatorial formulae and by using the integral form of the Kronecker delta we can show the following integral formula for $D_k(\mathbf{d})$, the details of the derivation appear in Appendix F

$$D_k(\mathbf{d}) = \frac{2}{\pi}\int_0^\pi d\theta \sin^2(\theta)\frac{\sin\left((2r+1)k\theta\right)}{\sin(k\theta)}\prod_{\ell=1}^{p}\frac{\sin\left(d_\ell\theta\right)}{\sin(\theta)}, \tag{5.11}$$

where r is an integer defined with the $[\cdot]$ floor function by

$$r \equiv \left[\frac{\Delta_p}{2k}\right] = \left[\frac{\sum_\ell (d_\ell - 1)}{2k}\right]. \tag{5.12}$$

This coefficient carries the quantum group generalization of (5.7), which is simply recovered in the regime $r = 0$. Thus, the notation used in (5.7) becomes clear: The dimension of the intertwiner space in the $SU(2)$ representation can be recovered by taking the limit $k \to \infty$ of the dimension of the intertwiner space of the $SU(2)_q$ quantum group representations.

Now, we go back to physics and use that $D_k(\mathbf{d})$ counts the number of states for a fixed puncture configuration of the horizon. Then, the microstate counting problem can be well-posed.

Remark: Before continuing we would like to make a side remark concerning $D_k(\mathbf{d})$. It will be used to motivate an alternative computation for the entropy in section 5.3.

Let us consider the following transformation on the labels of the punctures

$$j_\ell \to i s_\ell - \frac{1}{2} \quad \Longrightarrow \quad d_\ell \to i s_\ell, \tag{5.13}$$

with s_ℓ real. Now, combine this with the assumption of "large spin" $s_\ell \gg 1$ such that

$$\prod_{\ell=1}^{p} \sin\left(\frac{\pi d d_\ell}{k}\right) \to i^p \prod_{\ell=1}^{p} \sinh\left(\frac{2\pi d s_\ell}{k}\right) \approx \frac{i^p}{2^p} \exp\left(\frac{2\pi d}{k} \sum_{\ell=1}^{p} s_\ell\right). \tag{5.14}$$

In the Chern-Simons Isolated Horizon models the label k corresponds to a large macroscopic quantity, see Equation (4.40) or (4.58). Therefore, in the regime $k \gg 1$, the previous equation plugged in the expression for $D_k(\mathbf{d})$, (5.10), results in the following approximation

$$D_k(\mathbf{d}) \approx \frac{2}{k} \frac{i^p}{2^p} \sin^{2-p}(\pi) \exp\left(2\pi \sum_{\ell=1}^{p} s_\ell\right). \tag{5.15}$$

Now, there is an interesting fact about this expression. In the original Ashtekar self-dual variables the spectrum of the area-squared operator is negative (this is apparent in Equation (49) of [66]), for instance, on a surface pierced by a link with spin j_ℓ we have

$$\hat{a}_H^2 |j_\ell\rangle = -(8\pi \ell_p^2)^2 j_\ell (j_\ell + 1) |j_\ell\rangle, \tag{5.16}$$

let us apply the transformation (5.13) and also the "large spin" assumption

$$\hat{a}_H^2 |s_\ell\rangle = (8\pi \ell_p^2)^2 (s_\ell^2 + 1/4) |s_\ell\rangle \approx (8\pi \ell_p^2)^2 s_\ell^2 |s_\ell\rangle. \tag{5.17}$$

The framework which could support this possibility, and in particular give a meaning to the transformation (5.13), has been further explored in [67]. It opens a renewed interest on the old complex formulation of Loop Quantum Gravity [68]. Then, the area operator spectrum in this context would be simply $a_H \approx 8\pi \ell_p^2 \sum_{\ell=1}^{p} s_\ell$. The interesting observation is that, with the three ingredients: The transformation $d_\ell \to i s_\ell$, the large spin $s_\ell \gg 1$ and $k \gg 1$ approximations, there is a regime such that the degeneration is

$$D_k(\mathbf{d}) \sim \exp\left(2\pi \sum_{\ell=1}^{p} s_\ell\right) = \exp\left(\frac{a_H}{4\ell_p^2}\right). \tag{5.18}$$

where a factor, which is indeed complex and deserves further interpretation [67], has been omitted. Thus, the dimension of the Hilbert space of a fixed graph could reproduce by itself the standard Bekenstein-Hawking area law.

At this level the computation is precise, however, it requires further justification. For instance, the "large spin limit" is widely studied in Spin Foam models as an approximation needed to recover the semi-classical limit of quantum gravity theory [69]. On the other hand, the quantization procedure for Ashtekar self-dual variables is not fully understood. Nevertheless, here we will consider this remark to simply motivate the use of a degeneration that goes as the exponential of the area which will be explored in Section 5.3. Now, let us go back to the problem of the counting of microstates.

5.2 Asymptotics of the Number of States

Number of States

The quantum model a black hole horizon developed here admits a statistical analysis as it provides a detailed description for the quantum states. Here we start by adopting the microcanonical perspective and devote ourselves to the task of computing the asymptotic behaviour of the number of states that our models has. To do so, a few approximations will be done in order to make the technical problem tractable. In addition, classical inputs will be used, specifically we will assume a macroscopic area a_H and an angular momentum J for the black hole. They will be incorporated in the counting process as a global constraint and in the strength value of special *classical punctures* (explained below), respectively.

The total number of states associated with a given puncture configuration $\mathbf{d} = (d_1, \ldots,\ d_p)$ and a Chern-Simons level k, is given precisely by $D_k(\mathbf{d})$ computed in (5.11). Even if the macroscopic area of a given black holes is fixed, the number of punctures is not, because each puncture can contribute with a different area. Therefore, we should sum over all possible punctures as well as over all possible representations they can carry, while the macroscopic area is kept fix. To do that we use the Loop Quantum Gravity formula for the area spectrum in terms of the puncture strengths[2]

$$a_H = 8\pi\gamma\ell_p^2 \sum_{l=1}^{p} \sqrt{j_l(j_l+1)} = 4\pi\gamma\ell_p^2 \sum_{\ell=1}^{p} \sqrt{(d_\ell-1)(d_\ell+1)}, \tag{5.19}$$

The representations of the quantum group have finite dimension, in fact, when using the q parameter shown in (5.8) , the label runs $d_\ell \in [2, k+1]$.[3]

[2] Note that in this formula we are not using the quantum group modified Casimir operator of $SU(2)_q$. Instead, we keep the $SU(2)$ Casimir for simplicity and because for large k both coincide. However, it could be interesting to explore this direction.

[3] For $d_\ell = k+2$ the Verlinde coefficient is zero and for $d_\ell > k+2$ is negative, see (F.4)

Using (5.19) as a constraint, the total number of states in this model of quantum black hole horizons is

$$N(a) = \sum_{p=0}^{\infty} \sum_{\mathbf{d}=2}^{k+1} \delta \left(a - \frac{1}{2} \sum_{\ell=1}^{p} \sqrt{(d_\ell - 1)(d_\ell + 1)} \right) D_k(\mathbf{d}) \,, \tag{5.20}$$

with $a = a_H/(8\pi\gamma\ell_p^2)$ the normalized area. Note that for a fixed area there is a maximum number of punctures p allowed. Each puncture contributes with a minimum area $a_0 = 4\pi\sqrt{3}\gamma\ell_p^2$ so $p_{max} = a_H/a_0 = 2a/\sqrt{3}$. As we will see in a moment the formulae simplify if $p_{max} \to \infty$ is considered. Therefore, we will use the working assumption that the tail in the sum after p_{max} is negligible when we study the asymptotic $N(a)$ for $a \gg 1$.

To study the asymptotic behaviour of $N(a)$ we will use the Laplace method that is explained in what follows.

Laplace Method

The Laplace transform of the number of states is given by[4]

$$\widetilde{N}(s) = \int_0^\infty da\, e^{-as} N(a) \,, \tag{5.21}$$

by studying $\widetilde{N}(s)$ we can discriminate between different asymptotic behaviours of $N(a)$. For instance if $N(a)$ behaves as a polynomial, $N(s)$ converges for all s. If $N(a)$ grows faster than an exponential, $\widetilde{N}(s)$ diverges for all s. But, if there is some s such that $\widetilde{N}(s_c)$ is convergent for $s > s_c$ and divergent for $s < s_c$, it means that s_c is a critical coefficient and $N(a) \sim e^{s_c a}$ would be the asymptotic behaviour of the number of states.

If the leading asymptotic behaviour is in fact exponential, it is possible to go further and test the sub-leading behaviour by using the generalized Laplace transform [50]

$$\widetilde{N}_2(s,t) = \int_0^\infty da\, e^{-as} a^{-t} N(a). \tag{5.22}$$

It should be evaluated at the critical coefficient $s = s_c$, then, an equivalent analysis to find a second critical coefficient (exponent) $t = t_c$ can be

[4] Remark: Note that in Chapter 3 the statistical partition function was defined by the Laplace transform of the degeneration (number of states) in terms of the energy, see (3.2). Here we are doing exactly the same computation but considering the degeneration in terms of the area. If we consider the framework developed in Chapter 2, such that the area is a measure of the energy of the black hole, we are in fact, computing the partition function for that framework.

performed. If both coefficients are found the asymptotic behaviour of the number of states is

$$N(a) \sim e^{s_c a} \; a^{t_c - 1}. \tag{5.23}$$

And, from the number of states the microcanonical entropy of the system is (the Boltzmann constant is set to one)

$$S = \log N(a) = s_c a + (t_c - 1) \log a. \tag{5.24}$$

Thus, the Laplace method to compute the asymptotic behaviour of $N(a)$ provides a way to compute the leading and sub-leading order of the entropy.

Now, let us apply this technique in the counting of microscopic states for the models which incorporate angular momentum.

Microscopic Quantum Models for a Rotating Horizon

In Chapter 4 the model for a quantum rotating black hole was presented. In order to deal with angular momentum at the quantum level, the Chern-Simons formulation of horizon degrees of freedom was modified by adding two special punctures at poles. Each of them carries half of the classical angular momentum $j/2 = J/(2\ell_p^2)$ of the black hole. In the state counting formula a natural contribution from these new *classical punctures* can be obtained by a simple modification of (5.11)

$$D_k(\mathbf{d}) \;=\; \frac{2}{\pi} \int_0^\pi d\theta \, \frac{\sin((2r+1)k\theta)}{\sin(k\theta)} \sin^2(d_J\theta) \prod_{\ell=1}^{p} \frac{\sin(d_\ell\theta)}{\sin(\theta)}\,, \tag{5.25}$$

where two independent *special punctures* with a degeneration $d_J = 2(j/2)+1$ has been added

$$\frac{\sin^2(d_J\theta)}{\sin^2(\theta)}, \tag{5.26}$$

the special punctures also modify the coefficient r by

$$r = \left[\frac{\Delta_p}{2k}\right] = \left[\frac{\sum_{l=1}^{p}(d_l - 1) + j}{2k}\right], \tag{5.27}$$

in the rest of the calculation it will be useful to define a dimensionless rotation parameter

$$a^\star \equiv \frac{8\pi J}{a_H} = \frac{j}{a\gamma}, \tag{5.28}$$

which for the standard Kerr-Newmann solution controls the transition between regimes of the non-rotating, Schwarzschild solution—case $a^\star = 0$—and the extreme black hole solution—case $a^\star = 1$. In the previous chapter we also found as an appropriate Chern-Simons level the one that vanishes in the extreme black hole

$$k = \frac{a_H}{8\pi \ell_p^2} - \frac{J}{\ell_p^2} = \gamma a(1 - a^\star)\,. \tag{5.29}$$

The degeneration (5.25) is a complicated formula. To make it tractable we make the simplifying assumption that the spectrum of the area operator is *linear*. Specifically, a is taken as

$$a \approx \sum_{\ell=1}^{p} j_\ell = \frac{1}{2}\sum_{\ell=1}^{p}(d_\ell - 1)\,, \tag{5.30}$$

this is a weak assumption if the semi-classical regime of Loop Quantum Gravity is dominated by large spins as some results on Spin Foam models suggest [69]. Using the fact that $j_\ell \leq \sqrt{j_\ell(j_\ell+1)} \leq j_\ell + \frac{1}{2}$, it is possible to show that the integer r defined in (5.27) is bounded as

$$\left[\frac{a^\star}{2(1-a^\star)}\right] \leq r \leq \left[\frac{2+\gamma a^\star}{2\gamma(1-a^\star)}\right]. \tag{5.31}$$

We notice that for fixed $a^\star$ the parameter r can only take a finite amount of integer values. By using the linear area spectrum approximation the parameter r simplifies and becomes simply the upper bound

$$r = \left[\frac{2+\gamma a^\star}{2\gamma(1-a^\star)}\right], \tag{5.32}$$

which does not depend neither on the number of punctures p nor on the spin labels colouring the punctures d_ℓ.

The rest of this section is devoted to the explicit computation of the entropy for five similar models based on the previous considerations. The results are summarized in a table at the end (page 115). We start by considering a simple toy model where all punctures are kept fixed in its lower possible value $j_\ell = 1/2$ and the quantum group nature of the representation is ignored $r = 0$. Then, different assumptions are changed to test more complex and realistic situations.

Case: Two Special Punctures, $r = 0$, and $j_\ell = \frac{1}{2}$

Let us compute the number of states for a black hole with p puntures $j_\ell = 1/2$, two large punctures $j/2$ and $r = 0$, then $D_k(\mathbf{d})$ in (5.25) is reduced to

$$I^{(2)} = \frac{2}{\pi}\int_0^\pi \sin^2((j+1)\theta)\left(\frac{\sin(2\theta)}{\sin(\theta)}\right)^p \tag{5.33}$$
$$= 2^p\frac{2}{\pi}\int_0^\pi \left[1-\cos^2((j+1)\theta)\right](\cos(\theta))^p \tag{5.34}$$
$$= 2^p\frac{2}{\pi}(I_1 - I_2), \tag{5.35}$$

where, in this case, we call $D_k(\mathbf{d}) \to I^{(2)}$ with the superscript to remind us of the fact that there are two special punctures. As the puncture values are fixed, the area is proportional to p and the area constraint is trivial. Consequently, the Laplace method is not needed and the asymptotics can be computed directly. The large black hole limit we are interested in is reached for $a \gg 1$, or equivalently $p \gg 1$. The first term in (5.35) is

$$I_1 = \int_0^\pi d\theta(\cos(\theta))^p = \frac{(1+(-1)^p)2^p}{\Gamma(1-(1+p)/2)^2\ \Gamma(p+1)}$$
$$\approx (1+(-1)^p)\sqrt{\frac{\pi}{2}}\ p^{-1/2} + O(p^{-3/2}). \tag{5.36}$$

For the second term, I_2, we note that as far as $p \gg 1$ the integrand is concentrated on $\theta \approx 0$ and $\theta \approx \pi$. A Gaussian approximation $(\cos\theta)^p \approx e^{-p\frac{\theta^2}{2}}$ around $\theta = 0$ is accurate enough. As $2j+1$ is an integer the very same integral appears around $\theta = \pi$, but, an alternating sign should be taken into account

$$I_2 = \int_0^\pi \cos^2((2j+1)\theta)\,(\cos(\theta))^p \approx (1+(-1)^p)\int_0^\infty \cos^2((2j+1)\theta)e^{-p\frac{\theta^2}{2}}$$
$$= (1+(-1)^p)\frac{1}{2}\sqrt{\frac{\pi}{2}}p^{-1/2}\left(1+e^{-\frac{2(2j+1)^2}{p}}\right). \tag{5.37}$$

In the regime $p \gg 1$ the exponential term is negligible as $j \sim p$, then, the asymptotic behaviour is

$$I^{(2)} \approx (1+(-1)^p)\frac{1}{\sqrt{2\pi}}\ 2^p\ p^{-1/2}. \tag{5.38}$$

It does not depend on j, and it is zero if p is odd. To compute the entropy we will assume that p is even because this is the physically relevant contribution.[5] In this case the degeneration does not depends on k, further, there is a linear relation between the area and the number of punctures $a_H = 4\sqrt{3}\pi\gamma\ell_p^2\ p$.

[5] One can also argue that the real asymptote for $p \gg 1$ is a local average on the p values, by doing this one obtains the same result.

Hence, the number of states as written in (5.20) correspond simply to $I^{(2)}$, which expressed in terms of the dimensionless physical area, $A = a_H/\ell_p^2$, allows for a direct computation of the entropy $S(A) \equiv \log N(a) = \log I^{(2)}$. The result

$$S \approx \left(\frac{\log 2}{4\sqrt{3}\pi\gamma}\right) A - \frac{1}{2}\log A. \tag{5.39}$$

Notably, the entropy does no depend on the angular momentum for the leading and sub-leading orders.

Now, let us consider a more realistic case by relaxing the assumption of fixed small strength for the punctures.

Case: Two Special Punctures, $r = 0$, and j_ℓ free

First, we should replace $D_k(\mathbf{d})$ with $r = 0$ from (5.25) in (5.20). In this case we need to perform the Laplace transform of the number of microstates

$$\begin{aligned} \widetilde{N}(s) &= \frac{2}{\pi}\sum_{p=0}^{\infty}\sum_{\mathbf{d}=2}^{k+1}\int_0^{\pi} d\theta \left(\frac{e^{i\theta(1+\gamma a^\star a)} - e^{-i\theta(1+\gamma a^\star a)}}{2i}\right)^2 \left(\prod_{l=1}^{p}\frac{\sin d_l\theta}{\sin\theta}\right) e^{-sa}, \\ &= \frac{1}{2\pi}\int_0^{\pi} d\theta \left(2I_0 - e^{2i\theta}I_+ - e^{-2i\theta}I_-\right), \end{aligned} \tag{5.40}$$

in the first line the area is denoted by a for convenience, but, after the Laplace transform it should be replaced by its spectrum expression (linear in this case). The sine function of the special punctures (5.26) has been replaced through the Euler identity by exponentials, this trick allows us to reduce the complicated expression to a sum of three product series condensed in I_0, I_-, and I_+ computed below. Each of these terms is a geometric sum such that the following identity can be used[6]

$$\sum_{p=0}^{\infty}\sum_{\mathbf{d}=1}^{k}\prod_{\ell=1}^{p} f(d_\ell) = \sum_{p=0}^{\infty}\left(\sum_{d=1}^{k} f(d)\right)^p = \frac{1}{1-\sum_{d=1}^{k} f(d)}. \tag{5.41}$$

The result is

$$I_\lambda = \left[1 - \sum_{d=1}^{k}\frac{\sin((d+1)\theta)}{\sin\theta} e^{(-\frac{s}{2}+\lambda i\theta\gamma a^*)d}\right]^{-1}, \tag{5.42}$$

[6] In this specific step we are using the simplifying assumption $p_{\max} \to \infty$ commented before.

where $\lambda = 0, -1, +1$. Here, it is apparent that for each one of these functions there is a pole in $s = s_c$ such that

$$\sum_{d=1}^{k}(d+1)e^{-\frac{s_c}{2}d} = 1, \tag{5.43}$$

it appears exactly when $\theta = 0$. The sum is easily performed but the equation is algebraically involved and s_c cannot be explicitly computed. However, $k \gg 1$, and in the limit $k \to \infty$ the equation becomes trivial. By solving it we have a very good approximation for s_c

$$s_c \approx \log(6 + 4\sqrt{2}) \approx 2.46\,, \tag{5.44}$$

which is the coefficient of the exponential in the asymptote, $N(a) \sim e^{s_c a}$.

Let us now explore the sub-leading corrections. For that purpose, it would be nice to simplify the expression of the generalized Laplace transform $\widetilde{N}_2(s,t)$, (5.22), as we have just done for the Laplace transform. Unfortunately, the direct calculation of $\widetilde{N}_2(s,t)$ is much more complicated due to the presence of the multiple sum over $\mathbf{d}$ which does not reduce anymore to a product of geometric series when a polynomial term is considered. In order to circumvent this problem, we follow [50] and compute $\widetilde{N}_2(s,t)$ at $s \approx s_c$ as follows

$$\widetilde{N}_2(s_c, -t') \equiv \int_0^\infty da\, e^{-as_c} a^{t'} N(a) = \partial_\varepsilon^{t'} \widetilde{N}(s_c - \varepsilon)\Big|_{\varepsilon=0}\,. \tag{5.45}$$

The problem reduces to computing derivatives of the Laplace transform $\widetilde{N}(s_c - \varepsilon)$ or equivalently of the functions I_λ defined above (5.42). Note that in principle t' is any real number, thus, we are implicitly using a generalization of the differential operation "∂" known as *fractional derivative* [70]. We use $t' = -t$ for notation convenience. Due to the particular form of the function (5.42), it is sufficient to approximate I_λ for small ε, before computing the derivatives and evaluating them at $\varepsilon = 0$. This makes the calculation much easier. Further, we already know that one obtains the critical value t_c when one studies the behaviour of I_λ around $\theta = 0$, as it has been already the case for computing s_c. In summary, we have to compute the t'-derivatives with respect to ε of the integrand of (5.40) around $s = s_c - \varepsilon$ and $\theta \approx 0$. Around the pole the functions I_λ are

$$
\begin{aligned}
I_{\pm} &\approx \left[g_{\pm} - \varepsilon h_{\pm}\right]^{-1} \\
&= \left[1 - \sum_{d=1}^{k} (d+1) e^{-\frac{s_c}{2} d} \left(1 + \pm i\theta\gamma a^* d - \left(\frac{1}{6} d(d+2) + (\gamma a^* d)^2\right)\theta^2\right)\right. \\
&\qquad \left. \times \left(1 + \varepsilon \frac{d}{2}\right)\right]^{-1} \\
I_0 &\approx \left[g_0 - \varepsilon h_0\right]^{-1} \\
&= \left[1 - \sum_{d=1}^{k} (d+1) e^{-\frac{s_c}{2} d} \left(1 - \frac{1}{6} d(d+2)\theta^2\right)\left(1 + \varepsilon \frac{d}{2}\right)\right]^{-1}, \qquad (5.46)
\end{aligned}
$$

where the functions h_λ and g_λ are defined through the equalities. Now the derivative with respect to ε allows us to study the divergences of the generalized Laplace transform around the poles

$$
\partial_\varepsilon^{t'} I_{\pm}\Big|_{\varepsilon=0} = t'! \frac{h_{\pm}^{t'}}{g_{\pm}^{t'+1}} \sim \theta^{-t'-1} \qquad (5.47)
$$

$$
\partial_\varepsilon^{t'} I_0\Big|_{\varepsilon=0} = t'! \frac{h_0^{t'}}{g_0^{t'+1}} \sim \theta^{-2t'-2}, \qquad (5.48)
$$

the second term is the dominant one as it produces the higher critical exponent when $-2t'_c - 2 = -1$, and the logarithmic correction is $-t'_c - 1 = -1/2$.[7] The three terms behave similarly for $\theta \approx 0$, so it is important in the last computation to check directly that the sum of them goes indeed as $\theta^{-2t'-2}$, i.e., that there are not special cancellations of

$$
f = \partial_\varepsilon^{t'} \left(2I_0 - e^{2i\theta} I_+ - e^{-2i\theta} I_-\right)\Big|_{\varepsilon=0}. \qquad (5.49)
$$

This is a straightforward calculation that can even be performed by using $\theta \approx 0$ just at the end, the result is

$$
f \approx \frac{2(g_- g_+)^{t'+1} - (g_0 g_-)^{t'+1} - (g_0 g_+)^{t'+1}}{\left[g_0 g_+ g_-\right]^{t'+1}} t'! c^{2t'+2} \sim \theta^{-2t'-2} \qquad (5.50)
$$

where we use that $\lim_{\theta\to 0} h_{\pm} = \lim_{\theta \to 0} h_0 = \sum_{d=1}^{k} d(d+1) e^{-\frac{s_c}{2}} = constant$, i.e., it is the same constant (and they does not modify the θ-dependence for $\theta \approx 0$). Therefore, we have shown that there are not special cancellations between the terms in (5.49). Then, as stated before $t'_c = -1/2$, and the

[7] The first terms associated with $I_{\pm}$ would produce a logarithmic correction -1 which grows slower for large a and can be neglected.

logarithmic correction coefficient $-1/2$ is confirmed. The entropy is

$$S \approx \frac{\log(6+4\sqrt{2})}{8\pi\gamma} A - \frac{1}{2}\log A. \tag{5.51}$$

where again the have an interdependency of the angular momentum in the result.

Let us consider a further generalization to the case with the quantum group modification $r \neq 0$.

Case: Two Special Punctures, $r \neq 0$ and free j_ℓ

This case is a slight modification of the previous one, the starting point is

$$\begin{aligned}
&\widetilde{N}(s) = \\
&\frac{2}{\pi}\sum_{p=0}^{\infty}\sum_{\mathbf{d}=2}^{k+1}\int_0^\pi d\theta \left(\sum_{q=-r}^{r} e^{-2i\theta qk}\right)\left(\frac{e^{i\theta(1+\gamma a^\star a)} - e^{-i\theta(1+\gamma a^\star a)}}{2i}\right)^2 \left(\prod_{l=1}^{p}\frac{\sin d_l\theta}{\sin\theta}\right) e^{-sa}, \\
&\qquad = \frac{1}{2\pi}\int_0^\pi d\theta \sum_{q=-r}^{r}\left(2I_0' - e^{2i\theta}I_+' - e^{-2i\theta}I_-'\right), \qquad (5.52)
\end{aligned}$$

where we have used the relation

$$\frac{\sin((2r+1)k\theta)}{\sin(k\theta)} = \sum_{q=-r}^{r} e^{-2i\theta qk}, \tag{5.53}$$

to express the r-dependency in terms of exponentials. Again, this trick allows us to put all terms on the same footing and compute a geometric sum. We put a prime in the I_λ' to remember they are q-dependent

$$I_\lambda' \;=\; \left[1 - \sum_{d=1}^{k}\frac{\sin((d+1)\theta)}{\sin\theta} e^{(-\frac{s}{2}+i\theta\gamma_\lambda^*)d}\right]^{-1}, \tag{5.54}$$

with $\gamma_\lambda^* \equiv \gamma\left(q(1-a^*)+\lambda a^*\right)$. Around the pole

$$\begin{aligned}
&I_\lambda' \approx [g_\lambda' - \varepsilon h_\lambda']^{-1} \\
&\approx \left[1 - \sum_{d=1}^{k}(d+1)e^{-\frac{sc}{2}d}\left(1 + i\theta\gamma_\lambda^* d - \left(\frac{d}{6}(d+2) + (\gamma_\lambda^* d)^2\right)\theta^2\right)\left(1+\varepsilon\frac{d}{2}\right)\right]^{-1},
\end{aligned}$$

the pole is exactly at the same place $\theta = 0$ and it has the same previous value $s_c \approx \log(6+4\sqrt{2})$. The sub-leading term can be computed by the same method

$$\sum_{q=-r}^{r} \partial_\varepsilon^{t'} I'_\lambda = t'! \sum_{q=-r}^{r} \frac{h'^{t'}_\lambda}{g_\lambda^{t'+1}}, \tag{5.55}$$

the term $h'^{t'}_\lambda$ does not affect the θ-dependence as it approaches a constant for $\theta \to 0$. The relevant term in the asymptotic expansion appears in the q-sum of g_0 when $q = 0$, because for this value γ_0^* vanishes, consequently the linear term in the denominator of I'_0, and

$$\partial_\varepsilon^{t'} I'_0\Big|_{q=0} \sim \theta^{-2t'-2} \tag{5.56}$$

produces again the logarithmic correction $-1/2$. It means that the inclusion of the quantum group which in fact modifies slightly the computation, does not have any asymptotic consequences. This is an expected result because, as far as $k \sim a$, we are in the regime $k \gg 1$ and we know that in the exact limit $k \to \infty$ the parameter r should vanish.

Comment: In the previous model we have assumed that both special punctures carrying the angular momentum are completely independent one of the other. In the model of Chapter 4 they have a very special spacetime position given by the symmetry axis of the black hole. This can be interpreted as a restriction between them. To grasp this situation we study a model with only one special puncture where the underlying idea is that both of them are completely dependent—or "aligned"—such that they can be modelled, in fact, by just one special puncture. Again we start by the simplest model $r = 0$ and $j_\ell = 1/2$ for all the p punctures.

Case: One special puncture, $r = 0$, and $j_\ell = \frac{1}{2}$

Let us use the notation $D_k(\mathbf{d}) \to I^{(1)}$, as before this case is not dependent on k and $d_\ell = 2j_\ell + 1 = 2$ for all ℓ.

$$\begin{aligned} I^{(1)} &= \frac{2}{\pi}\int_0^\pi d\theta\ \sin^2(\theta)\left(\frac{\sin 2\theta}{\sin\theta}\right)^p \times \left(\frac{\sin d_J\theta}{\sin\theta}\right) \\ &= 2^p\frac{2}{\pi}\int_0^\pi d\theta\ \sin(\theta)\sin(d_J\theta)(\cos\theta)^p \qquad (5.57) \\ &= \frac{2^p}{\pi}(I_- - I_+), \qquad (5.58) \end{aligned}$$

with

$$
\begin{aligned}
I_\pm &= \int_0^\pi d\theta \;\cos(\theta(1\pm d_J))(\cos\theta)^p \\
&\approx \left(1+(-1)^{p+2j}\right)\int_0^\infty d\theta\;\cos(\theta(1\pm d_J))e^{-p\frac{\theta^2}{2}} \\
&\approx \left(1+(-1)^{p+2j}\right)\sqrt{\frac{\pi}{2}}\times p^{-1/2}e^{-\frac{(1\pm d_J)^2}{2p}} \qquad (5.59)
\end{aligned}
$$

where we have used the approximation $(\cos\theta)^p \approx e^{-p\frac{\theta^2}{2}}$ for $p \gg 1$ and $\theta \approx 0$ (similarly for $\theta \approx \pi$ with a sign change), and also that $d_J = 2j+1$ where j is either an integer or half-integer.

$$
\begin{aligned}
I^{(1)} &= \left(1+(-1)^{p+2j}\right)\frac{1}{\sqrt{2\pi}}p^{-1/2}2^p\left(e^{-\frac{(1-d_J)^2}{2p}} - e^{-\frac{(1+d_J)^2}{2p}}\right) \\
&= \left(1+(-1)^{p+2j}\right)\frac{1}{\sqrt{2\pi}}p^{-1/2}e^{(\log 2-(a^\star\gamma)^2/2)p}\left(1-e^{-2a^\star\gamma-\frac{2}{p}}\right).
\end{aligned}
$$

where we have used that $j = \gamma a^\star a = \gamma a^\star p/2$ in the linear area spectrum approximation. Note that as p is an integer and j an integer or half-integer, the combination $\gamma a^\star/2$ should be a particular rational number of order one.

The conclusions are that the number of states can be zero if $p+2j$ is odd, the leading order coefficient is modified by the presence of the new puncture to $\log 2 - (a^\star\gamma)^2/2$, and the sub-leading order exponent is the same $-1/2$. Now, let us assume $p+2j$ is even and write the entropy for this case in terms of the area $A = a_H/\ell_p^2$ and the angular momentum $j = J/\ell_p^2$

$$
S \approx \left(\frac{\log 2}{4\sqrt{3}\pi\gamma}\right)A - \left(2\pi\sqrt{3}\gamma\right)\frac{j^2}{A} - \frac{1}{2}\log A. \qquad (5.60)
$$

Notably, the leading coefficient gets modified by the presence of the angular momentum.

Remark: By considering a step back in the derivation of the degeneration (F.20), $I^{(1)}$ is equivalent to

$$
I^{(1)} = \sum_{M=-j}^{j}\sum_{\{m_1,\cdots,m_p\}}\left(\delta_{m_1+\cdots+m_p+M} - \delta_{m_1+\cdots+m_p+M+1}\right) \qquad (5.61)
$$

$$
= \sum_{M=-j}^{j}\left[\binom{p}{p/2-M} - \binom{p}{p/2-M-1}\right]. \qquad (5.62)
$$

It is clear that in this model the extra puncture we are adding—that carries the angular momentum of the black hole—is quantum, in the sense that we are summing over all its projections $M \in [-j, j]$, in principle this special puncture can have some effect on the asymptotic as far as $j_\ell = 1/2$, but if we sum over all possible j_ℓ and use the approximation that $k \gg 1$, the "special" puncture is no longer special as it acquires values in a range of the same order as all the other purely quantum punctures. However, observe that the quantized model we proposed in the previous chapter uses *classical punctures* to model the rotation. The classicality of the punctures can be thought as a selection of only the states with $M = j$ fixed. In standard quantum theory, the angular momentum states with a projection restricted to $M = j$ are called coherent states and stand for those which quantum spread is minimal (see the paragraph after Equation (4.82)). Thus, we can implement *classical punctures* by a simple restriction on the previous sum. Now, we briefly explore this possibility.

Sub-case: Keeping the Special Puncture classical

So, we are still in the case $r = 0$ and $j_\ell = 1/2$, let us analyse the asymptotics of the term with $M = j$ in formula (5.62)

$$\begin{aligned} I^{(1)*} &\equiv \binom{p}{p/2-j} - \binom{p}{p/2-j-1} \qquad (5.63)\\ &\approx \frac{2^p}{\sqrt{2\pi p[1-(2j/p)^2]}} \left(1-(2j/p)^2\right)^{-\frac{p}{2}} \left(\frac{1-2j/p}{1+2j/p}\right)^j \frac{2j+1}{p/2+j+1}, \end{aligned}$$

where we have used the Stirling approximation $p! \approx \left(\frac{p}{e}\right)^p \sqrt{2\pi p}$. Note that for the non-rotating case $j = 0$, the sub-leading exponent is $-3/2$, thus, the introduction of rotation modifies the logarithmic correction for this model. The entropy is

$$S \approx \frac{\log\left[2/\sqrt{1-(\gamma a^\star)^2}\right]}{4\sqrt{3}\pi\gamma} A + \log\left[\frac{1-\gamma a^\star}{1+\gamma a^\star}\right] j - \frac{1}{2}\log A, \qquad (5.64)$$

where we replaced $2j/p = \gamma a^\star$ as it is an number of order one in our asymptotic analysis. Therefore, the logarithmic correction is again $-1/2$. The coefficient of the leading term, that in the absence of the classical puncture is simply a constant gets modified into a more complicated j and A dependent expression.

Conclusion

In the following table we summarize the results for the entropy computations:

Rotating Models	Asymptotics
Simple cases: $r = 0$**,** $j_\ell = 1/2$	
2-*special puncts.* (p even)	$\frac{\log 2}{4\sqrt{3}\pi\gamma}A - \frac{1}{2}\log A$
1-*special punct.* ($p + 2j$ even)	$\frac{\log 2}{4\sqrt{3}\pi\gamma}A - 2\pi\sqrt{3}\gamma\frac{j^2}{A} - \frac{1}{2}\log A$
1-*special punct.* "kept rigid"	$\frac{\log\left[2/\sqrt{1-(\gamma a^\star)^2}\right]}{4\sqrt{3}\pi\gamma}A + \log\left[\frac{1-\gamma a^\star}{1+\gamma a^\star}\right]j - \frac{1}{2}\log A$
Free spin cases: j_ℓ **arbitrary**	
2-*special puncts.* $r = 0$	$\frac{\log(6+4\sqrt{2})}{8\pi\gamma}A - \frac{1}{2}\log A$
2-*special puncts.* $r \neq 0$	$\frac{\log(6+4\sqrt{2})}{8\pi\gamma}A - \frac{1}{2}\log A$
1-*special puncts.* "kept rigid" $r = 0$	*future work*

The clearest result is that all models share the logarithmic correction, which seems to be a robust property of the *classical punctures* approach to incorporate the angular momentum. The second observation is that when considering two special punctures the leading order is simply proportional to the area of the black hole, and the proportionality is indeed independent of the angular momentum. On the other hand, in both one special puncture cases, the leading order depends on the angular momentum. Interestingly in the one special puncture where we sum over the projection, the resulting angular momentum dependence is exactly the same as the one obtained in the model proposed in [55] where the angular momentum is implemented as a constraint on the projections of the quantum punctures (see Equation (26) in [55]). However, the j-dependence is in tension with the semi-classical Bekenstein-Hawking entropy.

The area law is found in all the two special punctures models. This result is in the same lines of most of the previous Loop Quantum Gravity computation, where the appearance of the Barbero-Immirzi parameter is fixed to get a 1/4 as the Bekenstein-Hawking entropy suggest. However, as observed before, in the particular cases of *free* j_ℓ—the physically relevant ones—the recovered area law is not a result of the implementation of rotation as far as for these models the special punctures considered are in the same footing as the quantum ones.

The last line in the table is a model that should naturally be computed in this context. However, the method of Laplace transform fails as the integral is highly osculating, therefore, we leave it for future work.

To finish the chapter on the entropy computation we present one further microscopic computation of the partition function for the black holes which follows completely different assumptions based in the lines of the quasilocal perspective developed in Chapters 2 and 3 plus the remark made at the end of section 5.1.

5.3 Holographic and Rotating Model

In the following we construct a canonical partition function out of the notion of quasilocal energy $E_{loc} = \frac{a_H}{8\pi\ell}$, the holographic degeneration suggested at the end of Section 5.1, $e^{a_H/(4\ell_p^2)}$, plus the linear area spectrum also discussed there $a_H = 8\pi\ell_p^2 \sum_{i=1}^{p} s_p$. We named this special degeneration holographic as it depends on the geometric area of the black hole.

The partition function for the canonical ensemble is

$$Z(\beta) = \sum_n e^{-\beta E_n} = \sum_E g(E) e^{-\beta E}. \tag{5.65}$$

Our interest is to incorporate the angular momentum to the model. To do so we will imagine the angular momentum as a quantity distributed over all punctures. A way to implement this is to restrict the projection of all the spins carried by the quantum punctures such that the sum of all the projections is the macroscopic angular momentum

$$\sum_{i=1}^{p} m_i = j, \tag{5.66}$$

this constraint should be considered in the counting. To do so, we re-express the sums with the occupation number n_{sm}, it counts the number of punctures with spin s and projection m. The angular momentum and the area becomes

$$j = \sum_{s,m} n_{sm} m, \quad a_H = 8\pi\ell_p^2 \sum_{s,m} n_{sm} s. \tag{5.67}$$

The angular momentum constraint can be imposed by using a Kronecker delta in its integral form

$$\delta_{m_1+m_2+\cdots+m_p-j} = \frac{1}{2\pi} \int_0^{2\pi} d\alpha \; e^{i\alpha\left(\sum_{s,m} n_{sm} m - j\right)}. \tag{5.68}$$

note that this is equivalent to the strategy used in (5.63) to keep the special punctures classical.

The partition function, with the angular momentum constraint left implicit, is

$$Z(\beta, j) = \sum_{p=1}^{\infty} \sum_{\{n_{sm}\}^*} \prod_{s,m} \frac{1}{n_{sm}!} \exp\left((2\pi\ell - \beta)s\ n_{sm}/\ell\right) \times (constraint) \quad (5.69)$$

where the sum on $\{n_{sm}\}^*$ is over all partition of p, i.e., $n_{s_1 m_1} + n_{s_2 m_2} + \cdots = p$, and the area spectrum have been distributed in the product. The combinatorial factor $n_{sm}!$ means that we are using *indistinguishable punctures.* In the standard Loop Quantum Gravity entropy computation—as the ones we performed in the previous section—the opposite criterion has been used: Distinguishable punctures. However, the model of punctured horizon does not provide a clear point of view on the issue and heuristic arguments can be constructed to justify both—contradictory—assumptions, let us briefly review them. For distinguishability it has been long argued that the way each puncture is connected to the rest of the spin-network in the bulk Hilbert space makes that the permutation between a pair of punctures essentially two different states. On the other hand, for indistinguishability, it can be argued that actually when splitting the whole Hilbert space of Loop Quantum Gravity into the horizon and the bulk part, as in (4.41), a sum should be performed over all different graphs of the bulk spin-network. This sum would erase the memory of the connection between any puncture and each particular spin-network making the former intrinsically indistinguishable. Here we explore the possibility of having indistinguishable punctures which seems more natural from a quantum fundamental frame.[8]

Let us define $\tilde{\beta} \equiv (\beta - 2\pi\ell)/\ell$ and plug the angular momentum constraint into the partition function

$$\begin{aligned} Z(\beta, j) &= \frac{1}{2\pi} \int_0^{2\pi} d\alpha\ e^{-i\alpha j} \sum_{p=1}^{\infty} \frac{1}{p!} \sum_{\{n_{sm}\}^*} \prod_{s,m} \frac{p!}{n_{sm}!} \left(e^{-\tilde{\beta}s + i\alpha m}\right)^{n_{sm}} \\ &= \frac{1}{2\pi} \int_0^{2\pi} d\alpha\ e^{-i\alpha j} \sum_{p=1}^{\infty} \frac{1}{p!} \left(\sum_{s,m} e^{-\tilde{\beta}s + i\alpha m}\right)^{p} \qquad (5.70) \\ &= \frac{1}{2\pi} \int_0^{2\pi} d\alpha\ e^{-i\alpha j} \exp\left(\sum_{s,m} e^{-\tilde{\beta}s + i\alpha m}\right), \qquad (5.71) \end{aligned}$$

[8] At this point it would be also interesting to explore different quantum statistics for the punctures: Bosonic or Fermionic? This model, as a first approximation, ignores the issue and uses a classical statistics.

where the multinomial theorem and the Taylor expression of the exponential were used. The transformation $j_\ell \to is_\ell - 1/2$ induces s_ℓ as a discrete label, here will will assume it runs as the SU(2) labels, i.e., $s_\ell = 1/2,\ 1,\ 3/2, \ldots$ while the projection $m_\ell \in [-s_\ell, s_\ell]$. Therefore

$$f(\alpha,\tilde{\beta}) \equiv \sum_{s,m} e^{-\tilde{\beta}s+i\alpha m} = \sum_{s=1/2}^{\infty}\sum_{m=-s}^{s} e^{-\tilde{\beta}s+i\alpha m} \tag{5.72}$$

$$= \frac{1+2\cos(\alpha/2)-e^{-\tilde{\beta}/2}}{2\cosh(\tilde{\beta}/2)-2\cos(\alpha/2)}, \tag{5.73}$$

as $f(\alpha-\pi,\tilde{\beta}) = f(-(\alpha-\pi),\tilde{\beta})$ the imaginary part of $Z(\beta,j)$ vanishes and the real part is

$$Z(\beta,j) = \frac{1}{2\pi}\int_0^{2\pi} d\alpha\ \cos(\alpha j)\exp\left(\frac{1+2\cos(\alpha/2)-e^{-\tilde{\beta}/2}}{2\cosh(\tilde{\beta}/2)-2\cos(\alpha/2)}\right). \tag{5.74}$$

The integrand diverges when

$$\cosh(\tilde{\beta}/2) = \cos(\alpha/2) \tag{5.75}$$

but $\cosh(x) \geq 1$, then, the only solution is $\tilde{\beta} = 0$. It means that the partition function is strongly concentrated on $\beta = 2\pi\ell$, this corresponds to the Unruh temperature which is the physical one from the near horizon stationary observers. Therefore, we focus on the regime $\tilde{\beta} \ll 1$.

The angular momentum is a macroscopic quantity $j \gg 1$, if we combine this regime with $\tilde{\beta} \ll 1$ the partition function is very *oscillating* (as j grows). However, if j is held fixed we can always look at an short enough interval of integration around zero such that $\cos(\alpha j) \approx 1$ and still we can describe the divergence that the integral has for $\tilde{\beta} = 0$. Because de divergence appears when $\alpha \approx 0$ the integral interval can be split to approximate the integrand around zero $[0,2\pi] = [0,\delta] \cup (\delta, 2\pi]$, for $\delta \ll 1$ but fixed in such a way that it is not affected by $\tilde{\beta}$

$$Z(\beta,j) = Z_\delta + Z_0. \tag{5.76}$$

and

$$Z(\beta \approx 2\pi\ell, j) \approx Z_\delta. \tag{5.77}$$

Note that $f(\alpha \ll 1, \tilde{\beta} \ll 1) \approx \frac{8}{\alpha^2+\tilde{\beta}^2}$, and the integrand simplifies

$$Z_\delta \approx \frac{1}{2\pi}\int_0^{\delta} d\alpha\ \exp\left(\frac{8}{\alpha^2+\tilde{\beta}^2}\right). \tag{5.78}$$

this function is not analytic around $\tilde{\beta}=0$ then we cannot use Taylor expansion before performing the integral. Instead, it is possible to estimate the asymptotic behaviour by using two natural bounds for the integral. As the asymptotic $\tilde{\beta}$-dependence is apparent, let us write

$$Z_\delta \approx \frac{1}{2\pi}\exp\left(\frac{8}{\tilde{\beta}^2}\right)\int_0^\delta d\alpha\ \exp\left(-\frac{8\alpha^2}{\alpha^2+\tilde{\beta}^2}\right). \tag{5.79}$$

Now, we show bounds for the integral

$$I_0 \equiv \int_0^\delta d\alpha\ \exp\left(-\frac{8\alpha^2}{\alpha^2+\tilde{\beta}^2}\right). \tag{5.80}$$

The inequalities $\frac{\alpha^2}{\delta^2+\tilde{\beta}^2} \leq \frac{\alpha^2}{\alpha^2+\tilde{\beta}^2} \leq \frac{\alpha^2}{\tilde{\beta}^2}$, allow us to conclude the following

$$4\int_0^\delta \exp\left(-\frac{8\alpha^2}{\tilde{\beta}^2}\right) \leq\ 4I_0\ \leq 4\int_0^\delta d\alpha \exp\left(-\frac{8\alpha^2}{\delta^2+\tilde{\beta}^2}\right)$$

$$\tilde{\beta}\int_0^{\frac{2\sqrt{2}\delta}{\tilde{\beta}}} e^{-t^2}dt \leq\ 4I_0\ \leq \sqrt{\tilde{\beta}^2+\delta^2}\int_0^{\frac{2\sqrt{2}\delta}{\sqrt{\tilde{\beta}^2+\delta^2}}} e^{-t^2}dt \tag{5.81}$$

$$\sqrt{\frac{\pi}{2}}\tilde{\beta}+o(\tilde{\beta}^2) \leq\ 4I_0\ \leq \sqrt{\frac{\pi}{2}}\left(\int_0^{2\sqrt{2}} e^{-t^2}dt\right)\delta + o(\tilde{\beta}^2) \tag{5.82}$$

in the last line we have used $\tilde{\beta} \ll 1$. In Fig. 5.3 the functions in (5.81) are plotted. From the upper bound we learn that the integral is finite for all $\tilde{\beta}$, while the lower bound goes simply as $\tilde{\beta}$.

Therefore, the integral I_0 does not affect the leading asymptotic behaviour of $Z(\beta \approx 2\pi\ell, j)$ but in fact it can be shown that it contributes to the subleading order with the same behaviour as the lower bound. The partition function turns out to be simply

$$\log Z(\beta \approx 2\pi\ell, j) \approx \frac{8}{\tilde{\beta}^2} + \log\tilde{\beta} = \frac{8\ell^2}{(\beta-2\pi\ell)^2} + \log[(\beta-2\pi\ell)/\ell], \tag{5.83}$$

in particular, this form implies that j does not play any role in the main thermodynamics behaviour.

In the canonical ensemble we can compute the average energy

$$\bar{E} = -\partial_\beta \log Z \approx \frac{16\ell^2}{(\beta-2\pi\ell)^3} - \frac{1}{\beta-2\pi\ell}, \tag{5.84}$$

which would diverge for the exact Unruh temperature. To estimate the departure from the Unruh temperature we can use the last equation but imposing

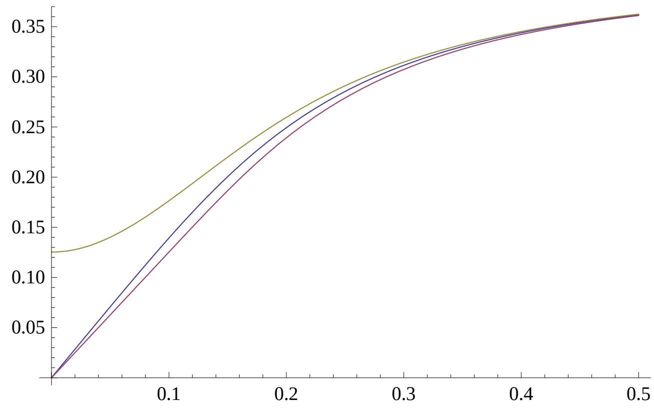

Fig. 5.1: Plot of (5.81) for $\tilde{\beta} \in [0, 0.5]$ and $\delta = 0.1$. The behaviour for $\tilde{\beta} \approx 0$ is the one given in (5.82).

de quasilocal notion of energy, i.e., the energy at which the system reaches the equilibrium $\bar{E} = E_{loc} = \frac{A}{8\pi\ell}$

$$\beta = 2\pi\ell + \varepsilon, \tag{5.85}$$

this gives an entropy

$$\begin{aligned} S &= \beta\bar{E} + \log Z &(5.86)\\ &\approx \frac{A}{4}, &(5.87)\end{aligned}$$

which is the Bekenstein-Hawking entropy. The leading term can be traced back to the simple relation $\beta_U E_{loc} = \frac{A}{4}$, i.e., the Unruh temperature and the quasilocal energy. Thus, this microscopic model can implement the known results in a consistent way. However, in the way it has been developed it cannot constitute a fundamental explanation of the Bekenstein-Hawking entropy as far as what directly produces the entropy it relies on the ingredients just stated. The missing piece in this approach is the computation of the temperature from first principles. Here, we are using the semi-classical result about the Unruh temperature for Rindler spacetime, which is the natural for the quasilocal framework, but, a complete microscopic approach should provide a notion of the temperature. It is expected that such a temperature is the Unruh one with a further correction ε which would allows us to compute the next order in the entropy. For a recent attempts in this direction see [71]. This open question is beyond the scope of the present work.

Appendices

Appendix A

$\mathcal{W}(\mathcal{O})$ FOR CHARGED KERR BLACK HOLE AND LOCAL SURFACE GRAVITY

Let us study the relation between the coordinates and the physical proper distance ℓ to the horizon which defines the surface $\mathcal{W}(\mathcal{O})$. The charged Kerr metric is

$$\begin{aligned} ds^2 &= -\left(\frac{\Delta - a^2\sin^2\theta}{\Sigma}\right)dt^2 - \frac{2a\sin^2\theta\,(r^2+a^2-\Delta)}{\Sigma}dtd\phi \\ &\quad + \left(\frac{(r^2+a^2)^2 - \Delta a^2\sin^2\theta}{\Sigma}\right)\sin^2\theta d\phi^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2, \quad \text{(A.1)} \\ \Delta &= r^2+a^2+q^2-2Mr = (r-r_+)(r-r_-), \quad \text{(A.2)} \\ \Sigma &= r^2 + a^2\cos^2\theta. \quad \text{(A.3)} \end{aligned}$$

We define our preferred family of observers as

$$u^a = \frac{\chi^a}{\sqrt{-\chi\cdot\chi}}, \quad \chi^a = \xi^a + \Omega_H\psi^a, \quad \Omega_H = \frac{a}{r_+^2+a^2}, \tag{A.4}$$

where ξ^a and ψ^a are the Killing vectors associated with staticity and axisymmetry of the metric.

The surface where this observers lay can be defined by a constant norm $\sqrt{-\chi\cdot\chi} = \sqrt{n} = constant$, but we can also do it with an arbitrary function of the norm

$$f(n) = constant, \tag{A.5}$$

the gradient of this function produces a vector orthogonal to the surface

$$\begin{aligned} df_a &= \frac{df}{dn}\frac{\partial n}{\partial r}dr_a + \frac{df}{dn}\frac{\partial n}{\partial\theta}d\theta_a, \quad \text{(A.6)} \\ (\partial_\lambda)^a &= g^{rr}\frac{df}{dn}\frac{\partial n}{\partial r}(\partial_r)^a + g^{\theta\theta}\frac{df}{dn}\frac{\partial n}{\partial\theta}(\partial_\theta)^a, \quad \text{(A.7)} \end{aligned}$$

with $\lambda(r,\theta)$ an arbitrary parameter. This arbitrary function is the standard gauge symmetry of the observers world-line. Let us fix this gauge by using

$\lambda = r$. This condition can be satisfied by fixing f such that $\frac{\partial r}{\partial \lambda} = g^{rr}\frac{df}{dn}\frac{\partial n}{\partial r} = 1$ and

$$(\partial_\lambda)^a = (\partial_r)^a + \frac{g^{\theta\theta}}{g^{rr}}\frac{\partial n/\partial\theta}{\partial n/\partial r}(\partial_\theta)^a, \tag{A.8}$$

the proper distance we are interested in is a coordinate independent quantity defined by

$$\begin{aligned} \ell &= \int_{\lambda_1}^{\lambda_2} \sqrt{g(\partial_\lambda, \partial_\lambda)}d\lambda & \text{(A.9)} \\ &= \int_{r_1}^{r_2} \sqrt{\frac{\Sigma}{\Delta}}\sqrt{1 + \frac{\partial n/\partial\theta}{\partial n/\partial r}}dr. & \text{(A.10)} \end{aligned}$$

Now let us choose $r_1 = r_+$ and $r_2 = r_+ + \varepsilon$, such that ℓ measures the proper distance for an observer near the horizon $\ell^2 \ll A$. In these coordinates small ℓ means $\varepsilon \ll r_+$. We can estimate this integral by simply approximating the integrand. An explicit calculation shows

$$\left.\frac{\partial n/\partial\theta}{\partial n/\partial r}\right|_{r=r_+ + \varepsilon} = o(\varepsilon), \tag{A.11}$$

Furthermore, we have

$$\begin{aligned} \Delta &= \varepsilon(r_+ + r_- + \varepsilon) & \text{(A.12)} \\ &= \varepsilon(r_+ + r_-) + o(\varepsilon^2), & \text{(A.13)} \end{aligned}$$

Then,

$$\begin{aligned} \ell &= \sqrt{\frac{\Sigma_+}{(r_+ - r_-)}}\int_0^\varepsilon \frac{1}{\sqrt{\epsilon}}(1 + o(\epsilon))d\epsilon & \text{(A.14)} \\ &= 2\sqrt{\varepsilon}\sqrt{\frac{\Sigma_+}{(r_+ - r_-)}} + o(\varepsilon^{3/2}), & \text{(A.15)} \end{aligned}$$

Or equivalently

$$\varepsilon(\theta) = \frac{\ell^2}{4}\frac{(r_+ - r_-)}{r_+^2 + a^2\cos^2\theta} + o(\ell^4), \tag{A.16}$$

thus, we learn that in this coordinates the r-separation from the horizon where our observers lays is θ-dependent for rotating horizons.

By definition $\bar{\kappa} = \frac{\kappa}{\sqrt{-\chi\cdot\chi}}$, to compute it for near horizon observers at a proper distance ℓ we evaluate the norm at $r_+ + \varepsilon(\theta)$, and we get a coordinate invariant expression to the order we are interested in,

$$\begin{aligned}\sqrt{-\chi\cdot\chi} &= \sqrt{-g_{tt}-2\Omega_H g_{t\phi}-\Omega_H^2 g_{\phi\phi}} && \text{(A.17)}\\ &\approx \kappa\ell+o(\ell^3) && \text{(A.18)}\end{aligned}$$

then

$$\bar{\kappa}=\frac{\kappa}{\sqrt{-\chi\cdot\chi}}=\frac{1}{\ell}+o(\ell), \tag{A.19}$$

just the first contribution is constant, and it is the only one relevant when $\ell^2 \ll A$.

Appendix B

EXTRINSIC CURVATURE TRACE FOR CHARGED KERR BLACK HOLE

Given a hypersurface Σ with intrinsic metric h_{ab} and normal n^a, the extrinsic curvature is defined by

$$K_{ab} \equiv \frac{1}{2}\mathscr{L}_n h_{ab}, \tag{B.1}$$

the trace of the extrinsic curvature, given by $K \equiv h^{ab}K_{ab}$, tells us how the surface is stretched with respect to the normal of the surface n^a. This interpretation comes directly from the explicit expression satisfied by K

$$\frac{\partial}{\partial n}\int_\Sigma d\Sigma = \int_\Sigma K d\Sigma, \tag{B.2}$$

where $d\Sigma$ is the differential element on the hypersurface Σ.

Now, using (A.16), the hypersurface Σ can be defined by

$$r = r_+ + \varepsilon(\theta), \quad du = dr - \varepsilon' d\theta, \quad \partial_n \sim \partial_u = g^{rr}\partial_r - \varepsilon' g^{\theta\theta}\partial_\theta \tag{B.3}$$

where du is the natural differential 1-form that allows us to compute an orthogonal vector ∂_u which can be normalized as

$$\partial_n = -\frac{1}{\sqrt{\partial_u \cdot \partial_u}}\partial_u = \frac{-g^{rr}\partial_r + \varepsilon' g^{\theta\theta}\partial_\theta}{\sqrt{g^{rr} + \varepsilon'^2 g^{\theta\theta}}}, \tag{B.4}$$

the sign is there to make $n^a \equiv (\partial_n)^a$ spacelike. The three-dimensional surface Σ can be parametrized as[1]

$$\sigma^1 = t,\ \sigma^2 = \phi,\ \sigma^3 = \theta, \tag{B.5}$$

such that

$$\begin{aligned} d\Sigma &= \sqrt{-g}n^a\varepsilon_{abcd}\frac{\partial x^b}{\partial\sigma^1}\frac{\partial x^c}{\partial\sigma^2}\frac{\partial x^d}{\partial\sigma^3}d\sigma^1 d\sigma^2 d\sigma^3 && \text{(B.6)} \\ &= \sqrt{-g(g^{rr} + \varepsilon'^2 g^{\theta\theta})}dt d\theta d\phi, && \text{(B.7)} \end{aligned}$$

[1] We choose the ordering to make $d\Sigma > 0$ given $\varepsilon_{tr\theta\phi} = +1$.

where the relation $r = r_+ + \varepsilon(\theta)$ determines $\frac{\partial r}{\partial \theta} = \varepsilon'$.

Because of time translation invariance we can replace $\int dt \to \Delta t$. If we consider (B.2) for an infinitesimal strip of the three-dimensional surface in the coordinate θ, we can also replace the integration $\int \theta \to \Delta\theta$, then

$$K = \frac{1}{\sqrt{-g}(g^{rr} + \varepsilon'^2 g^{\theta\theta})}(g^{rr}\partial_r - \varepsilon' g^{\theta\theta}\partial_\theta)\left(\sqrt{-g(g^{rr} + \varepsilon'^2 g^{\theta\theta})}\right). \quad \text{(B.8)}$$

The next step is to evaluate K on the surface $\mathcal{W}(\mathcal{O})$ given by $r = r_+ + \varepsilon(\theta)$. Let us use $\varepsilon(\theta)$ from (A.16) and neglect $o(\ell^3)$ terms. The result is

$$K = \frac{1}{\ell} + \ell f(\theta) + o(\ell^3), \quad \text{(B.9)}$$

$$f(\theta) = \frac{1}{8\Sigma_+^3}\left(4r_+^4 + 2r_+(r_+ - r_-)\Sigma_+ - \Sigma_+^2 - 4a^4\cos^2\theta\right). \quad \text{(B.10)}$$

For the Schwarzschild case it is reduced to

$$K|_{a=q=0} = \frac{1}{\ell} + \frac{5\ell}{32M^2} + o(\ell^3), \quad \text{(B.11)}$$

then, if we take $K_0 = \frac{1}{\ell}$, on the surface just over the horizon, we have

$$\begin{aligned} \frac{1}{8\pi}\int (K - K_0)d\Sigma &= \frac{\ell^2}{4}\int dt \int_0^\pi d\theta (r_+ - r_-)\sin(\theta) f(\theta) + o(\ell^4) \\ &\sim \ell^2 \Delta t. \end{aligned} \quad \text{(B.12)}$$

For Schwarzschild it is simply

$$\frac{1}{8\pi}\int (K - K_0)d\Sigma = \ell^2 \frac{5}{64M}\Delta t + o(\ell^4). \quad \text{(B.13)}$$

Appendix C

WICK ROTATION FOR KERR METRIC NEAR THE HORIZON

Let us consider the Kerr metric in the new coordinates

$$(t, r, \theta, \phi) \to (\tilde{t}, \ell, \theta, \phi), \tag{C.1}$$

such that

$$d\tilde{t} = \frac{1}{2}\left(dt + \frac{1}{\Omega_H}d\phi\right), \quad \Omega_H = \frac{a}{r_+^2 + a^2}, \tag{C.2}$$

$$d\tilde{\phi} = -\Omega_H dt + d\phi \tag{C.3}$$

$$d\ell = \sqrt{\frac{\Sigma_+}{(r - r_+)(r_+ - r_-)}}dr - 2a^2\sqrt{\frac{r - r_+}{(r_+ - r_-)\Sigma_+}}\cos\theta \sin\theta d\theta, \tag{C.4}$$

where we use the proper distance (A.15) as a coordinate.
We write the Kerr metric in these new coordinates, then, we evaluate it at $r = r_+ + \varepsilon(\theta)$, and keep just $o(\ell^2)$ terms. The result is

$$ds^2 = -\ell^2\kappa^2 d\tilde{t}^2 + d\ell^2 + \ell^2 g_{\tilde{t}\tilde{\phi}} d\tau d\tilde{\phi} + g_{\tilde{\phi}\tilde{\phi}} d\tilde{\phi}^2 + \ell g_{\theta\ell} d\theta d\ell + g_{\theta\theta} d\theta^2, \tag{C.5}$$

with $g_{\tilde{t}\tilde{\phi}}$, $g_{\tilde{\phi}\tilde{\phi}}$, $g_{\theta\ell}$, and $g_{\theta\theta}$ regular functions in the limit $\ell \to 0$. Note that in the previous expression the surface gravity κ has been introduced

$$\kappa = \frac{r_+ - r_-}{2(a^2 + r_+^2)} = \frac{\sqrt{M^2 - a^2}}{2M(M + \sqrt{M^2 - a^2})}. \tag{C.6}$$

Then, we have explicitly

$$\begin{aligned}
g_{\tilde{t}\tilde{\phi}} &= \kappa\left[(a^2+r_+^2)^2(r_+-r_-)+4a^2r_+(a^2+r_+^2)\sin^2\theta\right.\\
&\quad\left.-a^4(r_+-r_-)\sin^4\theta\right]\times\left[4a\Sigma_+^2\right]^{-1}\\
g_{\tilde{\phi}\tilde{\phi}} &= \frac{\sin^2\theta}{4\Sigma_+}(r_+^2+a^2)^2-\ell^2\frac{r_+-r_-}{64a^2\Sigma_+^3}\left[-(r_+-r_-)(a^2+r_+^2)\Sigma_+^2\right.\\
&\quad+2a^2(a^2+r_+^2)\left((r_-+3r_+)\Sigma_+-4r_+(r_+^2+a^2)\right)\sin^2\theta\\
&\quad\left.-a^4(r_+-r_-)\Sigma_+\sin^4\theta\right]\\
g_{\theta\ell} &= \frac{a^2}{\Sigma_+}\sin(2\theta)\\
g_{\theta\theta} &= \Sigma+\frac{a^4\ell^2}{2\Sigma_+^2}\sin(2\theta).
\end{aligned}\tag{C.7}$$

Now, let us perform a Wick rotation in the proper time $\tau = i\tilde{t}$

$$ds^2 = \ell^2\kappa^2 d\tau^2 + d\ell^2 - i\ell^2 g_{\tilde{t}\tilde{\phi}}d\tau d\tilde{\phi} + g_{\tilde{\phi}\tilde{\phi}}d\tilde{\phi}^2 + \ell g_{\theta\ell}d\theta d\ell + g_{\theta\theta}d\theta^2, \tag{C.8}$$

to avoid a conical singularity in the origin we should identify

$$\tau \to \tau + \frac{2\pi}{\kappa}, \quad \text{period } \beta = \frac{2\pi}{\kappa}. \tag{C.9}$$

Now we perform coordinate transformation

$$X = \ell\cos(\kappa\tau), \quad T = \ell\sin(\kappa\tau), \tag{C.10}$$

then

$$ds^2 = dT^2+dX^2-\frac{i}{\kappa}g_{\tilde{t}\tilde{\phi}}(XdT-TdX)d\tilde{\phi}+g_{\tilde{\phi}\tilde{\phi}}d\tilde{\phi}^2+g_{\theta\ell}(XdX+TdT)d\theta+g_{\theta\theta}d\theta^2, \tag{C.11}$$

as the transformation (C.10) shows the new variables are proportional to ℓ. We can think about X and T as moving in a small circle of radius ℓ. In the near horizon regime we can neglect cross terms, and, remarkably, the imaginary part of the metric

$$ds^2|_{\ell^2\ll A} \approx dT^2 + dX^2 + g_{\tilde{\phi}\tilde{\phi}}d\tilde{\phi}^2 + g_{\theta\theta}d\theta^2. \tag{C.12}$$

Therefore, we conclude that for the near horizon observers the metric that describes the surrounding spacetime admits a complex analytic continuation that keeps the metric real in the approximation used.

Appendix D

SYMPLECTIC STRUCTURE IN PALATINI VARIABLES

The action for General Relativity, in Hibert-Palatini formulation with tetrads [74], reads

$$S[e^I,\omega^{IJ}] = -\frac{1}{2\kappa}\int_{\mathcal{M}} \Sigma^{IJ}(e)\wedge F_{IJ}(\omega) + \frac{1}{2\kappa}\int_{\partial\mathcal{M}} \Sigma^{IJ}(e)\wedge \omega_{IJ}, \tag{D.1}$$

where $\kappa = 8\pi G$ with G the Newton constant (we set the speed of light $c = 1$). The veilbein enters through the definition

$$\Sigma^{IJ} = \star(e^I\wedge e^J) = \frac{1}{2}\varepsilon^{IJ}{}_{KL}e^K\wedge e^L, \tag{D.2}$$

where $\star$ is the Hodge dual. While the $SO(3,1)$ spin connection enters through the curvature

$$F^I{}_J = d\omega^I{}_J + \omega^I{}_K\wedge\omega^K{}_J. \tag{D.3}$$

The boundary term should be added in order to have a well-defined variational principle. The arbitrary variation of the action reads

$$\delta S = -\frac{1}{2\kappa}\int_{\mathcal{M}}\left(\delta\Sigma^{IJ}F_{IJ} + \delta\omega_{IJ}\; d_\omega\Sigma^{IJ}\right) + \frac{1}{2\kappa}\int_{\partial\mathcal{M}}\delta\Sigma^{IJ}\wedge\omega_{IJ}, \tag{D.4}$$

with $d_\omega\Sigma^{IJ} = d\Sigma^{IJ} + \omega^J{}_L\Sigma^{IL} - \omega_L{}^I\Sigma^{LJ}$.

For a introduction on the use of the symplectic structure in the context of General Relativity—and further introductory examples in Yang-Mills theories—see [72]. Here we use that the symplectic potential density can be directly read from the action variation [41], in consequence, we compute also the symplectic density

$$\begin{aligned}
\Theta(\delta) &= -\frac{1}{2\kappa}\delta\Sigma^{IJ}\wedge\omega_{IJ}\\
J(\delta_1,\delta_2) &\equiv \delta_1\Theta(\delta_2) - \delta_2\Theta(\delta_1)\\
J(\delta_1,\delta_2) &= \frac{1}{\kappa}\delta_{[1}\Sigma^{IJ}\wedge\delta_{2]}\omega_{IJ}.
\end{aligned} \tag{D.5}$$

Now, let us assume the tetrad is oriented in such a way that on each Cauchy surface M of our foliation we have $\underleftarrow{e}^0 = 0$, this is the *time gauge* choice. Then, the pullback of the density is reduced to

$$\underleftarrow{J}(\delta_1, \delta_2) = \frac{1}{\kappa}\delta_{[1}\underleftarrow{\Sigma_i} \wedge \delta_{2]}\underleftarrow{K}^i, \tag{D.6}$$

where $\Sigma_i = \varepsilon_{ijk}e^j \wedge e^k$ and $K^i = \omega^{0i}$.

Therefore, the gravity symplectic structure in these variables reads

$$\Omega(\delta_1, \delta_2) = \frac{1}{\kappa}\int_M \delta_{[1}\Sigma_i \wedge \delta_{2]}K^i, \tag{D.7}$$

where we omit the pullback symbol as it should be understood from the integration domain.

The linearized equations of motion imply $dJ = 0$. Now consider a slab of spacetime bounded by two Cauchy surfaces M_1, M_2 and a timelike cylinder at infinity τ_∞. The Stokes' theorem implies

$$\int_{M_1} J - \int_{M_2} J + \int_{\tau_\infty} J = 0, \tag{D.8}$$

if we assume asymptotic flatness on τ_∞, last term is zero and the symplectic structure defined above is independent of the Cauchy surface. Thus, for General Relativity in terms of Palatini variables and with asymptotically flat boundary conditions the symplectic structure 2-form—in the cotangent space of the phase space—is given by

$$\Omega(\delta_1, \delta_2) = \frac{1}{\kappa}\int_M \delta_{[1}\Sigma_i \wedge \delta_{2]}K^i, \tag{D.9}$$

is preserved. Note that no other boundary than τ_∞ encloses the Cauchy surfaces. In our application we deal with the Isolated Horizon as a second boundary.

Appendix E

ASHTEKAR-BARBERO CURVATURE ON A SCHWARZSCHILD HORIZON

To prove the equation

$$\underleftarrow{F}^i{}_{ab}(A) = -\frac{\pi(1-\gamma^2)}{a_H}\underleftarrow{\Sigma}^i{}_{ab}, \tag{E.1}$$

in the Schwarzschild spacetime we should solve the first Cartan equation, then, we use a specific Lorentz transformation such that the chosen tetrad becomes compatible with the Isolated Horizon null generator and so that the surface gravity produced by this generator coincide with the Schwarzschild surface gravity. At the end, we use Ashtekar-Barbero variables and compute the pullback of its curvature to the horizon.

In the first order formalism for gravity, the fundamental variables are the vielbein $e^I{}_a$ and the spin connection $\omega^I{}_{Ja}$. Both are naturally written as 1-forms

$$e^I = e^I{}_a dx^a, \quad \omega^I{}_J = \omega^I{}_{Ja} dx^a, \tag{E.2}$$

where dx^a is a basis of the cotangent space (1-forms) inherited from an arbitrary coordinate set. Here $a,\ b,\ c,\ d$ stand for the spacetime indices while $I,\ J,\ K,\ L$ stand for the internal indices over which the $SO(3,1)$ internal symmetry acts. In this notation the metric can be written as a tensor product

$$g = g_{ab} dx^a \otimes dx^b = e^I{}_a e^J{}_b \eta_{IJ} dx^a \otimes dx^b = e^I \otimes e^J \eta_{IJ}, \tag{E.3}$$

with $\eta_{IJ} = diag(-1,1,1,1)$. The curvature is naturally a 2-form defined by[1]

$$F^I{}_J(\omega) = d\omega^I{}_J + \omega^I{}_K \wedge \omega^K{}_J = \frac{1}{2} F^I{}_{Jab} dx^a \wedge dx^b, \tag{E.4}$$

now, we define the 2-form

$$\Sigma^{IJ} = e^I \wedge e^J = \frac{1}{2}\Sigma^{IJ}{}_{ab} dx^a \wedge dx^b, \tag{E.5}$$

[1] The convention for r-form components we use is $\omega = \frac{1}{r!}\omega_{a_1\cdots a_r} dx^{a_1}\cdots dx^{a_r}$ where complete antisymmetry is assumed in the lower indices [31].

combining both, we have the curvature written in components as in [40]

$$F^{IJ}{}_{ab}(\omega) = -\frac{1}{2} R_{ab}{}^{cd} \Sigma^{IJ}{}_{cd} \tag{E.6}$$

where $R^a{}_{cbd}$ are the components of the curvature Riemann tensor expressed in coordinates components. There is a proposal, in [40], to prove (E.1) by using directly the Isolated Horizon conditions, it takes as a starting point the last equation and then strongly uses the Newman-Penrose tetrad formalism which is specially adapted for this purposes, in particular for the computation of the Weyl tensor through the Weyl scalars. Here, as we will deal just with Schwarzschild spacetime it is possible to use directly the Schwarzschild metric (see appendix A in [40]). It is preferable to use Kruskal coordinates which are regular at the horizon

$$\begin{aligned} ds^2 &= \Omega^2(X,T)(-dT^2 + dX^2) + r^2(X,T)(d\theta^2 + \sin^2(\theta) d\phi^2), & \text{(E.7)} \\ \Omega^2 &= \frac{32M^3 e^{-\frac{r}{2M}}}{r}, & \text{(E.8)} \end{aligned}$$

where $r(X,T)$ is defined implicitly by the equation

$$F = \left(\frac{r}{2M} - 1\right) e^{\frac{r}{2M}} = X^2 - T^2. \tag{E.9}$$

therefore, the horizon worldsheet surface Δ is defined by the equation $X = \pm T$. If we perform a variation of the previous equation it can be shown that

$$\partial_X r|_\Delta = -\ \partial_T r|_\Delta = \frac{2X}{F'}. \tag{E.10}$$

A possible tetrad 1-form that can be chosen is

$$e^0 = \Omega(X,T)dT, \quad e^1 = \Omega(X,T)dX, \quad e^2 = r d\theta, \quad e^3 = r\sin(\theta)d\phi. \tag{E.11}$$

The first Cartan, torsionless, equation reads

$$de^I + \omega^I{}_J \wedge e^J = 0, \tag{E.12}$$

given the tetrad $e^I{}_a$ it can be solved in general [73]

$$\begin{aligned} \xi^I{}_{JK} &\equiv \partial_{[a} e^I{}_{b]} e_J{}^a e_K{}^b, \\ \omega_{IJK} &= \xi_{IJK} + \xi_{JKI} - \xi_{KIJ}, \\ \omega^{IJ}{}_a &= \omega_{LMK} \eta^{LI} \eta^{MJ} e^K{}_a. \end{aligned}$$

The 1-form connection $\omega^{IJ}=-\omega^{JI}$, turns out to be

$$\omega^{01} = \frac{1}{\Omega}\left(\partial_T\Omega\, dX+\partial_X\Omega\, dT\right), \quad \omega^{02}=\frac{\partial_T r}{\Omega}d\theta \tag{E.13}$$

$$\omega^{03} = \frac{\partial_T r}{\Omega}\sin(\theta)d\phi, \quad \omega^{12}=-\frac{\partial_X r}{\Omega}d\theta \tag{E.14}$$

$$\omega^{13} = -\frac{\partial_X r}{\Omega}\sin(\theta)d\phi, \quad \omega^{23}=-\cos(\theta)d\phi. \tag{E.15}$$

In the Isolated Horizon framework, the surface gravity is defined by

$$\ell^a\nabla_a\ell^b=\kappa_{IH}\ell^b. \tag{E.16}$$

As explained in Chapter 2, it is no uniquely defined, as the null vector ℓ^a can be rescaled arbitrarily. Here, we can use $\ell_a=\frac{1}{\sqrt{2}}\left(e^1{}_a-e^0{}_a\right)$ to express one of the vectors in the class (we define also $n_a=-\frac{1}{\sqrt{2}}\left(e^1{}_a+e^0{}_a\right)$), then, if we compute the surface gravity directly we realize that it does not match the standard value for Schwarzschild $\kappa=\frac{1}{4M}$. The reason is because κ is computed with a null normal at the horizon which is also a Killing vector normalized at infinity. To make them coincide we perform a special Lorentz transformation on the internal indices such that it rescales $\ell_a\to e^\alpha\ell_a$ at the horizon, then, we fix $\alpha=\alpha(X)$ with the condition $\kappa_{IH}=\kappa$. Such Lorentz transformation is

$$\Lambda^I{}_J=\begin{pmatrix}\cosh(\alpha(X)) & \sinh(\alpha(X)) & 0 & 0\\ \sinh(\alpha(X)) & \cosh(\alpha(X)) & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}, \tag{E.17}$$

by using the transformation rules

$$e'^I=\Lambda^I{}_Je^J, \quad \omega'^I{}_J=\Lambda^I{}_K\omega^K{}_L\Lambda^{-1}{}_J{}^L-d\Lambda^I{}_K\Lambda^{-1}{}^K{}_J, \tag{E.18}$$

we obtain

$$\begin{aligned} e'^0 &= \Omega\left(\cosh(\alpha)dT+\sinh(\alpha)dX\right)\\ e'^1 &= \Omega\left(\sinh(\alpha)dT+\cosh(\alpha)dX\right)\\ e'^2 &= e^2 \qquad (E.19)\\ e'^3 &= e^3, \qquad (E.20)\end{aligned}$$

and

$$
\begin{aligned}
\omega'^{01} &= -d\alpha(X) + d\log(\Omega(X,T)) \\
\omega'^{02} &= \frac{\partial_T r\cosh(\alpha) - \partial_X r\sinh(\alpha)}{\Omega} d\theta \\
\omega'^{03} &= \frac{\partial_T r\cosh(\alpha) - \partial_X r\sinh(\alpha)}{\Omega} \sin(\theta) d\phi \\
\omega'^{12} &= \frac{\partial_T r\sinh(\alpha) - \partial_X r\cosh(\alpha)}{\Omega} d\theta \\
\omega'^{13} &= \frac{\partial_T r\sinh(\alpha) - \partial_X r\cosh(\alpha)}{\Omega} \sin(\theta) d\phi \\
\omega'^{23} &= -\cos(\theta) d\phi.
\end{aligned}
\tag{E.21}
$$

Now, we compute the surface gravity after performing the Lorentz transformation on ℓ^a by projecting (E.16) along n^a, we also should remember that $\omega^{IJ}{}_a = e^I{}_b \nabla_a e^{Jb}$ and that everything should be evaluated on the horizon. The result is

$$
\begin{aligned}
\ell^a n_b \nabla_a \ell^b &= -\kappa_{IH} \\
\ell^a \omega'^{01}{}_a &= \kappa_{IH} \\
-\frac{(e^{-\alpha(X)})'}{\sqrt{2}} &= \kappa_{IH} = \frac{1}{4M} \\
e^{\alpha(X)} &= -\frac{2\sqrt{2}M}{X},
\end{aligned}
\tag{E.22}
$$

which is equivalent to (see equation A19 in [40])

$$
-\frac{1}{\sqrt{2}} = \frac{2X}{F'\Omega} e^{\alpha(X)} = \frac{\partial_X r}{\Omega} e^{\alpha(X)}. \tag{E.23}
$$

Before continuing, previous condition should be replaced, considering (E.10), in (E.21). At this point we are ready to compute the curvature in Ashtekar-Barbero variables defined by the connection

$$
A^i = \Gamma^i + \gamma K^i = -\frac{1}{2}\varepsilon^i{}_{jk}\omega'^{ij} + \gamma\omega'^{0i}, \tag{E.24}
$$

where the indexes $i, j, k = 1, 2, 3$. The curvature is

$$
F^i(A) = dA^i + \frac{1}{2}\varepsilon^i{}_{jk} A^j \wedge A^k. \tag{E.25}
$$

The pullback to the horizon worldsheet 3-surface is

$$
\underleftarrow{dr} = 0 \quad \longrightarrow \quad \underleftarrow{dT} = \underleftarrow{dX}. \tag{E.26}
$$

The pullback to the horizon 2-surfaces (with a fixed foliation) is

$$\underset{\Leftarrow}{dT} = \underset{\Leftarrow}{dX} = 0. \tag{E.27}$$

Finally, the pullback to the horizon of the curvature of the Ashtekar-Barbero variables is

$$\begin{aligned}
\underset{\Leftarrow}{F}^1(A) &= -\frac{1}{2}(1-\gamma^2)\sin(\theta)d\theta \wedge d\phi && \text{(E.28)}\\
\underset{\Leftarrow}{F}^2(A) &= 0 && \text{(E.29)}\\
\underset{\Leftarrow}{F}^3(A) &= 0, && \text{(E.30)}
\end{aligned}$$

we also have that $\underset{\Leftarrow}{e}^0 = \underset{\Leftarrow}{e}^1 = 0$, then $\Sigma^i = \varepsilon^i{}_{jk} e^j \wedge e^k$ becomes

$$\begin{aligned}
\underset{\Leftarrow}{\Sigma}^1 &= r^2 d\theta \wedge d\phi && \text{(E.31)}\\
\underset{\Leftarrow}{\Sigma}^2 &= 0 && \text{(E.32)}\\
\underset{\Leftarrow}{\Sigma}^3 &= 0, && \text{(E.33)}
\end{aligned}$$

and we conclude that

$$\underset{\Leftarrow}{F}^i(A) = -\frac{\pi(1-\gamma^2)}{a_H}\underset{\Leftarrow}{\Sigma}^i, \tag{E.34}$$

where we have introduced the area of the spherical horizon $a_H = 4\pi r^2$.

An important remark is that in this adapted framework to the spherically symmetric horizon, we have

$$\underset{\Leftarrow}{K}^i = \sqrt{\frac{2\pi}{a_H}}\,\underset{\Leftarrow}{e}^i \ , \tag{E.35}$$

from which is trivial to prove that

$$\varepsilon^i{}_{jk}\underset{\Leftarrow}{K}^j \wedge \underset{\Leftarrow}{K}^k = \frac{2\pi}{a_H}\underset{\Leftarrow}{\Sigma}^i, \tag{E.36}$$

this last equation implies that at the horizon there is the freedom to choose Ashtekar-Barbero-like variables with a parameter $\bar{\gamma}$ independent of the Immirzi parameter of the bulk variables (the proof is in Chapter 4). In the equivalent framework for the Kerr solution (rotating black hole) this equation is not valid any more. In particular, this prevent us from being able to prove (E.36) for rotating Isolated Horizons.

Appendix F

QUANTUM GROUP RECOUPLING THEORY: THE INTERTWINER SPACE

Here we reproduce in a minimal way the computation of the dimension of the Hilbert space $\mathscr{H}^k_{CS}$. We follow [50]. Specifically, we will compute explicitly the dimension of the space given by the invariant tensor part of the tensor product of p of the $SU(2)_q$ representations

$$N_k(\mathbf{j}) = \dim[\mathrm{Inv}(j_1 \otimes \cdots \otimes j_p)]. \tag{F.1}$$

with $\mathbf{j} = (j_1, \cdots, j_p)$.

The dimension of the Hilbert space is

$$N_k(\mathbf{j}) = \sum_{\ell_1, \cdots, \ell_p} \delta_{\ell_1,0}\delta_{\ell_{p+1},0} \prod_{i=1}^{p} Y(\ell_i, j_i, \ell_{i+1}), \tag{F.2}$$

where $Y(j_1, j_2, j_3) \in \{0, 1\}$ is one when (j_1, j_2, j_3) satisfy the triangular relation or zero otherwise.

The q-numbers are defined by

$$[x] = \frac{q^x - q^{-x}}{q - q^{-1}} = \frac{\sin\left(\frac{\pi}{k+2}x\right)}{\sin\left(\frac{\pi}{k+2}\right)}, \tag{F.3}$$

where in the second equality we have used the specific value of $q(k) = \exp\left(\frac{i\pi}{k+2}\right)$.

The Verlinde coefficients are defined by

$$S_{j_1 j_2} = [d_{j_1} d_{j_2}] = \frac{\sin\left(\frac{\pi d_{j_1} d_{j_2}}{k+2}\right)}{\sin\left(\frac{\pi}{k+2}\right)}, \tag{F.4}$$

with $d_{j_i} = 2j_i + 1$. The Verlinde coefficients satisfy two important properties that we state without proof. The *orthogonality relation*

$$\sum_{j_3} S_{j_1 j_3} S_{j_3 j_2} = \frac{\delta_{j_1 j_2}}{Z^2}, \quad \text{with } Z = \sqrt{\frac{2}{k+2}} \sin\left(\frac{\pi}{k+2}\right), \tag{F.5}$$

and the *recursive relation*, given by

$$\prod_{i=1}^{n-1} S_{j_i j_n} = [d_{n_j}]^{n-2} \sum_{\ell_1,\cdots,\ell_n} \delta_{\ell_1,0} \prod_{i=1}^{n-1} Y(j_i,\ell_i,\ell_{i+1}) S_{\ell_i j_n}, \tag{F.6}$$

using the two previous equations on the expression for $N(\mathbf{d})$ (with $\mathbf{d} = (d_1,\cdots,d_p)$, and $d_{j_i} = 2j_i+1$ is a simply change of notation), we have

$$N_k(\mathbf{d}) = \frac{2}{k+2}\sin^2\left(\frac{\pi}{k+2}\right)\sum_\ell [d_\ell]^{2-p}\prod_{i=1}^{p} S_{j_i\ell}, \tag{F.7}$$

by using the definitions of $S_{j_i\ell}$ and $[\cdot]$ explicitly we have

$$N_k(\mathbf{d}) = \frac{2}{k+2}\sum_{d=1}^{k+1}\sin^{2-p}\left(\frac{\pi d}{k+2}\right)\prod_{i=1}^{p}\sin\left(\frac{\pi d d_{j_i}}{k+2}\right). \tag{F.8}$$

This is already a useful closed formula. Now, we go a step further to obtain an integral expression useful for the asymptotic application we are interested in.

Let us use a redefinition $D_k(\mathbf{d}) = N_{k-2}(\mathbf{j})$, such that

$$D_k(\mathbf{d}) = \frac{2}{k}\sum_{d=1}^{k-1}\sin^2\left(\frac{\pi d}{k}\right)\prod_{i=1}^{p}\frac{\sin\left(\frac{\pi d d_{j_i}}{k}\right)}{\sin\left(\frac{\pi d_\ell}{k}\right)}. \tag{F.9}$$

Let use the trigonometric identities $\sin^2\theta = 1-\cos^2\theta$, $\cos\theta = \frac{\sin(2\theta)}{2\sin^2\theta}$; and define the function

$$B_k(\mathbf{d}) = \frac{2}{k}\sum_{d=0}^{k-1}\prod_{\ell=1}^{p}\frac{\sin\left(\frac{\pi d}{k}d_\ell\right)}{\sin\left(\frac{\pi d}{k}\right)}, \tag{F.10}$$

then, we can rewrite

$$D_k(\mathbf{d}) = B_k(\mathbf{d}) - \frac{1}{4}B_k(\mathbf{d}_+), \tag{F.11}$$

with $\mathbf{d}_+ = (d_1,\cdots,d_p,2,2)$. Note the identity

$$\frac{\sin\left(\frac{\pi d}{k}d_\ell\right)}{\sin\left(\frac{\pi d}{k}\right)} = e^{i\frac{\pi d}{k}(d_\ell-1)}\sum_{n_\ell=0}^{d_\ell-1} e^{-2i\frac{\pi d}{k}n_\ell}, \tag{F.12}$$

it allows us to write the function $B_k(\mathbf{d})$ as

$$B_k(\mathbf{d}) = \frac{2}{k}\sum_{d=0}^{k-1}\prod_{\ell=1}^{p}\sum_{n_\ell=0}^{d_\ell-1} e^{i\frac{\pi d}{k}(d_\ell-1-2n_\ell)} = \frac{2}{k}\sum_{d=0}^{k-1}\sum_{\{n_1,\cdots,n_p\}} e^{i\frac{\pi d}{k}(\Delta_p-2N)}, \tag{F.13}$$

with the definitions $\Delta_p = \sum_{\ell=1}^{p}(d_\ell - 1)$ and $N = \sum_{\ell=1}^{p} n_\ell$. The sums are finite and the interchange of the sum ordering is allowed

$$B_k(\mathbf{d}) = \frac{2}{k} \sum_{\{n_1,\cdots,n_p\}} \frac{1 - e^{i\pi(\Delta_p - 2N)}}{1 - e^{i\frac{\pi}{k}(\Delta_p - 2N)}} = \frac{2}{k} \sum_{\{n_1,\cdots,n_p\}} \frac{1 - e^{i\pi\Delta_p}}{1 - e^{i\frac{\pi}{k}(\Delta_p - 2N)}}, \tag{F.14}$$

from this formula a combinatorial form can be obtained, first note that if Δ_p is odd a further analysis shows that $D_k(\mathbf{d})$ vanishes. If Δ_p is even the sum is not zero because the denominator can also vanishes, this is equivalent to

$$B_k(\mathbf{d}) = 2 \sum_{\{n_1,\cdots,n_p\}} \delta_{\Delta_p - 2N[2k]}, \tag{F.15}$$

the quantity $\delta_{n[k]}$ is one if there exists an integer s such that $n = sk$ or zero otherwise.

Let us use the half integers given by $m_i = n_i - \frac{d_i - 1}{2} \in [\frac{1-d_i}{2}, \frac{1+d_i}{2}]$, then

$$B_k(\mathbf{d}) = 2 \sum_{\{m_1,\cdots,m_p\}} \delta_{m_1 + \cdots + m_p[k]}, \tag{F.16}$$

or, more explicitly, we use the floor function $[\cdot]$, to define

$$r \equiv \left[\frac{\Delta_p}{2k}\right] = \left[\frac{\sum_\ell (d_\ell - 1)}{2k}\right], \tag{F.17}$$

and we get a simpler expression

$$B_k(\mathbf{d}) = 2 \sum_{\{m_1,\cdots,m_p\}} \sum_{q=-r}^{r} \delta_{m_1 + \cdots + m_p - qk}. \tag{F.18}$$

Now, we replace this on the main formula for $D_k(\mathbf{d})$

$$D_k(\mathbf{d}) = \sum_{\{m_1,\cdots,m_p\}} \left(\sum_{q=-r}^{r} \delta_{m_1+\cdots+m_p-qk} - \frac{1}{4} \sum_{a,b \in \{-\frac{1}{2},\frac{1}{2}\}} \sum_{q=-s}^{s} \delta_{m_1+\cdots+m_p+a+b-qk} \right), \tag{F.19}$$

with $s = \left[\frac{\Delta_p + 2}{2k}\right]$.

Here we make a simplifying assumption: We have that $s \in \{r, r+1\}$, thus, we simply assume $r = s$ because as in our application k is a big number $s = r + 1$ will not happens. Therefore, we have

$$D_k(\mathbf{d}) = \sum_{\{m_1,\cdots,m_p\}} \sum_{q=-r}^{r} \left(\delta_{m_1+\cdots+m_p-qk} - \frac{1}{2}\delta_{m_1+\cdots+m_p-qk+1} - \frac{1}{2}\delta_{m_1+\cdots+m_p-qk-1} \right). \tag{F.20}$$

To get at the integral formula we use the integral version of the Kronecker delta

$$\delta_{m_1+\cdots+m_p+a} = \frac{1}{2\pi}\int_0^{2\pi} d\theta\ e^{i\theta(m_1+\cdots+m_p+a)}, \tag{F.21}$$

and because $m_i \in [\frac{1-d_\ell}{2}, \frac{1+d_\ell}{2}]$, we have

$$\sum_{\{m_1,\cdots,m_p\}} \delta_{m_1+\cdots+m_p+a} = \frac{1}{2\pi}\int_0^{2\pi} d\theta\ \cos(a\theta)\prod_{\ell=1}^{p}\frac{\sin\left(d_\ell\frac{\theta}{2}\right)}{\sin\frac{\theta}{2}}, \tag{F.22}$$

and

$$\sum_{q=-r}^{r}\cos(\theta qk) = \frac{\sin\left((r+\frac{1}{2})k\theta\right)}{\sin\frac{k\theta}{2}}, \tag{F.23}$$

then we get the desired formula

$$\begin{aligned} D_k(\mathbf{d}) &= \frac{1}{\pi}\int_0^{2\pi} d\theta \sin^2\left(\frac{\theta}{2}\right)\frac{\sin\left((r+\frac{1}{2})k\theta\right)}{\sin\frac{k\theta}{2}}\prod_{\ell=1}^{p}\frac{\sin\left(d_\ell\frac{\theta}{2}\right)}{\sin\frac{\theta}{2}} \\ &= \frac{2}{\pi}\int_0^{\pi} d\theta \sin^2(\theta)\frac{\sin\left((2r+1)k\theta\right)}{\sin(k\theta)}\prod_{\ell=1}^{p}\frac{\sin\left(d_\ell\theta\right)}{\sin(\theta)}, \end{aligned} \tag{F.24}$$

this is the starting point to study the dimension of the Hilbert space $\mathscr{H}^k_{CS}$. We remember that this is the dimension of the space of the invariant tensors of the tensor space obtained out of the coupling (tensor product) of p-punctures. Were each puncture carries a j_i quantum $SU(2)$ irreducible representation. The physically interesting consequences of this model should be obtained on the regime where all this microscopic structure mimics a semiclassical black hole.

Symbols

Chapter 2

Γ	:	phase space
$\Omega(\delta_1, \delta_2)$	:	Symplectic structure
H_t	:	Hamiltonian flow by δ_t
$\mathcal{M}$	:	four dimensional manifold
M	:	Cauchy surface
Δ	:	Isolated Horizon
S_Δ	:	topological 2-sphere on an Isolated Horizon
H	:	topological 2-sphere on a Killing Horizon
$\mathcal{O}$	:	near horizon observers
$\mathcal{W}(\mathcal{O})$	:	worldsheet for near horizon observers
S	:	black hole entropy
M	:	black hole mass
E	:	black hole energy defined by quasilocal observers
A	:	area of the black hole horizon
J	:	black hole angular momentum
$a = J/M$	:	dimensionless angular momentum
Q	:	black hole charge
T_H	:	Hawking black hole temperature
Ω_H	:	horizon angular velocity w.r.t a static observer at infinity
ϕ_H	:	electric potential at the horizon (w.r.t. a zero potential fixed at infinity)
ℓ	:	proper length to the horizon
κ	:	surface gravity
$\bar{\kappa}$	:	local surface gravity
κ_{IH}	:	surface gravity for isolated horizons, depends on ℓ^a
θ	:	expansion
σ^{ab}	:	shear
ω^{ab}	:	twist
$[\ell]$ or $[\chi]$	:	class of null vectors defined over Δ
ξ^a	:	time symmetry Killing field
ϕ^a	:	axial symmetry Killing field

Chapter 3

ϕ :	a generic field variable
$S[\phi]$:	action
$S_E[\phi]$:	Wick rotated action
Z_e :	partition function (Euclidean)
Z_s :	partition function (statistical)
$\mathcal{W}(\mathcal{O})$:	worldsheet for near horizon observers
E_{loc} :	quasilocal energy

Chapter 4

$\kappa = 8\pi G$:	with G Newton constant
γ :	Immirzi parameter
$\bar{\gamma}$:	Immirzi-like parameter for variables at the horizon
$\Omega_M(\delta_1, \delta_2)$:	symplectic structure
M :	Cauchy surface
Δ :	Isolated Horizon worldsheet, three dimensional surface
$H = \Delta \cap M$:	horizon surface
e^I :	tetrad 1-form
${\omega^I}_J$:	$SO(3,1)$ connection 1-form
$\Sigma^i = {\varepsilon^i}_{jk} e^j \wedge e^k$:	densitized triad 2-form
$K^i = \omega^{0i}$:	extrinsic curvature 1-form
$\Gamma^i = -\frac{1}{2}{\varepsilon^i}_{jk}\omega^{jk}$:	$SO(3)$ connection on M, 1-form
$A^i = \Gamma^i + \gamma K^i$:	Ashtekar-Barbero connection
$\bar{A}^i = \Gamma^i + \bar{\gamma} K^i$:	Ashtekar-Barbero-like connection at the horizon
a_H :	horizon area
$\underset{\leftarrow}{}$:	pullback to the Cauchy three-surface M
$\underset{\Leftarrow}{}$:	pullback to the horizon two-surface H
$\mathscr{G}$:	graph
Φ :	scalar variable
$\mathcal{J}$:	2-form representing angular momentum density
J :	angular momentum
$j = J/\ell_p^2$:	dimensionless angular momentum

Chapter 5

$\mathscr{H}_{CS}^{k}$	:	Hilbert space of the quantum Chern-Simons horizon model
a_H	:	horizon area
$A = a_H/\ell_p^2$	:	dimensionless horizon area
$a = a_H/(8\pi\gamma\ell_p^2)$	:	normalized area

References

[1] S. W. Hawking, "Particle Creation By Black Holes," Commun. Math. Phys. **43** (1975) 199.

[2] R. M. Wald, "General Relativity," Chicago, Usa: Univ. Pr. (1984) 491p.

[3] C. Rovelli, F. Vidotto, "Introduction to Loop Quantum Gravity" *in preparation.*

[4] E. Frodden, A. Ghosh, A. Perez, "Quasilocal first law for black hole thermodynamics," Phys. Rev. D 87, 121503(R) (2013), arXiv:gr-qc/1110.4055.

[5] E. Frodden, K. Noui and A. Perez "Black hole entropy for rotating black holes in LQG" *in preparation.*

[6] A. Ashtekar, V. Taveras, M. Varadarajan and , "Information is Not Lost in the Evaporation of 2-dimensional Black Holes" Phys. Rev. Lett. **100** (2008) 211302, arXiv:gr-qc/0801.1811.

[7] A. Ashtekar, F. Pretorius and F. Ramazanoglu, "Surprises in the Evaporation of 2-Dimensional Black Holes" Phys. Rev. Lett. **106**, 161303 (2011), arXiv:gr-qc/1011.6442.

[8] B. Krishnan, "Quasi-local black hole horizons" arXiv:gr-qc/1303.4635.

[9] A. Ashtekar, C. Beetle, O. Dreyer, S. Fairhurst, B. Krishnan, J. Lewandowski, J. Wisniewski, "Isolated horizons and their applications" Phys. Rev. Lett. **85** (2000) 3564-3567, arXiv:gr-qc/0006006.

[10] D. Sudarsky and R. M. Wald, "Extrema of mass, stationarity, and staticity, and solutions to the Einstein Yang-Mills equations" Phys. Rev. D **46** (1992) 1453.

[11] B. Carter, "Black Holes, Les Astres Occlus" edited by C. DeWitt and B. S. DeWitt, Gordon and Breach Science Publishers (1973).

[12] A. Ashtekar, S. Fairhurst, B. Krishnan and , "Isolated horizons: Hamiltonian evolution and the first law" Phys. Rev. D **62** (2000) 104025, arXiv:gr-qc/0005083.

[13] A. Ashtekar, C. Beetle, J. Lewandowski, "Geometry of generic isolated horizons" Class. Quant. Grav. **19**, 1195-1225 (2002), arXiv:gr-qc/0111067.

[14] A. Ashtekar, C. Beetle, and J. Lewandowski, "Mechanics of rotating isolated horizons" Phys. Rev. D **64** (2001) 044016, arXiv:gr-qc/0103026.

[15] J. Engle, K. Noui, A. Perez, and D. Pranzetti, "Black hole entropy from an SU(2)-invariant formulation of Type I isolated horizons" Phys. Rev. D **82**, 044050 (2010), arXiv:gr-qc/10060634.

[16] V. P. Frolov and K. S. Thorne, "Renormalized Stress - Energy Tensor Near The Horizon Of A Slowly Evolving, Rotating Black Hole" Phys. Rev. D **39**, 2125 (1989).

[17] A. Ghosh and A. Perez, "Black hole entropy and isolated horizons thermodynamics" Phys. Rev. Lett. **107** (2011) 241301, arXiv:gr-qc/11071320.

[18] A. Ghosh and A. Perez, "The scaling of black hole entropy in loop quantum gravity" arXiv:gr-qc/12102252.

[19] S. W. Hawking and G. F. R. Ellis, "The Large scale structure of space-time" Cambridge University Press, Cambridge, 1973.

[20] R. M. Wald, "Quantum Field Theory in Curved Spacetime and Black Hole Thermodynamics" Chicago, Usa: Univ. Pr. (1994) 202p.

[21] E. Poisson, "A Relativist's Toolkit: The Mathematics of Black-Hole Mechanics" Cambridge Univ. Pr. (2004) 233p.

[22] A. J. Amsel, D. Marolf and A. Virmani, "The Physical Process First Law for Bifurcate Killing Horizons" Phys. Rev. D **77** (2008) 024011, arXiv:gr-qc/0708.2738.

[23] K. S. Thorne, (Ed.), R. H. Price, (Ed.), D. A. Macdonald, (Ed.), "Black Holes: The Membrane Paradigm" NEW HAVEN, USA: YALE UNIV. PR. (1986) 367p.

[24] B. Krishnan, "The spacetime in the neighborhood of a general isolated black hole" Class. Quant. Grav. **29** (2012) 205006, arXiv:gr-qc/1204.4345.

[25] J. Lewandowski and T. Pawlowski, “Geometric characterizations of the Kerr isolated horizon” Int. J. Mod. Phys. D **11** (2002) 739, arXiv:gr-qc/0101008.

[26] C. W. Misner, K. S. Thorne and J. A. Wheeler, “Gravitation” San Francisco 1973, 1279p.

[27] R. K. Pathria, “Statistical Mechanics”.

[28] W. G. Unruh, “Notes on black hole evaporation” Phys. Rev. D **14** (1976) 870.

[29] G. W. Gibbons and S. W. Hawking, “Action Integrals and Partition Functions in Quantum Gravity” Phys. Rev. D **15**, 2752 (1977).

[30] S. W. Hawking, “Euclidean Quantum Gravity” NATO Adv. Study Inst. Ser. B Phys. **44** (1979) 145.

[31] M. Nakahara, “Geometry, topology and physics” Boca Raton, USA: Taylor Francis (2003) 573 p.

[32] J. W. York, Jr., “Role of conformal three geometry in the dynamics of gravitation” Phys. Rev. Lett. **28** (1972) 1082.

[33] G. W. Gibbons, “The Einstein Action of Riemannian Metrics and Its Relation to Quantum Gravity and Thermodynamics” Phys. Lett. A **61** (1977) 3.

[34] M. Bañados, “Notes on black holes and three-dimensional gravity” AIP Conf. Proc. **490** (1999) 198, arXiv:hep-th/9903244.

[35] S. N. Solodukhin, “Entanglement entropy of black holes” Living Rev. Rel. **14** (2011) 8, arXiv:hep-th/1104.3712.

[36] D. V. Fursaev and S. N. Solodukhin, “On the description of the Riemannian geometry in the presence of conical defects” Phys. Rev. D **52** (1995) 2133, arXiv:hep-th/9501127.

[37] S. W. Hawking and C. J. Hunter, “The Gravitational Hamiltonian in the presence of nonorthogonal boundaries” Class. Quant. Grav. **13** (1996) 2735, arXiv:gr-qc/9603050.

[38] S. Carlip and C. Teitelboim, “The Off-shell black hole” Class. Quant. Grav. **12** (1995) 1699, arXiv:gr-qc/9312002.

[39] A. Ashtekar, J. C. Baez, and K. Krasnov, “Quantum geometry of isolated horizons and black hole entropy” Adv. Theor. Math. Phys. **4** (2000) 1, arXiv:gr-qc/0005126.

[40] J. Engle, A. Perez, K. Noui, and D. Pranzetti, “Black hole entropy and SU(2) Chern-Simons theory” Phys. Rev. Lett. **105** (2010) 031302 arXiv:gr-qc/0905.3168.

[41] R. M. Wald and A. Zoupas, “A General definition of ’conserved quantities’ in general relativity and other theories of gravity” Phys. Rev. D **61** (2000) 084027, arXiv:gr-qc/9911095.

[42] A. Perez, “Introduction to loop quantum gravity and spin foams” arXiv:gr-qc/0409061.

[43] C. Rovelli, “Quantum gravity” Cambridge, UK: Univ. Pr. (2004) 455 p.

[44] A. Corichi and E. Wilson-Ewing, “Surface terms, Asymptotics and Thermodynamics of the Holst Action” Class. Quant. Grav. **27**, 205015 (2010), arXiv:gr-qc/1005.3298.

[45] T. Thiemann, “Modern canonical quantum general relativity” Cambridge, UK: Cambridge Univ. Pr. (2007) 819 p, arXiv:gr-qc/0110034.

[46] J. Zanelli, “Lecture notes on Chern-Simons (super-)gravities. Second edition (February 2008)” arXiv:hep-th/0502193.

[47] J. Zanelli, “Chern-Simons Forms in Gravitation Theories” Class. Quant. Grav. **29** (2012) 133001, arXiv:hep-th/1208.3353.

[48] S. Alexandrov, M. Geiller and K. Noui, “Spin Foams and Canonical Quantization” SIGMA **8** (2012) 055, arXiv:gr-qc/1112.1961.

[49] H. Sahlmann and T. Thiemann, “Chern-Simons expectation values and quantum horizons from LQG and the Duflo map” Phys. Rev. Lett. **108** (2012) 111303, arXiv:gr-qc/1109.5793.

[50] J. Engle, K. Noui, A. Perez, and D. Pranzetti, “The SU(2) Black Hole entropy revisited” JHEP **1105**, 016 (2011), arXiv:gr-qc/1103.2723.

[51] A. Ashtekar, J. Engle and C. Van Den Broeck, “Quantum horizons and black hole entropy: Inclusion of distortion and rotation” Class. Quant. Grav. **22** (2005) L27, arXiv:gr-qc/0412003.

[52] A. Perez and D. Pranzetti, "Static isolated horizons: SU(2) invariant phase space, quantization, and black hole entropy" Entropy **13**, 744 (2011), arXiv:gr-qc/10112961.

[53] C. Röken, "SU(2) Connection Variable Formulations of Kerr Isolated Horizon Geometries for Loop Quantum Gravity" *In preparation.*

[54] K. Krasnov, "Quanta of geometry and rotating black holes" Class. Quant. Grav. **16** (1999) L15, arXiv:gr-qc/9902015.

[55] E. Bianchi, "Black Hole Entropy, Loop Gravity, and Polymer Physics" Class. Quant. Grav. **28** (2011) 114006, arXiv:gr-qc/10115628.

[56] E. R. Livine and D. R. Terno, "Entropy in the Classical and Quantum Polymer Black Hole Models" arXiv:gr-qc/12055733.

[57] S. W. Hawking, G. T. Horowitz and S. F. Ross, "Entropy, area, and black hole pairs" Phys. Rev. D **51** (1995) 4302, arXiv:gr-qc/9409013.

[58] A. Ashtekar, J. Baez, A. Corichi, and K. Krasnov, "Quantum geometry and black hole entropy" Phys. Rev. Lett. **80** (1998) 904, arXiv:gr-qc/9710007.

[59] T. Jacobson, D. Marolf and C. Rovelli, "Black hole entropy: Inside or out?" Int. J. Theor. Phys. **44** (2005) 1807, arXiv:hep-th/0501103.

[60] J. D. Brown and M. Henneaux, "Central Charges in the Canonical Realization of Asymptotic Symmetries: An Example from Three-Dimensional Gravity" Commun. Math. Phys. **104** (1986) 207.

[61] A. Strominger, "Black hole entropy from near horizon microstates" JHEP **9802** (1998) 009, arXiv:hep-th/9712251.

[62] M. Guica, T. Hartman, W. Song and A. Strominger, "The Kerr/CFT Correspondence" Phys. Rev. D **80** (2009) 124008, arXiv:hep-th/0809.4266.

[63] C. Rovelli, "Black hole entropy from loop quantum gravity" Phys. Rev. Lett. **77** (1996) 3288, arXiv:gr-qc/9603063.

[64] A. Sen, "Logarithmic Corrections to Schwarzschild and Other Non-extremal Black Hole Entropy in Different Dimensions" arXiv:hep-th/1205.0971.

[65] V. Chari and A. Pressley, "A guide to quantum groups" Cambridge, UK: Univ. Pr. (1994) 651 p.

[66] C. Rovelli and L. Smolin, "Discreteness of area and volume in quantum gravity" Nucl. Phys. B **442** (1995) 593 [Erratum-ibid. B **456** (1995) 753], arXiv:gr-qc/9411005.

[67] E. Frodden, M. Geiller, K. Noui and A. Perez, "Black Hole Entropy from complex Ashtekar variables" arXiv:gr-qc/1212.4060.

[68] A. Ashtekar, "Lectures on nonperturbative canonical gravity" Adv. Ser. Astrophys. Cosmol. **6** (1991) 1.

[69] A. Pérez, "The Spin Foam Approach to Quantum Gravity" arXiv:gr-qc/1205.2019.

[70] K. Miller and B. Ross, "An Introduction to the fractional calculus and fractional differential equations" John Wiley & Sons, Inc. (1993) 364p.

[71] E. Bianchi, "Entropy of Non-Extremal Black Holes from Loop Gravity" arXiv:gr-qc/1204.5122.

[72] C. Crnkovic and E. Witten, in 'Three hundred years of gravitation'; ed. S. Hawking, W. Israel. J. Lee and R. M. Wald, "Local symmetries and constraints" J. Math. Phys. **31** (1990) 725.

[73] R. A. Bertlmann, "Anomalies in quantum field theory" Oxford, UK: Clarendon (1996) 566 p. (International series of monographs on physics: 91).

[74] P. Peldan, "Actions for gravity, with generalizations: A Review" Class. Quant. Grav. **11** (1994) 1087, arXiv:gr-qc/9305011.

Abstract

Black holes are studied from a theoretical point of view. The thermodynamics and quantum properties are addressed from a new perspective. A range of logically connected problems are explored: Starting from the laws of black hole mechanics, going through the Euclidean partition function, to the microscopic quantum granular models.
The approach is supported by two guiding principles: What is physically relevant for black hole thermodynamics lays close to the horizon and the quantum geometry of the spacetime is coarse-grained.
The first law of black hole mechanics is reviewed from the new quasilocal perspective based on near horizon observers. It turns out that the first law can be reformulated as variations of the area of the horizon. On the same grounds, the semiclassical Euclidean partition function is reviewed from the new quasilocal perspective. The framework reproduces the classic Bekenstein-Hawking entropy and the newly introduced quasilocal energy.
The quasilocal approach can also be addressed by using Isolated Horizons. The quantization procedures are explored for the rotating Isolated Horizon starting from a symplectic structure analysis, and using the Loop Quantum Gravity Hilbert space.
Finally, through a statistical analysis, the macroscopic consequences of the quantum granular model based on the Loop Quantum Gravity approach are studied. Special emphasis is put on the rotating quantum black hole model, however the results are not conclusive as several assumptions should be made on the way. Nevertheless, the perspective is promising as some of the semiclassical results, for instance the entropy, can be reproduced.

Résumé

Les trous noirs sont étudiés d'un point de vue théorique. Les propriétés thermodynamiques et quantiques des trous noirs sont abordées à travers des nouvelles perspectives. On explore différents problèmes logiquement reliés: depuis les lois de la mécanique des trous noirs, en passant par la function partition Euclidienne des trous noirs, jusqu'aux modèles microscopiques quantiques et granulaires.
L'approche repose sur deux principes: la thermodynamique importante pour les trous noirs se situe près de l'horizon et la géométrie quantique de l'espace-temps est granuleuse.
On examine la première loi de la mécanique des trous noirs avec une perspective quasilocal basée sur des observateurs près de l'horizon. Il s'avère que la première loi peut être simplement reformulée comme la variation de l'aire de l'horizon. Ensuite, on examine la fonction de partition Euclidienne à partir de la nouvelle perspective quasilocal, et on reproduit l'entropie de Bekenstein-Hawking ainsi que l'energie quasilocal nouvellement introduite.
L'approche quasilocal peut être abordée par un point de vue basé sur les Horizons Isolés. Dans ce cadre, on explore la quantification de l'Horizon Isolé rotatoire, en analysant la structure symplectique, et en utilisant l'espace de Hilbert de la Gravitation Quantique à Boucles. Finalement, on étudie les conséquences macroscopiques du modèle quantique granulaire basé sur la Gravitation Quantique à Boucles. L'accent est mis sur le modèle de trou noir en rotation, les résultats ne sont pas concluants du fait que plusieurs hypothèses doivent être posées. Cependant, la perspective est prometteuse. Certains des résultats, comme l'entropie, peuvent être reproduits.