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Spacetime torsion

by

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Índice general

Introduction

We begin this introduction with a brief survey on Einstein-Cartan Theory, the underlying framework on which this thesis has been developed.

In Special Relativity Theory (SRT), the underlying Minkowski spacetime admits, as its group of automorphisms, the full Poincaré group, consisting of translations and Lorentz transformations. It follows from the first Noether theorem that classical, special relativistic field equations, derived from a variational principle, give rise to conservation laws of energy-momentum and angular momentum. Using Cartesian coordinates (x^μ), abbreviating $\varphi_{,\rho} \equiv \frac{\partial \varphi}{\partial x^\rho}$ and denoting by $\mathcal{T}^{\mu\nu}$ and $\mathcal{S}^{\mu\nu\rho} = -\mathcal{S}^{\nu\mu\rho}$ the tensors of energy-momentum and of intrinsic angular momentum (spin), respectively, one can write the conservation laws in the form

$$\mathcal{T}^{\mu\nu}{}_{,\nu} = 0 \tag{1}$$

and

$$(x^\mu \mathcal{T}^{\nu\rho} - x^\nu \mathcal{T}^{\mu\rho} + \mathcal{S}^{\mu\nu\rho})_{,\rho} = 0. \tag{2}$$

In the presence of spin, the tensor $\mathcal{T}^{\mu\nu}$ need not be symmetric,

$$\mathcal{T}^{\mu\nu} - \mathcal{T}^{\nu\mu} = \mathcal{S}^{\mu\nu\rho}{}_{,\rho}. \tag{3}$$

Belinfante and Rosenfeld have shown that the tensor

$$\mathfrak{T}^{\mu\nu} = \mathcal{T}^{\mu\nu} + \frac{1}{2}(\mathcal{S}^{\nu\mu\rho} + \mathcal{S}^{\nu\rho\mu} + \mathcal{S}^{\mu\rho\nu})_{,\rho} \tag{4}$$

is symmetric and its divergence vanishes.

In quantum theory, the irreducible, unitary representations of the Poincaré group correspond to elementary systems such as stable particles; these representations are labeled by the mass and spin.

In Einstein's General Relativity Theory (GRT), the spacetime M is curved; the Lorentz group - but not the Poincaré group - appears as the structure group acting on orthonormal frames in the tangent spaces of M. The energy-momentum tensor $\mathfrak{T}$ appearing on the right side of the Einstein equation is necessarily symmetric. In GRT there is no room for translations and the tensors $\mathcal{T}$ and $\mathcal{S}$.

By introducing torsion and relating it to $\mathcal{S}$, Cartan restored the role of the Poincaré group in relativistic gravity: this group acts on the affine frames in the tangent spaces of M. Curvature and torsion are the surface densities of Lorentz transformations and translations, respectively. In a space with torsion, the Ricci tensor need not be symmetric so that an asymmetric energy-momentum tensor can appear on the right side of the Einstein equation.

Sciama and Kibble showed that the equation of motion for such a theory is

$$\tilde{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \tilde{R} = 8\pi \mathcal{T}^{\mathrm{eff}}_{\mu\nu}. \tag{5}$$

Here, $\tilde{R}_{\mu\nu}$ and $\tilde{R}$ are, respectively, the Ricci tensor and scalar formed from g. Neglecting indices one can write symbolically

$$\mathcal{T}^{\mathrm{eff}} = \mathfrak{T} + \mathcal{S}^2. \tag{6}$$

The symmetric tensor $\mathfrak{T}$ is

$$\mathfrak{T}^{\mu\nu} = \mathcal{T}^{\mu\nu} + \frac{1}{2} \tilde{\nabla}_\rho (\mathcal{S}^{\nu\mu\rho} + \mathcal{S}^{\nu\rho\mu} + \mathcal{S}^{\mu\rho\nu}). \tag{7}$$

It is remarkable that the Belinfante-Rosenfeld symmetrization of the canonical energy-momentum tensor appears as a natural consequence of Einstein-Cartan Theory (ECT). From the physical point of view, the second term on the right side of (6), can be thought of as providing a spin-spin contact interaction, reminiscent of the one appearing in the Fermi theory of weak interactions.

It is clear from (5), (6) and (7) that whenever terms quadratic in spin can be neglected - in particular in the linear approximation - ECT is equivalent to GRT. To obtain essentially new effects, the density of spin squared should be comparable to the density of mass. For example, to achieve this, a nucleon of mass m should be squeezed so that its radius r_{Cart} be such that

$$\left(\frac{\ell^2}{r^3_{\mathrm{Cart}}} \right)^2 \approx \frac{m}{r^3_{\mathrm{Cart}}}, \tag{8}$$

where $\ell^2 \approx 10^{-33}\,\mathrm{cm}$ is the Planck lenght (in general relativistic units so $G = c = 1$ and $\hbar = \ell^2$ so mass and energy are measured in centimeters). Introducing the Compton wavelength $r_{\mathrm{Compt}} = \frac{\ell^2}{m} \approx 10^{-13}\,\mathrm{cm}$, one can write

$$r_{\mathrm{Cart}} \approx (\ell^2 r_{\mathrm{Compt}})^{1/3}. \tag{9}$$

The "Cartan radius" of the nucleon, $r_{\mathrm{Cart}} \approx 10^{-26}\,\mathrm{cm}$, so small when compared to its physical radius under normal conditions, is much larger than the Planck length. Curiously enough, the energy ℓ^2/r_{Cart} is of the order of the energy at which, according to some estimates, the grand unification of interactions is presumed to occur.

Having said this let us summarize the content of our work:

- In Chapter I the chiral anomaly in the context of the Loop Quantum Gravity (LQG) canonical formulation of gravity with fermions has been studied. In particular we have focused our attention in the so called “non-minimal” coupling formulation for fermions, motivated mainly by recent investigations about how to reconcile the nonvanishing torsion tensor generated by the presence of fermionic degrees of freedom and the starting point of LQG, which is the Holst modification of the Hilbert-Palatini action. A brief comment on the Atiyah-Singer index theorem as a way of calculating the anomaly is made along with a naive ansatz for getting further “torsional topological invariants” in arbitrary even spacetime dimensions.

- In Chapter II we canonically analize gauge theories in the presence of spacetime torsion. In the literature it is usually assumed that gauge bosons cannot couple to spacetime torsion because it would spoil the gauge invariance of the action leading to a disaster at the quantum level. However, some 30 years ago Hojman et al. found a way of reconciling the two usually assumed **principles** of nature, namely, minimal coupling and gauge invariance. Here we arrive at the same conclusion in the Maxwell case and discuss a bit why this fails in the case of Yang-Mills theory but also how it can somehow be “cured”. We also show that the radiation equation of state is not modified in the presence of spacetime torsion. Finally we make a little analysis on how the “tlaplon” field (a dynamical source of torsion) could modify the current understanding of the behavior of LQG in the presence of spacetime torsion.

- In Chapter III we present our main proposal. It is known that fermions are not irreducible representations of $GL(4, R)$ but $SO(3, 1)$. This implies that in curved spacetime it is mandatory to use the vielbein formalism along with the equivalence principle in order to have a well defined Dirac operator. Thus we are led to a local gauge theory of gravity for the group $SO(3, 1)$ with a corresponding “spin gauge connection”. As Cartan understood, it is arbitrary that the vielbein (metric) will be the only independent field of the gravitational theory because the metric and affine properties of space need not be related. The spin connection should be taken more seriously since in analogy with electromagnetism, when a field couples to this connection it acquires a “charge” in the Noether sense. We show that this charge is nothing but the torsion generated by this coupling. The question then arises of why bosons could not do the same. All forms of matter generate spacetime curvature through their energy-momentum content as Einstein taught us. So why the generation of spacetime torsion should be an exclusive feature of fermions? We propose a new kind of bosonic fields ($(p-1)$-forms) in arbitrary dimensions that do generate torsion. Apart of being an academic exercise, we try to realize what would be the implications of having such fields in situations of physical interest. These “toy models” should not be taken so seriously as they are first approximations of what this mathematical framework has to offer.

- Finally we present several appendices in order to make the discussion self-contained and as a source of future reference.

Capítulo 1

Chiral Anomaly in Loop Quantum Gravity and spacetime torsion

1.1. Non-minimally coupled Dirac fermion and Holst action for gravity

Let us consider four-dimensional "Holst action" [1], which at the Hamiltonian level reduces to the Ashtekar-Barbero formalism for gravity [3][4][5]. Recall that in this setup for the gravitational interaction, its connection belongs to a $SU(2)$ algebra (universal covering of $SO(3)$). This connection is necessary in order to preserve the local gauge freedom of the triad under rotations in the foliation leaves of spacetime. Besides we add the contribution of a massless uncharged Dirac field which couples non-minimally to curved spacetime and creates nonvanishing torsion [2]. The total action is

$$\begin{aligned} S[e,\omega,\Psi] &= S_G[e,\omega] + S_F[e,\omega,\Psi] \\ &= \tfrac{1}{16\pi G}\left(\int_{\mathscr{M}} d^4x|e|e^\mu_I e^\nu_J R^{IJ}_{\mu\nu}(\omega) - \tfrac{1}{\gamma}\int_{\mathscr{M}} d^4x|e|e^\mu_I e^\nu_J {}^*R^{IJ}_{\mu\nu}(\omega)\right) \\ &+ \tfrac{i}{2}\int_{\mathscr{M}} d^4x|e|\left[\bar{\Psi}\gamma^I e^\mu_I\left(1-\tfrac{i}{\alpha}\gamma_5\right)\nabla_\mu\Psi - \overline{\nabla_\mu\Psi}\left(1-\tfrac{i}{\alpha}\gamma_5\right)\gamma^I e^\mu_I\Psi\right], \end{aligned} \tag{1.1}$$

where ω^{IJ}_μ is the Lorentz connection and $R^{IJ}_{\mu\nu}(\omega) = 2\partial_{[\mu}\omega^{IJ}_{\nu]} + [\omega_\mu,\omega_\nu]^{IJ}$ its curvature. ${}^*R^{IJ}_{\mu\nu} = \frac{1}{2}\epsilon^{IJ}{}_{KL}R^{KL}_{\mu\nu}$ is a Hodge dual and $\nabla_\mu \equiv \partial_\mu - \frac{i}{2}\omega^{IJ}_\mu\sigma_{IJ}$ is the Lorentz covariant derivative, where σ_{IJ} are the Lorentz algebra generators. In the spinorial representation $\sigma_{IJ} = \frac{i}{4}[\gamma_I,\gamma_J]$, so $\nabla_\mu = \partial_\mu + \frac{1}{4}\omega^{IJ}_\mu\gamma_{[I}\gamma_{J]}$ is the appropiate covariant derivative for fermions (with γ_I being the Dirac matrices in Minkowski spacetime) which is defined through $[\nabla_\mu,\nabla_\nu] = \frac{1}{4}R^{IJ}_{\mu\nu}\gamma_{[I}\gamma_{J]}$. $\gamma \in \Re$ is the so-called Immirzi parameter [6][7] of Loop Quantum Gravity (LQG)[8][9][10][11]. $\alpha \in \Re$ is a non-minimal coupling for fermions [2][12]. Minimal coupling is recovered taking $\alpha \to \infty$. We define $\not\nabla = \gamma^I e^\mu_I\nabla_\mu = \gamma^I e^\mu_I\partial_\mu + \frac{1}{4}\gamma^I e^\mu_I\omega^{JK}_\mu\gamma_{[J}\gamma_{K]}$.

1.2. Fujikawa method for the evaluation of the Anomaly

1.2.1. Minimal coupling

Let us follow the standard method due to Fujikawa for the evaluation of the Chiral Anomaly [15]. Gamma matrices in chiral representation are

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \tag{1.2}$$

so ${\gamma^0}^2 = 1$ and ${\gamma^i}^2 = -1$. Besides ${\gamma^0}^\dagger = \gamma^0$ and ${\gamma^i}^\dagger = -\gamma^i$ so ${\gamma^I}^\dagger = \gamma^0\gamma^I\gamma^0$, with $I = 0, 1, 2, 3$. We now perform the following Wick rotation in the local Lorentz frame:

$$e_0^\mu \to ie_4^\mu, \qquad e \to -ie, \qquad \gamma^0 \to -i\gamma^4, \qquad \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \gamma^4\gamma^1\gamma^2\gamma^3. \tag{1.3}$$

With such a change we get the "inner" metric $G^{IJ} = \mathrm{diag}(-1, -1, -1, -1)$. $\nabla\!\!\!/$ operator is Hermitian, $\nabla\!\!\!/^\dagger = \nabla\!\!\!/$, so we can consider a complete basis set $\varphi_n(x)$ which satisfies

$$\nabla\!\!\!/ \varphi_n(x) = \lambda_n \varphi_n(x), \tag{1.4}$$

$$\int d^4x |e| \varphi_n^\dagger(x) \varphi_m(x) = \delta_{n,m}. \tag{1.5}$$

Proof of Hermiticity is easy (see Appendix A).

Let us now consider local chiral transformations on the Dirac field

$$\begin{aligned} \Psi(x) \to& \Psi'(x) \equiv e^{i\epsilon(x)\gamma_5}\Psi(x), \\ \bar{\Psi}(x) \to& \bar{\Psi}'(x) \equiv \bar{\Psi}(x) e^{i\epsilon(x)\gamma_5}. \end{aligned} \tag{1.6}$$

We can expand these fields in the complete basis in the form

$$\Psi(x) \equiv \sum_n a_n \varphi_n(x) = \sum_n a_n \langle x|n\rangle \tag{1.7}$$

$$\bar{\Psi}(x) \equiv \sum_n \varphi_n^\dagger(x) \bar{b}_n = \sum_n \langle n|x\rangle \bar{b}_n, \tag{1.8}$$

where a_n and b_n are Grassmannian coefficients. Then,

$$\Psi'(x) \equiv \sum_n a'_n \varphi_n(x) = \sum_n a_n e^{i\epsilon(x)\gamma_5} \varphi_n(x),$$

so

$$\begin{aligned} a'_m &= \sum_n \int d^4x |e| \varphi_m^\dagger(x) e^{i\epsilon(x)\gamma_5} \varphi_n(x) a_n \\ &= \sum_n C_{m,n} a_n. \end{aligned} \tag{1.9}$$

According with Grassmannian nature, $\mathscr{D}\Psi'(x) = \prod_m da'_m = [\mathrm{det}C_{m,n}]^{-1}\prod_n da_n$ and $[\mathrm{det}C_{m,n}]^{-1} = \mathrm{det}[\delta_{m,n} + i\int d^4x|e|\epsilon(x)\varphi^\dagger_m(x)\gamma_5\varphi_n(x)]^{-1}$. Using Jacobi's formula, $\mathrm{det}X = \exp(\mathrm{Tr}(\ln X))$, we get that

$$
\begin{aligned}
[\mathrm{det}C_{m,n}]^{-1} &= \exp\left[-i\sum_n\int d^4x|e|\epsilon(x)\varphi^\dagger_n(x)\gamma_5\varphi_n(x)\right] \\
&\equiv \exp\left[-\frac{i}{2}\int d^4x\epsilon(x)\mathscr{A}(x)\right] \qquad (1.10)
\end{aligned}
$$

where the anomaly $\mathscr{A}(x)$ has been defined as

$$
\mathscr{A}(x) \equiv 2\sum_n |e(x)|\varphi^\dagger_n(x)\gamma_5\gamma_n(x) = 2\mathrm{Tr}\gamma_5\cdot\delta(0), \qquad (1.11)
$$

which is an ill-defined quantity. The Jacobian for $\mathscr{D}\bar{\Psi}$ gives an identical factor so $\mathscr{D}\mu \to \mathscr{D}\mu\exp[-i\int d^4x\epsilon(x)\mathscr{A}]$ where $\mathscr{D}\mu$ is the total integration measure of the path integral. With the standard regularization $\mathscr{A}(x)$ is

$$
\begin{aligned}
\mathscr{A}(x) &= \lim_{\beta\to 0} 2\sum_n |e(x)|\varphi^\dagger_n(x)\gamma_5\exp(\beta\lambda_n^2)\varphi_n(x) \\
&= \lim_{\beta\to 0}\lim_{x'\to x} 2\mathrm{Tr}\gamma_5|e|e^{\beta\not\nabla^2}\sum_n\varphi_n(x)\varphi_n(x')^\dagger. \qquad (1.12)
\end{aligned}
$$

The regulator β need not be taken to zero in order to regulate the trace. The reason is that for each nonzero eigenvalue of $\not\nabla^2$, there are two states of opposite chirality and therefore they cancel pairwise in the trace. The only remaining contribution comes from the zero modes and on those states the exponential of $\not\nabla^2$ is just the identity. Thus the anomaly equals the number of right-handed (ν_+) minus the number of left-handed (ν_-) zero modes, or

$$
\int\mathscr{A}(x) = \nu_+ - \nu_-, \qquad (1.13)
$$

so the anomaly is the index of the Dirac operator in the sense of the Atiyah-Singer theorem [16][17].

We recall that the commutator of two covariant derivatives for the group of diffeomorphisms of a manifold in a coordinate basis is

$$
[\nabla_\mu, \nabla_\nu]V^A = -T^\lambda{}_{\mu\nu}\nabla_\lambda V^A + R^A{}_{B\mu\nu}V^A, \qquad (1.14)
$$

where V^A represents any tensor (or spinor) under diffeomorpshims or under the group of tangent rotations, and $R^A{}_B$ is the curvature tensor in the corresponding representation [18]. Here curvature and torsion play quite different roles: $T^\lambda{}_{\mu\nu}$ is the "structure function" for the diffeomorphism group and $R^A{}_{B\mu\nu}$ is a "central charge". The square of the Dirac operator acting on a spinor is given by

$$
\begin{aligned}
\not\nabla^2\Psi &= \gamma^\mu\nabla_\mu(\gamma^\nu\nabla_\nu\Psi) \\
&= \gamma^I\gamma^J e^\mu_I e^\nu_J\nabla_\mu\nabla_\nu\Psi \\
&= (\gamma^{(I}\gamma^{J)} + \gamma^{[I}\gamma^{J]})e^\mu_I e^\nu_J\nabla_\mu\nabla_\nu\Psi \\
&= \nabla^\mu\nabla_\mu\Psi + \sigma^{IJ}e^\mu_I e^\nu_J[\nabla_\mu, \nabla_\nu]\Psi,
\end{aligned}
$$

so we have that [19][22]

$$\nabla\!\!\!/^2 = \nabla^\mu \nabla_\mu - e^\mu_I e^\nu_J e^\lambda_K \sigma^{IJ} T^K_{\mu\nu} \nabla_\lambda + \frac{1}{2} e^\mu_I e^\nu_J \sigma^{IJ} \sigma^{KL} R_{KL\mu\nu}. \tag{1.15}$$

The anomaly is

$$\mathscr{A}(x) = \lim_{y\to x} \lim_{\beta\to 0} 2\mathrm{Tr}[\gamma_5 \exp(\beta \nabla\!\!\!/^2)] \delta(x,y) \tag{1.16}$$

where $\delta(x,y)$ is the generalized Dirac delta in curved spacetime

$$\delta(x,y) = \int \frac{d^4k}{(2\pi)^4} e^{ik^\mu \nabla_\mu \Sigma(x,y)} \tag{1.17}$$

and $\Sigma(x,y)$ is the geodesic biscalar [23], a generalization of the quantity $\frac{1}{2}(x-y)^2$ in flat spacetime. $\Sigma(x,y)$ has the following properties:

1. $\Sigma(x,y) = \frac{1}{2} g^{\mu\nu}(x) \nabla_\mu \Sigma(x,y) \nabla_\nu \Sigma(x,y)$
2. $\Sigma(x,x) = 0$
3. $\lim_{y\to x} \nabla_\mu \nabla^\nu \Sigma(x,y) = g_\mu{}^\nu = \delta_\mu{}^\nu$.

The integral over the "wave vector" k^μ requires some careful handling. The spacetime manifold over which the anomaly is evaluated was taken to be a compact Euclidean space (e.g., S^4) with a typical lenght scale ℓ often called "the radius of the Universe" (this ensures that the tangent space symmetry $SO(4)$ can be embedded into $SO(5)$). Thus, k must be quantized in multiples of the inverse radius, $k^\mu \sim \frac{2\pi n^\mu}{\ell}$ with $n^\mu \in \mathbb{Z}$. Now, since ℓ is supposed to be very large, the wave vectors k^μ can be approximated by a continuous variable. This means that the integrations over k yield inverse powers of ℓ, which we normalize as

$$\int \frac{d^4k}{(2\pi)^4} = \ell^{-4}, \qquad \int \frac{d^4k}{(2\pi)^4} k^\mu k^\nu = \ell^{-6} g^{\mu\nu}, \qquad \text{etc.} \tag{1.18}$$

Applying the operator $\exp(\beta\nabla\!\!\!/^2)$ on the delta, taking the limit $y \to x$ and tracing over spinor indices, one finds

$$\mathscr{A}_\beta = \frac{1}{8\pi^2}\left[-2\left(\frac{\beta}{\ell^2}\right)\ell^{-2} e\cdot R\cdot e + \left(\frac{\beta}{\ell^2}\right)^2 R^{ab}\wedge R_{ab} + 2\left(\frac{\beta}{\ell^2}\right)^2 \ell^{-2} T\cdot T\right] + O\left[\left(\frac{\beta}{\ell}\right)^{-2}\right]. \tag{1.19}$$

In the standard calculations, the lenght scales $\ell \sim \beta^{1/2}$ are identified with M^{-1}. This means that only the second term would be finite, while the first and third diverge like M^2 and the terms $O\left[\left(\frac{\beta}{\ell}\right)^{-2}\right] \sim M^{-2}$ are neglected. In our case, we see that if one identifies β with ℓ^2 the expression for the anomaly is finite to all orders and the first three terms are

$$\frac{1}{8\pi^2}\left[R^{ab}\wedge R_{ab} + \frac{2}{\ell^2}(T^a \wedge T_a - R_{ab}\wedge e^a \wedge e^b)\right], \tag{1.20}$$

which is the Chern class for $SO(5)$ (See Appendix B). The above result has been a source of controversy [20][21] and it has been argued that it is not conceptually right. We think the line of thought is correct. Moreover, other authors have obtained the same result using other methods [24][25][26][27].

We see that the relevant Dirac operator that entered in the regulator could be written as $\bar{e}^{\mu}_{a}\gamma^{a}\nabla_{\mu}$, where $\bar{e}^{a}_{\mu} = \ell^{-1}e^{a}_{\mu}$, which is the way the vielbein enters in the embedding $(\omega, e) \to \mathcal{W}$. In agreement with this, the anomaly is the second Chern class for $SO(5)$, instead of being the second Chern class for $SO(4)$.

In terms of the "physical" fields ω and $\bar{e}$, the regulator $\beta = \ell^2$ drops out the trace before the limit $\beta \to 0$ is performed. In other words, the result should be correct to all orders in powers of β. This is because the limit $\beta \to 0$ is actually unnecessary: as we mentioned before, the trace erases all β-dependence. Thus the result should be independent of β before the limit is performed. It should be stressed that the choice $\beta = \ell^2$ is the only one needed to yield a β-independent result, and there seems to be no other similarly simple adjustment that would do the trick. For example, if one had chosen $\beta' = \vartheta\ell$, with ϑ an arbitrary constant, the result would not be an exact form because this would change the relative factor between the two terms in the Nieh-Yan form.

1.2.2. Non-minimal coupling

Let us now consider the non-minimal coupling. Before we had that $\gamma^{\mu} = -\gamma^{\mu\dagger}$ and that implied that $\not{\nabla}^{\dagger} = \not{\nabla}$. Now, $\not{\nabla}' = \gamma^{I}e^{\mu}_{I}\left(1 - \frac{i}{\alpha}\gamma_5\right)\nabla_{\mu}$ will be Hermitian as long as $\gamma^{\mu}\left(1 - \frac{i}{\alpha}\gamma_5\right)$ is too. (The proof can be found in Appendix A). Using the fact that $\{\gamma_5, \gamma^{I}\} = 0$ and ${\gamma_5}^2 = 1$, we obtain that in this case

$$\not{\nabla}'^2\Psi = \left(1 + \frac{1}{\alpha^2}\right)\not{\nabla}^2\Psi \equiv \chi\not{\nabla}^2\Psi, \tag{1.21}$$

where $\not{\nabla}^2$ stands for the minimal Dirac operator. So we have that $\exp(\beta\not{\nabla}^2) \to \exp(\beta\chi\not{\nabla}^2) \equiv \exp(\beta_\chi\not{\nabla}^2)$ and this leads us to

$$\begin{aligned}\mathscr{A}_{\beta_\chi}(x) =& \frac{1}{8\pi^2}\left[-2\chi\left(\frac{\beta}{\ell^2}\right)\ell^{-2}e\cdot R\cdot e + \chi^2\left(\frac{\beta}{\ell^2}\right)^2 R^{ab}\wedge R_{ab} + 2\chi^2\left(\frac{\beta}{\ell^2}\right)^2\ell^{-2}T\cdot T\right] \\ &+ O\left[\chi^{-2}\left(\frac{\beta}{\ell}\right)^{-2}\right].\end{aligned} \tag{1.22}$$

Now it all depends on the identification we will do.

$\underline{\beta = \ell^2}$

If we insist in taking $\beta = \ell^2$, we get

$$\mathscr{A}_{\beta_\chi}(x) = \frac{1}{8\pi^2}\left[\chi^2 R^{ab}\wedge R_{ab} + 2\chi^2\ell^{-2}T\cdot T - 2\chi\ell^{-2}e\cdot R\cdot e\right] + O\left[\chi^{-2}\ell^{-2}\right]. \tag{1.23}$$

The anomaly has to be of a topological character because it is a quantum effect not present at the classical level. In order to achieve the Nieh-Yan four-form, it is necessary that the second and third terms in the above expression be of the same order for the Nieh-Yan topological invariant is $\mathcal{N} = d(e_a \wedge T^a) = T^a \wedge T_a - e_a \wedge e_b \wedge R^{ab}$. In order to achieve this we take $\chi^2 = \chi$, implying that $\chi = 0$ or $\chi = 1$. The first solution would lead us to $\alpha = \pm i$ but we need it to be real-valued (If not, $\nabla\!\!\!/'$ would not be Hermitian and $\nabla\!\!\!/'^2 = 0$). The second solution implies that $\alpha = \infty$, which leads us to the minimal coupling scenario.

$\underline{\beta\chi = \ell^2}$

We could "save" the situation if we make the identification $\beta\chi = \ell^2 \equiv \frac{1}{|\Lambda|}$ where Λ is the "cosmological constant". Then we would get the following anomaly

$$\begin{aligned}\mathscr{A}_{\beta_\chi}(x) &= \frac{1}{8\pi^2}\left[R^{ab} \wedge R_{ab} + \frac{2}{\beta\chi}(T^a \wedge T_a - R_{ab} \wedge e^a \wedge e^b)\right] \\ &= \frac{1}{8\pi^2}\left[R^{ab} \wedge R_{ab} + \frac{2}{\ell^2}(T^a \wedge T_a - R_{ab} \wedge e^a \wedge e^b)\right]. \end{aligned} \tag{1.24}$$

Now, if the regulator was M (as in the standard Fujikawa method), $\exp(-\nabla\!\!\!/^2/M^2) \to \exp(-\chi\nabla\!\!\!/^2/M^2) \equiv \exp(-\nabla\!\!\!/^2/{M_\chi}^2)$, where we have defined ${M_\chi}^2 = \frac{M^2}{\chi}$. Then one gets

$$\begin{aligned}\mathscr{A}_{M_\chi}(x) &= \frac{1}{8\pi^2}\left[R^{ab} \wedge R_{ab} + 2{M_\chi}^2(T^a \wedge T_a - R_{ab} \wedge e^a \wedge e^b)\right] \\ &= \frac{1}{8\pi^2}\left[R^{ab} \wedge R_{ab} + \frac{2M^2}{\chi}(T^a \wedge T_a - R_{ab} \wedge e^a \wedge e^b)\right]. \end{aligned} \tag{1.25}$$

If we perform the non-trivial transformation

$$e^a \to \tilde{e}^a = \frac{e^a}{M_\chi \ell} = \frac{\chi^{1/2} e^a}{M\ell} \tag{1.26}$$

we arrive at a finite anomaly

$$\begin{aligned}\mathscr{A}(x) &= \frac{1}{8\pi^2}\left[R^{ab} \wedge R_{ab} + \frac{2}{\ell^2}(T^a \wedge T_a - R_{ab} \wedge e^a \wedge e^b)\right] \\ &= \frac{1}{8\pi^2}\left[R^{ab} \wedge R_{ab} + 2|\Lambda|(T^a \wedge T_a - R_{ab} \wedge e^a \wedge e^b)\right]. \end{aligned} \tag{1.27}$$

Even if the transformation (1.26) is arbitrary we are performing the calculations on a given background spacetime without dynamics so it has no physical consequences and it is purely formal. In the analysis e is an external classical background field. One could view the rescaling of the vielbein as an invariance of the action, provided the Dirac field is suitably rescaled as well. However, in order for this invariance of the action to be interpreted as a symmetry generated by charges acting on the fields, one should include a scale-invariant Lagrangian for e. The vielbein has units of (mass)^0 and is, therefore, not of the same canonical dimension as the connection. If $\ell^{-1}e$ is to be regarded as part of a connection of $SO(5)$, the limit $M \to \infty$ keeping ℓ fixed could be interpreted as a way to turn the $SO(4)$-invariant action into that for a spinor minimally coupled to an $SO(5)$ connection. In this case, the chiral anomaly is then given by $P_4[SO(5)]$ as we have seen.

It has been shown that when $\alpha = \gamma$ a classical effect of the Immirzi parameter through spacetime torsion is avoided [2][12] since the usual "extra-term" of the Holst modification compared to Hilbert-Einstein becomes the Nieh-Yan topological density multiplied by $\frac{1}{2\gamma}$ and hence cannot affect the classical behaviour of the system [13]. Since there is not known way of giving quantum dynamics to the gravitational field (the calculation of the anomaly is performed on a classical spacetime background) we did not success in finding a link between α and γ. However if we stick with the first approximation we took for the non-minimal coupling case, $\alpha = \infty$, we should accept that γ will have classical implications and could in principle be measured through spacetime torsion [28][29].

Surprisingly during the final stage of this thesis another closely related calculation has been published [30]. The authors of this reference perform the calculation of the chiral anomaly using Schwinger's proper-time formalism and the Seeley-DeWitt heat-kernel expansion [31][32][33]. Based in the primary, old-fashioned idea of Sakharov's induced gravity and gauge interactions [34] they found that

$$\gamma = \frac{(N_0 + N_{\frac{1}{2}} - 4N_1)}{3N_L}, \tag{1.28}$$

where N_0 is the number of minimal scalar degrees of freedom (dof), $N_{\frac{1}{2}}$ is the number of two-component fermion fields, N_1 is the number of gauge fields (half the number of gauge dof) and N_L is the number of chiral left handed modes. In the framework of the Standard Model, they take $N_0 = 4$ (Higgs), $N_{\frac{1}{2}} = 45$, $N_1 = 12$, $N_L = 3$ (neutrinos), yielding $\gamma = \frac{1}{9} \approx 0{,}11$ which is quite close to the (a bit obsolete) Ashtekar-Baez-Corichi-Krasnov value [35][36] $\gamma_{ABCK} = \frac{\ln 2}{\pi\sqrt{3}} \approx 0{,}13$ (see [37] for a better estimation). Their starting point however was an action consisting in the Hilbert-Einstein plus Nieh-Yan terms.

We can adapt our calculation to these results. In the first approach we took, we should conclude that dynamics for fermions must be of minimal coupling nature and the Immirzi parameter is a real constant multiplying the Nieh-Yan topological term in the gravitational sector of the action so γ does not appear in the classical equations of motion. This alternative is attractive since it resembles the so called θ_{QCD} angle ambiguity of Quantum Chromodynamics [38][39] in the sense that a classical canonical transformation would not be unitarilly implemented at the quantum level so the spectrum of geometric operators depends explicitly on γ [40]. This suggests that the Ashtekar-Barbero canonical formulation of gravity represents a non-trivial extension of the Einstein-Cartan theory. Specifically, the presence in the action of the Nieh-Yan invariant, introduces into the theory also information about the global structure of the local gauge group [41] and should not be regarded just as an ad-hoc way of getting rid of classical effects of the Immirzi parameter in the presence of torsional matter. (See Appendix C)

If we stick with the second approach, we will be led to a "rescaled Immirzi parameter" γ_χ that reads

$$\gamma_\chi = \frac{(N_0 + N_{\frac{1}{2}} - 4N_1)}{3N_L}\chi, \qquad \chi = 1 + \frac{1}{\alpha^2}. \tag{1.29}$$

In this way, reconciling Hawking's semiclassical black hole entropy formula along with the particle content of the Standard Model would require that chiral left handed modes (such as neutrinos) couple non-minimally to curved spacetime. Suppose we are dealing with an action describing a massive neutrino.

The equation of motion in a spacetime background would be

$$(i\not{\nabla}' - m)\Psi = 0. \tag{1.30}$$

Multiplying by $(i\not{\nabla}' + m)$ we get

$$(\not{\nabla}'^2 + m^2)\Psi = (\chi\not{\nabla}^2 + m^2)\Psi = 0. \tag{1.31}$$

Now let us go to a local frame where gravity is neglected in virtue of the equivalence principle so we set $e^a_\mu = \delta^a_\mu$ and $\omega^{ab} = 0$. Then, the equation of motion reads

$$(\chi\not{\partial}^2 + m^2)\Psi = 0, \tag{1.32}$$

which is equivalent to the dispersion relation

$$E^2 = p^2 + m_\chi^2 \tag{1.33}$$

where $m_\chi \equiv \frac{m}{\chi}$. So we see that the naive "classical relativistic rest mass" is redefined in the presence of a non-minimal coupling parameter.

1.3. Calculation of the Anomaly via Index Theorem

For a fermionic theory with gravitational and Yang-Mills gauge fields we have to consider the following Dirac operator:

$$\not{D} = e^\mu_a \gamma^a (\partial_\mu + A_\mu + \omega_\mu), \tag{1.34}$$

with the Yang Mills one-form $A = A_\mu dx^\mu = A^i_\mu T^i dx^\mu$ and spin connection $\omega = \omega_\mu dx^\mu = \frac{1}{2}\omega_{ab\mu}\sigma^{ab}dx^\mu$. The Atiyah-Singer index theorem tells us that

$$\text{index}\, D_+ = \int_{M_{2n}} \mathfrak{Ch}(F)\hat{A}(M), \tag{1.35}$$

where D_+ is the Weyl operator $D_+ = i\not{D}P_+$ and $P_+ = \frac{1}{2}(1+\gamma_5)$ as usual [43].
Here $\mathfrak{Ch}(F)$ stands for the Chern character defined as

$$\begin{aligned} \mathfrak{Ch}(F) &= \operatorname{tr}\exp\left[\frac{i}{2\pi}F\right] \\ &= r + \frac{i}{2\pi}\operatorname{tr} F + \frac{1}{2!}\left(\frac{i}{2\pi}\right)^2 \operatorname{tr} F^2 + \ldots, \end{aligned} \tag{1.36}$$

where r is the dimension of the group and F is the curvature 2-form, $F = dA + A\wedge A$.
$\hat{A}(M)$ is the Dirac genus defined by

$$\hat{A}(M) = \prod_a \frac{x_a/2}{\sinh x_a/2}. \tag{1.37}$$

The quantities x_a denote the skew eigenvalues of the curvature 2-form

$$\frac{R_{ab}}{2\pi} = \begin{pmatrix} 0 & x_1 & & & \\ -x_1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & x_n \\ & & & -x_n & 0 \end{pmatrix}, \tag{1.38}$$

which we consider as a matrix in the Lie algebra of $SO(2n)$. Each x_a expresses a 2-form. The Dirac genus $\hat{A}(M)$ can be expanded to arbitrary order in R so it represents a sum of invariant polynomials in the curvature 2-form to a given finite order depending on the dimension of the manifold [42]. (See Appendix D)

In our expression for the AS theorem $\mathfrak{Ch}(F)\hat{A}(M)$ means the wedge product of the Chern character with the Dirac genus in a given order corresponding to the dimension of the manifold. Mixed terms occur only in dimensions higher than or equal to 6 so in particular in $n = 4$ there are not mixed anomaly contributions. Explicitly, for $n = 4$ the index of the Weyl operator is

$$\operatorname{index} D_+ = \frac{1}{(2\pi)^2} \int_{M_4} \left[-\frac{1}{2} \operatorname{tr} F^2 + \frac{r}{48} \operatorname{tr} R^2 \right]. \tag{1.39}$$

In the presence of torsion the relevant tangent group of rotations is $SO(5)$ instead of $SO(4)$ [19] (see Appendix B) so we have to consider the Pontryagin density associated with the curvature 2-form

$$\mathcal{R}_{AB} \wedge \mathcal{R}^{AB} = R_{ab} \wedge R^{ab} + \frac{2}{\ell^2}(T_a \wedge T^a - e_a \wedge e_b \wedge R^{ab}). \tag{1.40}$$

Taking account of this fact, the index we should seek for is

$$\begin{aligned} \operatorname{index} D_+^T &= \frac{1}{(2\pi)^2} \int_{M_4} \left[-\frac{1}{2} \operatorname{tr} F^2 + \frac{r}{48} \operatorname{tr} \mathcal{R}^2 \right] \\ &= \frac{1}{(2\pi)^2} \int_{M_4} \left[-\frac{1}{2} \operatorname{tr} F^2 + \frac{r}{48} \operatorname{tr} \left\{ R^2 + \frac{2}{\ell^2}(T^2 - e^2 R) \right\} \right], \end{aligned} \tag{1.41}$$

where D_+^T stands for the Weyl operator in the presence of torsion.

A naive ansatz would be that in order to obtain possible torsional contributions to the chiral anomaly in even n-dimensional spacetimes for a fermionic theory with Yang-Mills and gravitational gauge fields, all we have to do is to replace the $SO(n)$ curvature 2-form by a $SO(n+1)$ curvature 2-form (as in (1.40)) in the expansion of the Dirac genus. This will be carefully analized and discussed elsewhere.

Capítulo 2

Gauge Theories in the presence of spacetime torsion

2.1. Maxwell Field in the presence of spacetime torsion

Let us consider the action for Maxwell electrodynamics in curved spacetime

$$S[g,\Gamma,A] = -\frac{1}{4}\int d^4x\sqrt{-g}F_{\mu\nu}F^{\mu\nu}. \tag{2.1}$$

If we take minimal coupling as a true principle of nature then

$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu + T^\lambda{}_{\mu\nu}A_\lambda, \tag{2.2}$$

recalling that $T^\lambda{}_{\mu\nu} = \Gamma^\lambda{}_{\mu\nu} - \Gamma^\lambda{}_{\nu\mu}$. This of course spoils gauge invariance in its usual form. Since gauge invariance is a key feature for the renormalizability of Quantum Electrodynamics, the usual approach is to take Γ as the Levi-Civita connection and deny the possibility that "photons" can couple to spacetime torsion. Let us take a stubborn attitude and stick with minimal coupling as a true principle of nature. Canonical analysis will show us the way of reconciliation with gauge invariance.

We will perform the ADM Hamiltonian decomposition. Without loss of generality one takes the metric to be

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -N^2dt^2 + q_{ij}(dx^i + N^i dt)(dx^j + N^j dt). \tag{2.3}$$

N and N^i are called the *Lapse Function* and the *Shift Vector*, respectively [44][45]. They are Lagrange multipliers associated with the Hamiltonian and Vector constraints of General Relativity. q_{ij} is the 3-metric induced in the foliation of spacetime we are performing in order to achieve a Hamiltonian analysis. Using the fact that in this setup $\sqrt{-g} = N\sqrt{q}$, we define the canonical momenta associated with the spatial A field as

$$\Pi^i = \frac{\delta S}{\delta \mathring{A}_i} = -N\sqrt{q}F^{0i}, \tag{2.4}$$

and where we define $\mathring{A}_i \equiv \partial_0 A_i + T^j_{\ 0i} A_j$. (See Appendix E for a better understanding of this choice, [43][46]).

With these facts we can write (2.1) in the following $3+1$ form

$$S[q,\Gamma,A] = \int d^3x \int dt \left\{ \Pi^i \mathring{A}_i - \frac{1}{2}\Pi^i F_{0i} + A_0 D_i \Pi^i - \frac{N\sqrt{q}}{4} F_{ij} F^{ij} \right\}, \tag{2.5}$$

where $D_i \Pi^i \equiv \partial_i \Pi^i + T^0_{\ 0i}\Pi^i$ is a covariant derivative. This deviation of the standard result is natural when we realize that, as always, A_0 has no dynamics so it enters in the action as a Lagrange multiplier of the first class constraint of the $U(1)$ gauge theory for electromagnetism, Gauss constraint. One can prove that

$$F_{0i} = \frac{N}{\sqrt{q}} q_{ij}\Pi^j + N^j F_{ji}\,. \tag{2.6}$$

We then get the final form

$$S[q,\Gamma,A] = \int d^3x \int dt \left\{ \Pi^i \mathring{A}_i + A_0 D_i \Pi^i - \frac{1}{2} N^j \Pi^i F_{ji} - N\left(\frac{1}{2\sqrt{q}} \Pi^i \Pi^j q_{ij} + \frac{\sqrt{q}}{4} F_{ij} F^{ij} \right) \right\}. \tag{2.7}$$

Given a function f on phase space, we may associate with it a vector field V on phase space by the requirement that for any function g on phase space, we have $V(g) = \{f,g\}_{\mathrm{PB}}$, where the *Poisson bracket* $\{f,g\}_{\mathrm{PB}}$ is defined by

$$\{f,g\}_{\mathrm{PB}} = \int_{\Sigma_t} \left(\frac{\partial f}{\partial q}\frac{\partial g}{\partial \pi} - \frac{\partial g}{\partial q}\frac{\partial f}{\partial \pi} \right). \tag{2.8}$$

One may verify that the vector field V associated in this manner with the "constraint function" $f = \int_{\Sigma_t} \chi^i \phi_i$ (where χ is an arbitrary function on Σ_t, there is no summation on i and $\phi_i \approx 0$ is a first class constraint) is just the infinitesimal generator of the one-parameter family of transformations on phase space associated with the gauge transformations of the theory. The constraint "generates" the gauge transformations. By restricting to the "constraint submanifold" $\phi_i = 0$ and to the space of orbits of V on this submanifold, we obtain a consistent, constraint-free Hamiltonian formulation on a "reduced phase space" [47][48]. So considering the Gauss first class constraint

$$D_i \Pi^i \approx 0, \tag{2.9}$$

we get that

$$\delta A_\mu = D_\mu \chi = \partial_\mu \chi + T^0_{\ 0\mu}\chi. \tag{2.10}$$

We note that in particular $\delta A_0 = \partial_0\chi$ under a gauge transformation. Given this transformation we must impose that the strength field be gauge invariant. So we require that

$$\delta F_{\mu\nu} = \partial_\mu T^0{}_{0\nu}\,\chi + T^0{}_{0\nu}\,\partial_\mu\chi - \partial_\nu T^0{}_{0\mu}\,\chi - T^0{}_{0\mu}\,\partial_\nu\chi + T^\rho{}_{\mu\nu}\,\partial_\rho\chi + T^\rho{}_{\mu\nu}\,T^0{}_{0\rho}\,\chi = 0. \tag{2.11}$$

This is equivalent to the system

$$\partial_\mu T^0{}_{0\nu} - \partial_\nu T^0{}_{0\mu} + T^\rho{}_{\mu\nu}\,T^0{}_{0\rho} = 0, \tag{2.12}$$

$$T^0{}_{0\nu}\,\delta^\rho{}_\mu - T^0{}_{0\mu}\,\delta^\rho{}_\nu + T^\rho{}_{\mu\nu} = 0. \tag{2.13}$$

The second equation implies that $T^\rho{}_{\mu\nu}\,T^0{}_{0\rho} = 0$, so the first one is satisfied if

$$T^0{}_{0\mu} = \partial_\mu\varphi, \tag{2.14}$$

with $\varphi(x)$ some scalar field. Finally,

$$T^\rho{}_{\mu\nu} = \delta^\rho{}_\nu\partial_\mu\varphi - \delta^\rho{}_\mu\partial_\nu\varphi, \tag{2.15}$$

where $\varphi(x)$ is the "tlaplon" field of Hojman et al. (HRRS theory, [49][50]), which in the literature has been identified with the dilaton. In our case we have that

$$\delta A_\mu = D_\mu\chi = \partial_\mu\chi + \chi\partial_\mu\varphi, \quad \mathring{A}_i = \dot{A}_i + \dot{\varphi}A_i. \tag{2.16}$$

HRRS theory arrives to the same form of torsion when taking $\delta A_\mu = e^\varphi\partial_\mu\Lambda$. In any case it is clear that when $\varphi(x)$ vanishes we get the usual theory. So as it was stressed before for the Maxwell field, minimal coupling and gauge invariance are consistent with nonvanishing torsion of a particular type. Dynamics for the $\varphi(x)$ field is incorporated into the Hilbert-Einstein action if we consider the Ricci scalar $R(\Gamma)$ as a function of the full connection $\Gamma^\rho{}_{\mu\nu}$ which includes torsion. The Hamiltonian for the Maxwell theory is

$$H_M = \int d^3x\left\{-A_0 D_i\Pi^i + \frac{1}{2}N^j\Pi^i F_{ji} + N\left(\frac{1}{2\sqrt{q}}\Pi^i\Pi^j q_{ij} + \frac{\sqrt{q}}{4}F_{ij}\,F^{ij}\right)\right\}, \tag{2.17}$$

a linear combination of first class constraints. Energy density is defined in this context as

$$\rho = \frac{1}{\sqrt{q}}\frac{\delta H_M}{\delta N} = \frac{1}{2q}\Pi^i\Pi^j q_{ij} + \frac{1}{4}F_{ij}\,F^{ij}. \tag{2.18}$$

On the other hand, pressure is defined as (see Appendix F)

$$P = -\frac{2}{3N\sqrt{q}}q_{ab}\frac{\delta H_M}{\delta q_{ab}} = \frac{1}{3}\left(\frac{1}{2q}\Pi^i\Pi^j q_{ij} + \frac{1}{4}F_{ij}\,F^{ij}\right), \tag{2.19}$$

so, as always,

$$P = \frac{1}{3}\rho, \tag{2.20}$$

and the presence of spacetime torsion does not affect the radiation equation of state [46].

2.2. Yang-Mills Fields in the presence of spacetime torsion

It is natural to generalize the above results for the case of Yang-Mills fields. In this case the field strenght will be

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g_{\rm YM} f^a{}_{bc} A^b_\mu A^c_\nu + T^\lambda{}_{\mu\nu} A^a_\lambda, \tag{2.21}$$

where $f^a{}_{bc}$ are the structure constants of the Lie algebra where the connection belongs which we will assume is $SU(N)$. $a, b, c = 1, \dots, N^2-1$, are internal group indices and $g_{\rm YM}$ stands for a coupling constant. The Yang-Mills action is

$$S[g, \Gamma, A^a] = -\frac{1}{4}\int d^4x \sqrt{-g} F^a_{\mu\nu} F^{a\mu\nu}. \tag{2.22}$$

Going through the same ADM decomposition we get to the following form of the $3+1$ action

$$S[q, \Gamma, A^a] = \int d^3x \int dt \Bigg\{ \Pi^i_a \mathring{A}^a_i + A^a_0 D_i \Pi^i_a - \frac{1}{2} N^j \Pi^i_a F^a_{ji} - N\left(\frac{1}{2\sqrt{q}} \Pi^i_a \Pi^j_a q_{ij} + \frac{\sqrt{q}}{4} F^a_{ij} F^{aij} \right) \Bigg\}, \tag{2.23}$$

where again $\mathring{A}^a_i = \partial_0 A^a_i + T^j{}_{0i} A^a_j$ and the Gauss constraint reads

$$D_i \Pi^i_a \equiv \partial_i \Pi^i_a + T^0{}_{0i} \Pi^i_a - g_{\rm YM} f^c{}_{ba} A^b_i \Pi^i_c \approx 0. \tag{2.24}$$

The infinitesimal gauge transformation of the connection will be

$$\delta A^a_\mu = D_\mu \chi^a \equiv \frac{1}{g_{\rm YM}} \partial_\mu \chi^a + \frac{1}{g_{\rm YM}} T^0{}_{0\mu} \chi^a + f^a{}_{bc} A^b_\mu \chi^c. \tag{2.25}$$

We now demand that under such transformation the Yang Mills field strenght transforms homogeneously. By this we mean that at the infinitesimal level,

$$\delta F^a_{\mu\nu} = \chi^c f^a{}_{bc} F^b_{\mu\nu}. \tag{2.26}$$

Applying (2.25) to (2.21) and demanding (2.26) we must impose the system

$$\partial_\mu T^0{}_{0\nu} \delta^a{}_b - \partial_\nu T^0{}_{0\mu} \delta^a{}_b + g_{\rm YM} f^a{}_{bc} T^0{}_{0\mu} A^c_\nu - g_{\rm YM} f^a{}_{bc} T^0{}_{0\nu} A^c_\mu + T^\lambda{}_{\mu\nu} T^0{}_{0\lambda} \delta^a{}_b = 0, \tag{2.27}$$

$$T^0{}_{0\nu} \delta^\rho{}_\mu - T^0{}_{0\mu} \delta^\rho{}_\nu + T^\rho{}_{\mu\nu} = 0. \tag{2.28}$$

Multiplying the second equation by $T^0{}_{0\rho}$ we get that, as before, $T^\rho{}_{\mu\nu} T^0{}_{0\rho} = 0$, so we are left with

$$\partial_\mu T^0{}_{0\nu} \delta^a{}_b - \partial_\nu T^0{}_{0\mu} \delta^a{}_b + g_{\rm YM} f^a{}_{bc} T^0{}_{0\mu} A^c_\nu - g_{\rm YM} f^a{}_{bc} T^0{}_{0\nu} A^c_\mu = 0. \tag{2.29}$$

Now multiplying (2.28) by A^c_ρ we can write (2.27) as

$$\partial_\mu T^0{}_{0\nu} \delta^a{}_b - \partial_\nu T^0{}_{0\mu} \delta^a{}_b + g_{\rm YM} f^a{}_{bc} T^\rho{}_{\mu\nu} A^c_\rho = 0. \tag{2.30}$$

Let us recall that for compact semisimple Lie groups we can always define a rank two symmetric tensor as

$$g_{ab} = f^d{}_{ac} f^c{}_{bd}, \tag{2.31}$$

which serves as a metric and defines an inner product in the group. We can always diagonalize this metric so it is proportional to the identity tensor. If we define $f_{dbc} \equiv g_{da} f^{a}{}_{bc}$, using the Jacobi identity

$$f^{a}{}_{bc} f^{d}{}_{ae} + f^{a}{}_{ce} f^{d}{}_{ab} + f^{a}{}_{eb} f^{d}{}_{ac} = 0, \tag{2.32}$$

one can prove that the structure constants f_{abc} are completely antisymmetric [51][52]. Such is the case of $SU(N)$. So taking $a \neq b \neq c$ in (2.30) one is led to

$$g_{\mathrm{YM}} g^{ad} f_{dbc} T^{\rho}{}_{\mu\nu} A^{c}_{\rho} = 0. \tag{2.33}$$

So we see that in order to keep gauge invariance of the action equation (2.33) demands us to set the torsion tensor $T^{\rho}{}_{\mu\nu} = 0$.

For completeness we must stress that Mukku et al. found that a modification of the covariant derivative and the Yang-Mills field strength [53], namely

$$D_{\mu} = \partial_{\mu} - i g_{\mathrm{YM}} e^{-\varphi} A_{\mu} \cdot \Theta, \tag{2.34}$$

$$F^{a}_{\mu\nu} = \partial_{\mu} A^{a}_{\nu} - \partial_{\nu} A^{a}_{\mu} + g_{\mathrm{YM}} e^{-\varphi} f^{a}{}_{bc} A^{b}_{\mu} A^{c}_{\nu} - A^{a}_{\sigma} T^{\sigma}{}_{\mu\nu}, \tag{2.35}$$

where $\varphi(x)$ is the tlaplon field and the torsion tensor takes the same form as in the case of the abelian theory. The net effect of torsion on the gauge field interactions is essentially to define an effective coupling constant which is a function of the space-time point at which the interaction takes place.

Finally, it has been argued [54] that if we take the approach that the gauge coupling is a spacetime function then $D_{\mu} \sim \partial_{\mu} - i g(x) A_{\mu}$, recalling that $F_{\mu\nu} \sim [D_{\mu}, D_{\nu}]$ one can find that the torsion tensor is in fact the one proposed by Hojman et al. and $g(x) \sim g_{\mathrm{YM}} e^{-\varphi}$. Moreover, if for example we take a product group like $SU(2)_L \times U(1)_Y$ where the covariant derivative is $D_{\mu} = \partial_{\mu} - i g T^{a} A^{a}_{\mu} - i \frac{g'}{2} B_{\mu}$ it can be shown that since the tlaplon field is the only "torsional degree of freedom" of the theory then in this setup the coupling constants must converge. This is quite interesting since torsional effects of spacetime are expected to become important just around the Grand Unification Theory (GUT) scale. The nontrivial assumption that the coupling could be a a spacetime function could be somehow justified in the light of the so called "String Landscape" of Susskind and others [55]. Of course the stringent bounds on these couplings remember us the fact that the universe we observe right now is Riemannian and torsion could only have played a role in the distant past. The relation between a transition from Riemann-Cartan to Riemann spacetime and a GUT (like $SU(5)$ or $SO(10)$, [56][57]) to the Standard Model of particle physics could set an unknown bridge between Quantum Field Theory and Gravitation.

2.3. Tlaplon field in Loop Quantum Gravity

Nowadays, the covariant starting point of Loop Quantum Gravity is Dirac canonical quantization program applied to the four-dimensional Lorentzian action known as the Holst modification of the

Hilbert-Palatini action for the gravitational field,

$$S[e,\omega] = -\frac{1}{2}\int_{M^4} d^4x\, e\, e^\mu_a e^\nu_b \left(R^{ab}{}_{\mu\nu} - \frac{\beta}{2}\epsilon^{ab}{}_{cd}R^{cd}{}_{\mu\nu}\right). \tag{2.36}$$

Here $R^{ab}{}_{\mu\nu} = 2\partial_{[\mu}\omega^{ab}{}_{\nu]} + 2\omega^a{}_{c[\mu}\omega^{cb}{}_{\nu]}$ and β is the Immirzi parameter of LQG which could be promoted to a pseudoscalar field. We will split the Lorentz spin connection in a torsionless part $\tilde{\omega}^{ab}$ (Ricci connection, which obeys the homogeneous structure equation) plus the contorsion one-form C^{ab} so $\omega^{ab}{}_\mu = \tilde{\omega}^{ab}{}_\mu + C^{ab}{}_\mu$, where the contorsion tensor is $C^{ab}{}_\mu = e^a_\nu e^b_\rho C^{\nu\rho}{}_\mu$, $C^{\nu\rho}{}_\mu = -C^{\rho\nu}{}_\mu$, is related to the torsion tensor $T^\nu{}_{\rho\mu} = -T^\nu{}_{\mu\rho}$ by $C^\nu{}_{\rho\mu} = \frac{1}{2}(T^\nu{}_{\rho\mu} - T_\rho{}^\nu{}_\mu - T_\mu{}^\nu{}_\rho)$. We are considering a torsion tensor of the form $T^\rho{}_{\mu\nu} = \delta^\rho{}_\nu\partial_\mu\varphi - \delta^\rho{}_\mu\partial_\nu\varphi$, so the "trace" vector is $T^\alpha{}_{\mu\alpha} \equiv T_\mu = 3\partial_\mu\varphi$ and we get that

$$\begin{aligned} S &= -\frac{1}{2}\int d^4x\, e \left\{ e^\mu_a e^\nu_b \tilde{R}_{\mu\nu}{}^{ab} - \frac{2}{3}T_\mu T^\mu \right\}, \\ &= -\frac{1}{2}\int d^4x\, e \left\{ e^\mu_a e^\nu_b \tilde{R}_{\mu\nu}{}^{ab} - 6\partial_\mu\varphi\partial^\mu\varphi \right\}. \end{aligned} \tag{2.37}$$

So we see that if the torsion tensor is of "tlaplonic" form the Immirzi parameter has not classical implications. However if we consider charged fermions coupled to curved spacetime where the fermionic part of the action reads

$$S_f[e,\omega,\psi,\bar{\psi}] = \frac{i}{2}\int d^4x\, e\, e^\mu_a \left(\bar{\psi}\gamma^a\nabla_\mu\psi - \overline{\nabla_\mu\psi}\gamma^a\psi \right), \tag{2.38}$$

it has been shown that torsional trace and axial vectors arise [12][13][14], namely

$$T^\rho \sim \frac{\beta}{1+\beta^2}J^\rho_{(A)}, \qquad S^\sigma \sim \frac{1}{1+\beta^2}J^\sigma_{(A)}, \tag{2.39}$$

where $J^\rho_{(A)} = \bar{\psi}\gamma^\rho\gamma^5\psi$ is the axial fermionic current.

Taking the Holst modification with a tlaplonic torsion tensor along with a Dirac charged particle and the Maxwell action in the HRRS form will give us interaction terms between the fermionic axial current and the tlaplon mediated by the Immirzi parameter (so it does not violate parity) besides the usual interactions of the Einstein-Cartan-HRRS theory. This new term would be of the form $\sim \frac{\beta}{1+\beta^2}\bar{\psi}\gamma^\alpha\gamma^5\psi\partial_\alpha\varphi$, and we see that the parity odd nature of the Immirzi parameter gives us the theoretical chance of coupling trace and axial torsional vectors so a Dirac field could not only feel axial torsion but also of tlaplonic nature.

Capítulo 3

$(p-1)$-forms as bosonic spacetime torsion sources

3.1. The Action

Let us consider the following action:

$$S[e,\omega,\phi,p,q] = -\frac{\lambda}{2}\int \star(\nabla\phi_{a_1\dots a_q})\wedge\nabla\phi^{a_1\cdots a_q}, \tag{3.1}$$

where $\phi_{a_1\dots a_q}$ is a Lorentz valued $(p-1)$-form and $\{a_1,\dots,a_q\}$ is a completely antisymmetric set of indices. As an abstract operator, $\nabla = d + [\omega,\ ]$ where d is the exterior derivative and ω is the spin gauge connection one-form of gravity. It is evident that $1\le p\le n$ and $0\le q\le n$ in a n-dimensional spacetime which for simplicity will be taken as Euclidean and compact. In doing so the inner space group becomes $SO(n)$ instead of $SO(n-1,1)$. λ could be an n-dependent constant. If we define the "field strenght" p-form as $F_{a_1\dots a_q} = \nabla\phi_{a_1\dots a_q}$, our action reads

$$\begin{aligned} S[e,\omega,\phi,p,q] &= -\frac{\lambda}{2}\int \star F_{a_1\dots a_q}\wedge F^{a_1\dots a_q} \\ &= -\frac{\lambda}{2p!p!}\int F_{a_1\dots a_q b_1\dots b_p}F^{a_1\dots a_q}{}_{c_1\dots c_p}\star(e^{b_1}\wedge\cdots\wedge e^{b_p})\wedge e^{c_1}\wedge\cdots\wedge e^{c_p} \\ &= -\frac{\lambda}{2p!p!(n-p)!}\int F_{a_1\dots a_q b_1\dots b_p}F^{a_1\dots a_q}{}_{c_1\dots c_p}\epsilon^{b_1\dots b_p}{}_{b_{p+1}\dots b_n}e^{b_{p+1}}\wedge\cdots\wedge e^{b_n}\wedge e^{c_1}\wedge\cdots\wedge e^{c_p}. \end{aligned} \tag{3.2}$$

3.2. The Currents

3.2.1. Energy-momentum current

The definition of the energy-momentum $(n-1)$-form $\star U$ goes as follows [58]: Given a matter Lagrangian $\mathscr{L}_M[e]$ we replace e by $e+f$ and calculate the term linear in f in the variation

$$\delta_e \mathscr{L}_M[e] \equiv \mathscr{L}_M[e+f] - \mathscr{L}_M[e] = \star U_a \wedge f^a + \mathcal{O}(f^2). \tag{3.3}$$

Here, $\star U_a$ is a vector-valued $(n-1)$-form, the "energy-momentum current of matter". Integrated over a $(n-1)$-dimensional spacelike hypervolume it yields the energy-momentum of matter included in this hypervolume. Our Lagrangian depends on the orthonormal frame only via the Hodge star. This we will make it explicit writting $\star|_e$ for the Hodge star associated with the metric described by the orthonormal frame e. Its dependence on e is implicit and the variation not straightforward. We will derive the "Maxwell case" in 4-D, i.e., F shall be a 2-form with no inner space indices. The general case is a mere generalization. Let us begin from the following identity

$$\star|_e F \wedge e^a \wedge e^b = \frac{1}{2}\epsilon^{ab}{}_{cd} F \wedge e^c \wedge e^d, \tag{3.4}$$

Now making the replacement $e \to e+f$, considering the variation and neglecting terms quadratic in f we get

$$\star|_e F \wedge f^a \wedge e^b + \star|_e F \wedge e^a \wedge f^b + (\star|_{e+f} F - \star|_e F) \wedge e^a \wedge e^b = \epsilon^{ab}{}_{cd} F \wedge f^c \wedge e^d. \tag{3.5}$$

We now multiply by $-\frac{\lambda}{4}F_{ab}$ and contract. The result of doing so is

$$\mathscr{L}_M[e+f] - \mathscr{L}_M[e] = \left\{ -\frac{\lambda}{2} F_{ab} \star F \wedge e^b + \frac{\lambda}{4} F_{cb}\, \epsilon^{cb}{}_{ad} F \wedge e^d \right\} \wedge f^a, \tag{3.6}$$

and we recognize the energy-momentum 3-form. The generalization is straightforward. Being $F_{a_1 \dots a_q}$ a p-form with q indices in the Lorentz algebra its associated energy-momentum $(n-1)$-form is

$$\begin{aligned} \star U_i[e,\omega,\phi,p,q] = {} & \frac{\lambda}{2(p-1)!} F_{a_1 \dots a_q b_1 \dots b_{p-1} i} \star F^{a_1 \dots a_q} \wedge e^{b_1} \wedge \cdots \wedge e^{b_{p-1}} \\ & - \frac{(-1)^{p(n-p)}\lambda}{2p!(n-p-1)!} F_{a_1 \dots a_q b_1 \dots b_p}\, \epsilon^{b_1 \dots b_p}{}_{b_{p+1} \dots b_{n-1} i} F^{a_1 \dots a_q} \wedge e^{b_{p+1}} \wedge \cdots \wedge e^{b_{n-1}}. \end{aligned} \tag{3.7}$$

We can write this expression in a more compact way by means of the contraction operator [17]. Recalling that $I_{e_i} e^j = \delta^j{}_i$, we get that

$$\star U_i[e,\omega,\phi,p,q] = \left(\frac{(-1)^{p-1}\lambda}{2} \right) \star F^{a_1 \dots a_q} \wedge I_{e_i} F_{a_1 \dots a_q} + \left(\frac{(-1)^{n-p}\lambda}{2p!} \right) F^{a_1 \dots a_q} \wedge I_{e_i} \star F_{a_1 \dots a_q}. \tag{3.8}$$

What we usually call the energy-momentum tensor is defined as $\mathcal{T}^i_{\ k} = \star(e^i \wedge \star U_k)$.

3.2.2. Spin current

Now if we vary the action with respect to the spin connection one-form we can define the spin-torsion $(n-1)$-form $\star J_{ab}$. This is not as involved as the vielbein case. Explicitly,

$$\delta_\omega S[e,\omega,\phi,p,q] = -\lambda \int \star F_{a_1 \dots a_q} \wedge \delta_\omega F^{a_1 \dots a_q}. \tag{3.9}$$

We have that $\delta_\omega F^{a_1 \dots a_q} = \delta_\omega \nabla \phi^{a_1 \dots a_q}$. Now let us recall that for a p-form $V^a_{\ b}$ the covariant derivative is defined in such a way that $\nabla V^a_{\ b} = d V^a_{\ b} + \omega^a_{\ c} \wedge V^c_{\ b} - (-1)^p V^a_{\ c} \wedge \omega^c_{\ b}$ [59]. Therefore, we have that

$$\nabla \phi^{a_1 \dots a_q} = d\,\phi^{a_1 \dots a_q} + \omega^{a_1}{}_c \wedge \phi^{c a_2 \dots a_q} + \dots + \omega^{a_q}{}_c \wedge \phi^{a_1 \dots a_{q-1} c}. \tag{3.10}$$

This implies that

$$\delta_\omega \nabla \phi^{a_1 \dots a_q} = (-1)^{p-1} \phi^{c a_2 \dots a_q} \wedge \delta\omega^{a_1}{}_c + \dots + (-1)^{p-1} \phi^{a_1 \dots a_{q-1} c} \delta\omega^{a_q}{}_c. \tag{3.11}$$

Finally,

$$\delta_\omega S[e,\omega,\phi,p,q] = (-1)^p \lambda q \int \star F_{a_1 \dots a_q} \wedge \phi^{a_1 \dots a_{q-1}}{}_c \wedge \delta\omega^{a_q c}, \tag{3.12}$$

so

$$\star J_{mn}[e,\omega,\phi,p,q] = (-1)^p \lambda q \star F_{a_1 \dots a_{q-1} m} \wedge \phi^{a_1 \dots a_{q-1}}{}_n. \tag{3.13}$$

The better known spin tensor is defined as $\mathcal{S}^i{}_{jk} = \star(e^i \wedge \star J_{jk})$.

3.3. Curvature as an abstract operator

From the definition $F_{a_1 \dots a_q} = \nabla \phi_{a_1 \dots a_q}$, taking the exterior derivative and recalling that $R^{ab} = d\omega^{ab} + \omega^a{}_c \wedge \omega^{cb}$ we get a "kind of Bianchi identity", namely

$$R_{a_1}{}^c \wedge \phi_{c \dots a_q} + \dots + R_{a_q}{}^c \wedge \phi_{a_1 \dots a_{q-1} c} = \nabla F_{a_1 \dots a_q}. \tag{3.14}$$

We could shorten these expressions abstractly as

$$\begin{aligned} \mathbb{F} &= \nabla \Phi, \\ [\mathbb{R}, \Phi] &= \nabla \mathbb{F} = \nabla^2 \Phi, \end{aligned} \tag{3.15}$$

so we are lead to a definition of ∇^2 as an abstract operator,

$$\nabla^2 = [\mathbb{R},\ ]. \tag{3.16}$$

3.4. The Symmetries

Let us now consider (3.1) in a curved Riemann-Cartan *background*. Its variation induced by arbitrary variations $\delta e^a, \delta\omega^{ab}$ and $\delta\phi^{a_1\cdots a_q}$ is

$$\delta S[e,\omega,\phi,p,q] = \int \Big\{ \star U_a \wedge \delta e^a + \star J_{ab} \wedge \delta\omega^{ab} \Big\}. \tag{3.17}$$

The coefficient of $\delta\phi^{a_1\cdots a_q}$ is zero because of the equations of motion [60].

3.4.1. Lorentz symmetry

We know the fact that this action is Lorentz invariant, $\delta_L S = 0$. For the gravitational fields these transformations are

$$\begin{aligned} \delta_L\, e_a &= \delta\varepsilon_a{}^b e_b, \\ \delta_L\, \omega^{ab} &= -\nabla\delta\varepsilon^{ab}, \end{aligned} \tag{3.18}$$

where ε^{ab} is an arbitrary antisymmetric 0-form. Using this in (3.17) we get the conservation law

$$\nabla \star J_{ab} + (-1)^{n-1} \star U_{[a} \wedge e_{b]} = 0. \tag{3.19}$$

In Einstein-Cartan theory local Lorentz symmetry does not imply a vanishing antisymmetric piece of the energy-momentum tensor $\mathcal{T}_{[\mu\nu]}$. Instead it is proportional to the divergence of the spin tensor, $\nabla_\lambda \mathcal{S}^\lambda{}_{\mu\nu} \propto \mathcal{T}_{[\mu\nu]}$.

3.4.2. Diffeomorphism symmetry

It is well known that if we consider a diffeomorphism $x^\mu \to x'^\mu = x^\mu + \xi^\mu$ (a general coordinate transformation) in Riemannian geometry and impose $\delta_{\text{diff}} S = 0$ we get Bianchi identity as the local conservation of the energy-momentum tensor $\mathcal{T}^{\alpha\beta}{}_{;\beta} = 0$. Let us derive the analog in our case. Under a diffeomorphism a p-form transforms with the Lie derivative as an operator through Cartan's magic formula $\pounds_\xi = dI_\xi + I_\xi d$, where d in the exterior derivative and I_ξ is the contraction operator . Hence,

$$\begin{aligned} \delta_{\text{diff}}\, e^a &= -\pounds_\xi e^a, \\ \delta_{\text{diff}}\, \omega^{ab} &= -\pounds_\xi \omega^{ab}. \end{aligned} \tag{3.20}$$

It is not hard to show that the following identities hold

$$\pounds_\xi e^a = \nabla\xi^a + I_\xi T^a - I_\xi(\omega^a{}_b) e^b, \tag{3.21}$$

$$\pounds_\xi \omega^{ab} = \nabla(I_\xi \omega^{ab}) + I_\xi R^{ab}. \tag{3.22}$$

The conservation law associated with this symmetry is, using (3.19),

$$\nabla(\star U_a)\xi^a + (-1)^n \star U_a \wedge I_\xi T^a + (-1)^n \star J_{ab} \wedge I_\xi R^{ab} = 0. \tag{3.23}$$

In the last expression we cannot isolate immediately the arbitrary vector field ξ^χ. However it can be shown that it reduces to

$$\mathcal{T}^\tau{}_{\chi;\tau} + (-1)^{n+1}\mathcal{T}^\alpha_\beta\, T^\beta{}_{\chi\alpha} + (-1)^{n+1}\mathcal{S}^\lambda{}_{\alpha\beta} R^{\alpha\beta}{}_{\chi\lambda} = 0. \tag{3.24}$$

Here ; stands for the total covariant derivative which includes torsion. Of course when torsion and spin tensors are set to zero we recover the usual Riemannian covariant energy-momentum tensor conservation law.

3.4.3. Conformal Weyl symmetry

Finally there is a conformal symmetry in this action, namely, if we consider the following transformation

$$\begin{aligned} \delta_\Lambda e^a &= -\delta\Lambda(x) e^a, \\ \delta_\Lambda \omega^{ab} &= 0, \end{aligned} \tag{3.25}$$

our action is left invariant as long as

$$\star U_a \wedge e^a = 0. \tag{3.26}$$

This is nothing but the known fact that conformal symmetry implies a vanishing trace of the energy-momentum tensor, $\mathcal{T}^\mu_\mu = 0$.

3.5. The equations of motion

The equations of motion, $\delta_\phi S = 0$, are “simply”

$$\nabla(\star F_{a_1 \ldots a_q}) = \nabla(\star\nabla\phi_{a_1 \ldots a_q}) = 0. \tag{3.27}$$

3.6. Torsion as a Noether current

3.6.1. $(p-1)$-forms

Let us apply Noether theorem to this family of actions [61]. An arbitrary variation is

$$\begin{aligned}\delta_\phi S[e,\omega,\phi,p,q] &= -\lambda\int \star(\nabla\phi_{a_1\dots a_q})\wedge\nabla\delta\phi^{a_1\dots a_q}\\ &= (-1)^{n-p}\lambda\int \nabla(\star(\nabla\phi_{a_1\dots a_q}))\wedge\delta\phi^{a_1\dots a_q}\\ &+ (-1)^{n-p+1}\lambda\int d(\star(\nabla\phi_{a_1\dots a_q})\wedge\delta\phi^{a_1\dots a_q}).\end{aligned} \tag{3.28}$$

The first term vanishes on-shell. The second term is the boundary $\int d\Theta$ (see Appendix G). It is somehow clear that the conserved current will be $\star J_{ab}$ defined in (3.13). The associated symmetry is that of rotations in Lorentz inner space. As always, the conserved charge is the generator of the symmetry. So we see that in analogy to electromagnetism and Yang-Mills theories, when matter couples to the connection it acquires a kind of "gravitational charge", in this case related to the spacetime torsion that it creates. Let us see this in detail.
The rotation in Lorentz inner space is

$$\begin{aligned}\phi'^{a_1\dots a_q} &= \Lambda^{a_1}{}_{b_1}\Lambda^{a_2}{}_{b_2}\dots\Lambda^{a_{q-1}}{}_{b_{q-1}}\Lambda^{a_q}{}_{b_q}\phi^{b_1b_2\dots b_{q-1}b_q}\\ &= (\mathbb{1}+\varepsilon\theta)^{a_1}{}_{b_1}(\mathbb{1}+\varepsilon\theta)^{a_2}{}_{b_2}\dots(\mathbb{1}+\varepsilon\theta)^{a_{q-1}}{}_{b_{q-1}}(\mathbb{1}+\varepsilon\theta)^{a_q}{}_{b_q}\phi^{b_1b_2\dots b_{q-1}b_q},\end{aligned} \tag{3.29}$$

where ε is an infinitesimal parameter and θ^{ab} are the generators of Lorentz algebra, which is

$$[\theta_{ab},\theta_{cd}] = \eta_{cb}\theta_{ad} - \eta_{ca}\theta_{bd} + \eta_{db}\theta_{ca} - \eta_{da}\theta_{cb}. \tag{3.30}$$

So we see that neglecting higher powers of $\varepsilon \ll 1$, the symmetry transformation is

$$\delta_{\text{sym}}\phi^{a_1\dots a_q} = \varepsilon\theta^{a_1}{}_{b_1}\phi^{b_1\dots a_q} + \dots + \varepsilon\theta^{a_q}{}_{b_q}\phi^{a_1\dots a_{q-1}b_q}. \tag{3.31}$$

Analogously,

$$\delta_{\text{sym}}\phi_{a_1\dots a_q} = -\varepsilon\theta_{a_1}{}^{b_1}\phi_{b_1\dots a_q} - \dots - \varepsilon\theta_{a_q}{}^{b_q}\phi_{a_1\dots a_{q-1}b_q}. \tag{3.32}$$

Now let us calculate the $\int d\Omega$ boundary term (see Appendix G). We have to take the difference $S'[\phi'] - S[\phi]$. Explicitly

$$\begin{aligned}&S'[\phi'] - S[\phi] =\\ &-\frac{\lambda}{2}\int \star\nabla(\phi_{a_1\dots a_q} - \varepsilon\theta_{a_1}{}^{b_1}\phi_{b_1\dots a_q} - \dots - \varepsilon\theta_{a_q}{}^{b_q}\phi_{a_1\dots a_{q-1}b_q})\wedge\nabla(\phi^{a_1\dots a_q} + \varepsilon\theta^{a_1}{}_{b_1}\phi^{b_1\dots a_q} + \dots + \varepsilon\theta^{a_q}{}_{b_q}\phi^{a_1\dots a_{q-1}b_q})\\ &+\frac{\lambda}{2}\int \star(\nabla\phi_{a_1\dots a_q})\wedge\nabla\phi^{a_1\dots a_q} = 0,\end{aligned} \tag{3.33}$$

where we have neglected powers of ε greater than unity. So we find that in this case $\Omega = 0$. The Noether current is just $\star J = -\Theta$. Thus we are led to

$$\begin{aligned}
\star J &= (-1)^{n-p}\lambda \star (\nabla\phi_{a_1\ldots a_q}) \wedge \delta_{\mathrm{sym}}\phi^{a_1\ldots a_q} \\
&= (-1)^{n-p}\lambda \star (F_{a_1\ldots a_q}) \wedge (\varepsilon\theta^{a_1}{}_{b_1}\phi^{b_1\ldots a_q} + \cdots + \varepsilon\theta^{a_q}{}_{b_q}\phi^{a_1\ldots a_{q-1}b_q}) \\
&= (-1)^{n-p}\varepsilon\lambda q \star (F_{a_1\ldots a_q}) \wedge \phi^{a_1\ldots a_{q-1}}{}_{b_q}\theta^{a_q b_q} \\
&= (-1)^{n}\varepsilon \star J_{ab}\theta^{ab}.
\end{aligned} \tag{3.34}$$

As it is well known, assuming that the spacetime manifold has a topology $R \times \Sigma$, being Σ the spatial section, there is a conserved charge $Q = \int_\Sigma \star J$ which is the generator of the symmetry in the sense of Poisson brackets, $\delta_{\mathrm{sym}}(\cdot) = \{\cdot, Q\}_{\mathrm{PB}}$.

3.6.2. Fermions

For completeness we show that this conclusion is general. The fermionic action for a Dirac field is

$$S_f[e,\omega,\bar\psi,\psi] = \frac{i}{2}\int \star e_a \wedge (\bar\psi\gamma^a\nabla\psi - \overline{\nabla\psi}\gamma^a\psi). \tag{3.35}$$

Here, $\nabla\psi = d\psi - \frac{i}{4}\omega^{ab}\sigma_{ab}\psi$, $\overline{\nabla\psi} = d\bar\psi + \frac{i}{4}\bar\psi\sigma_{ab}\omega^{ab}$ with $\sigma_{ab} = \frac{i}{2}[\gamma_a, \gamma_b]$. Varying with respect to the spin connection and remembering that $\{\gamma^a, \sigma^{bc}\} = 2\epsilon^{abc}{}_d\gamma_5\gamma^d$, the definition

$$\delta_\omega S_f[e,\omega,\bar\psi,\psi] = \int \star J_{ab} \wedge \delta\omega^{ab}, \tag{3.36}$$

gives us

$$\star J_{ab}[e,\bar\psi,\psi] = \frac{1}{4}\epsilon_{abcd} \star e^c j_A^d, \qquad j_A^d = \bar\psi\gamma_5\gamma^d\psi, \tag{3.37}$$

where j_A^d is the axial fermionic current. Varying $S_f[e,\omega,\bar\psi,\psi]$ with respect to $\bar\psi$ we get Dirac equation and a surface term. Explicitly,

$$\begin{aligned}
\delta_{\bar\psi}S_f[e,\omega,\bar\psi,\psi] = {}& \frac{i}{2}\int \delta\bar\psi \star e_a \wedge \gamma^a\nabla\psi - \frac{i}{2}\int \delta\bar\psi\nabla(\star e_a\gamma^a\psi) \\
&+ \frac{i}{2}\int \nabla(\delta\bar\psi \star e_a\gamma^a\psi).
\end{aligned} \tag{3.38}$$

From this we get Dirac equation as

$$i \star e_a \wedge \gamma^a\nabla\psi - \frac{i}{2}\nabla(\star e_a)\gamma^a\psi = 0. \tag{3.39}$$

This, in the usual form, is

$$i\gamma^\mu\mathring{\nabla}_\mu\psi = 0, \tag{3.40}$$

where $\mathring{\nabla}_\mu = \nabla_\mu - \frac{1}{2}T^\nu{}_{\nu\mu}$.

Let us analyze the surface term. Under Lorentz transformations, the spinor transforms as $\psi(x) \to \exp\left(-\frac{i}{4}\varepsilon^{ab}(x)\sigma_{ab}\right)\psi(x)$. This implies that $\delta\psi(x) = -\frac{i}{4}\delta\varepsilon^{ab}\sigma_{ab}\psi(x)$ so $\delta\bar{\psi}(x) = \frac{i}{4}\bar{\psi}(x)\sigma_{ab}\delta\varepsilon^{ab}$. The surface $\mathscr{B}$ is then

$$\mathscr{B} = -\frac{1}{8}\int \nabla\left(\bar{\psi}\sigma_{bc}\delta\varepsilon^{bc} \star e_a\gamma^a\psi\right). \tag{3.41}$$

Varying with respect to ψ, we get the adjoint Dirac equation and another surface term $\mathbb{B}$. Explicitly

$$\mathbb{B} = -\frac{1}{8}\int \nabla\left(\star e_a\bar{\psi}\gamma^a\delta\varepsilon^{bc}\sigma_{bc}\psi\right). \tag{3.42}$$

The total surface is then

$$\begin{aligned}\mathscr{B} + \mathbb{B} &= -\frac{1}{8}\int \nabla\left(\bar{\psi}\sigma_{bc}\delta\varepsilon^{bc} \star e_a\gamma^a\psi\right) - \frac{1}{8}\int \nabla\left(\star e_a\bar{\psi}\gamma^a\delta\varepsilon^{bc}\sigma_{bc}\psi\right)\\ &= -\int d\,(\delta\varepsilon^{bc}\star J_{bc}).\end{aligned} \tag{3.43}$$

3.7. Explicit separation of Riemannian and torsional contributions

It is easy to split the action into Riemannian and torsional parts. The result of doing so is

$$\begin{aligned}S[e,\omega,\phi,p,q] &= -\frac{\lambda}{2}\int \star\tilde{F}_{a_1\ldots a_q}\wedge\tilde{F}^{a_1\ldots a_q} + \int \star J_{ab}\wedge C^{ab}\\ &+ \lambda q\left(q-\frac{1}{2}\right)\int \star(C_{a_q}{}^{f}\wedge\phi_{a_1\ldots a_{q-1}f})\wedge C^{a_q}{}_{d}\wedge\phi^{a_1\ldots a_{q-1}d} - \frac{\lambda q(q-1)}{2}\int \star(C_{a_q}{}^{f}\wedge\phi_{a_1\ldots a_{q-1}f})\wedge C^{a_{q-1}}{}_{d}\wedge\phi^{a_1\ldots d a_q},\end{aligned} \tag{3.44}$$

where the quantities with $\tilde{}$ stand for Riemannian ones (i.e., dependent on the torsion-free Levi-Civita spin connection) and C^{ab} is the contorsion one-form such that $T^a = C^a{}_b \wedge e^b$.

Let us just quote that the analogue separation in the fermionic case gives

$$\begin{aligned}S_f[e,\omega,\bar{\psi},\psi] &= \frac{i}{2}\int \star e_a\wedge(\bar{\psi}\gamma^a\nabla\psi - \overline{\nabla\psi}\gamma^a\psi)\\ &= \frac{i}{2}\int \star e_a\wedge(\bar{\psi}\gamma^a\tilde{\nabla}\psi - \overline{\tilde{\nabla}\psi}\gamma^a\psi) + \int \star J_{ab}\wedge C^{ab}.\end{aligned} \tag{3.45}$$

3.8. Gravitational field

3.8.1. Holst gravity for $n = 4$.

Let us consider the following action for a theory of gravity in four dimensions coupled with arbitrary forms of matter $\Psi(x)$:

$$S_{\text{Holst}}[e,\omega,\Psi,\beta] = -\frac{1}{2\kappa}\int \star(e_a\wedge e_b)\wedge R^{ab} - \frac{1}{2\kappa}\int \beta(x)\,e_a\wedge e_b\wedge R^{ab} + S_{\text{matter}}[e,\omega,\Psi]. \tag{3.46}$$

Here, $\kappa = 8\pi G$ and $\beta = \beta(x)$ is the Barbero-Immirzi (BI) pseudoscalar field. If it instead were a parameter (fixed value) we are led to the action that in the absence of matter, at the Hamiltonian level, corresponds to the canonical approach to gravity of Ashtekar and others known as Loop Quantum Gravity (LQG) [8][9].

Let us define the tensor field ${P_{ab}}^{cd}(x) = \frac{1}{2}{\epsilon_{ab}}^{cd} + \beta(x)\delta_{ab}^{[cd]}$ so our action reads

$$S_{\text{Holst}}[e, \omega, \Psi, \beta] = -\frac{1}{2\kappa}\int {P_{ab}}^{cd}\, e_c \wedge e_d \wedge R^{ab} + S_{\text{matter}}[e, \omega, \Psi]. \tag{3.47}$$

If we vary this action with respect to the vielbein, $\delta_e S_{\text{Holst}} = 0$, we find the analogous of Einstein's equations,

$$ {P_{cd}}^{ab} R^{cd} \wedge e_b = -\kappa \star U^a, \tag{3.48}$$

where $\star U^a$ is the energy-momentum 3-form of the matter fields.
If we vary with respect to the spin connection $\omega^{ab}, \delta_\omega S_{\text{Holst}} = 0$, we find using Palatini's identity $\delta_\omega R^{ab} = \nabla \delta\omega^{ab}$ and integrating by parts that

$$\nabla({P_{ab}}^{cd}\, e_c \wedge e_d) = -2\kappa \star J_{ab}. \tag{3.49}$$

Let us recall Cartan's structure equations:

$$T^a = de^a + {\omega^a}_b \wedge e^b \equiv \nabla e^a, \tag{3.50}$$

$$R^{ab} = d\omega^{ab} + {\omega^a}_c \wedge \omega^{cb}, \tag{3.51}$$

and their respective consistency conditions, the Bianchi identities,

$${R^a}_b \wedge e^b = \nabla T^a, \tag{3.52}$$

$$\nabla R^{ab} = 0. \tag{3.53}$$

Since the equation defining the torsion 2-form is a first order differential equation for the vielbein, we can always find an algebraic solution $\omega^{ab}[e, \Psi] = \tilde{\omega}^{ab}[e] + C^{ab}[e, \Psi]$ so $\tilde{\omega}^{ab}$ is the Riemannian Levi-Civita connection, solution of the homogeneous equation $de^a + {\tilde{\omega}^a}_b \wedge e^b = 0$ and $T^a = {C^a}_b \wedge e^b$. Using this in (3.49) we get that

$${P_{ab}}^{cd}\, {C_c}^f \wedge e_f \wedge e_d = -\frac{1}{2} d\beta \wedge e_a \wedge e_b - \kappa \star J_{ab}. \tag{3.54}$$

3.8.2. Nieh-Yan gravity for $n = 4$

Using Cartan's first structure equation and its consistency condition it is easy to prove that

$$d(e_a \wedge T^a) = T_a \wedge T^a - e_a \wedge e_b \wedge R^{ab}. \tag{3.55}$$

The integral of this 4-form over a compact space is known as the Nieh-Yan topological invariant. Let us then consider the following alternative to Holst action:

$$\begin{aligned}
S_{\rm NY}[e,\omega,\Psi,\beta] = &-\frac{1}{2\kappa}\int \star(e_a\wedge e_b)\wedge R^{ab} - \frac{1}{2\kappa}\int \beta(x)\, e_a\wedge e_b\wedge R^{ab} + \frac{1}{2\kappa}\int \beta(x)\, T^a\wedge T_a \\
&+ S_{\rm matter}[e,\omega,\Psi], \\
=&-\frac{1}{2\kappa}\int \star(e_a\wedge e_b)\wedge R^{ab} + \frac{1}{2\kappa}\int \beta(x)\, d(e_a\wedge T^a) + S_{\rm matter}[e,\omega,\Psi], \\
=&-\frac{1}{2\kappa}\int \star(e_a\wedge e_b)\wedge R^{ab} - \frac{1}{2\kappa}\int d\beta\wedge\, e_a\wedge T^a + S_{\rm matter}[e,\omega,\Psi] \qquad (3.56)
\end{aligned}$$

Varying with respect to the vielbein, $\delta_e S_{\rm NY}=0$, gives

$$\star R_{ab}\wedge e^b = d\beta\wedge T_a - \kappa\star U_a. \qquad (3.57)$$

Varying with respect to the spin connection, $\delta_\omega S_{\rm NY}=0$, gives

$${\epsilon_{ab}}^{cd} {C_c}^f \wedge e_f\wedge e_d = -d\beta\wedge e_a\wedge e_b - 2\kappa\star J_{ab}. \qquad (3.58)$$

3.9. Substituting solutions for algebraic equations of motion within the action

First order gravity à la Palatini was developed in first place as a computational tool only. It is a trick to vary in a quick and easy way the Hilbert-Einstein action to get Einstein's field equations. Einstein-Cartan theory is however, inequivalent to Einstein's theory in the presence of fermionic matter as it is a torsion source because it couples with the spin connection. It is a well known fact that the vielbein and the spin connection are independent fields from a geometrical point of view. The vielbein defines a notion of metricity and the connection that of affinity of space. These properties are not necessarily linked as Cartan understood. However, in the theories of gravity we are considering, the equation of motion for the spin connection is algebraic so it can be solved and then we are allowed to put this back into the original action leaving us an equivalent action at least at the classical level.
For instance we can split the curvature 2-form as

$$R^{ab} = \tilde{R}^{ab} + \bar{R}^{ab}, \qquad (3.59)$$

$$\tilde{R}^{ab} = d\tilde{\omega}^{ab} + {\tilde{\omega}^a}_c\wedge\tilde{\omega}^{cb}, \qquad (3.60)$$

$$\bar{R}^{ab} = \nabla C^{ab} - {C^a}_c\wedge C^{cb}, \qquad (3.61)$$

where as before, $\tilde{\omega}^{ab}$ is the Levi-Civita spin connection and C^{ab} is the contorsion 1-form such that $T^a = {C^a}_b\wedge e^b$, T^a being the torsion 2-form. Here $\tilde{R}^{ab}$ is the Riemannian part of the curvature 2-form such that ${\tilde{R}^a}_b\wedge e^b=0$. With these expressions we will be able to easily separate the Riemannian and torsional contributions in the gravitational sector.

For completeness let us recall the theorem behind the fact that algebraic equations of motion can be pulled back into the action giving a completely equivalent theory [62][48]:

Let $S(q_i, Q_j)$ be an action depending on two sets of dynamical variables, q_i and Q_j. The solutions of the dynamical equations are extrema of the action with respect to both sets of variables. If the dynamical equations $\frac{\partial S}{\partial q_i} = 0$ have a unique solution, $q_i^{(0)}(Q_j)$ for each choice of Q_j, then the pull-back $S(q_i(Q_j), Q_j)$ of the action to the set of solutions has the property that its extrema are precisely the extrema of the total action $S(q_i, Q_j)$.

3.10. Toy models

In this section we review some "toy models" which can serve as an insight about the behavior of the objects we have just defined. They are not to be taken so seriously and in some cases they are just mere statements for future work. The actual calculations that will survive severe judgement and criticism will be further discussed elsewhere [73].

3.10.1. Particular case: $n = 4$, $p = 1$, $q = 1$

Let us consider a Lorentz valued 0-form, ϕ^a. Now let us perform a Wick rotation so our spacetime becomes Euclid and compact and our forms are now $SO(n)$ valued. We can always come back to the Lorentzian case by reverting the Wick rotation. The action for this object is ($\lambda = 1$),

$$S[e, \omega, \phi, 1, 1] = -\frac{1}{2} \int \star(\nabla\phi_a) \wedge \nabla\phi^a, \tag{3.62}$$

its associated energy-momentum 3-form is

$$\star U_i[e, \omega, \phi, 1, 1] = \frac{1}{2} F_{ai} \star F^a + \frac{1}{4} F_{ab}\, \epsilon^b{}_{fgi} F^a \wedge e^f \wedge e^g, \tag{3.63}$$

and its associated spin-torsion 3-form is

$$\begin{aligned} \star J_{ab}[e, \omega, \phi, 1, 1] &= - \star F_{[a}\phi_{b]} \\ &= -\frac{1}{2}\{\star(\nabla\phi_a)\phi_b - \star(\nabla\phi_b)\phi_a\}. \end{aligned} \tag{3.64}$$

$\beta = 0$ case

Let us see what this implies in the Einstein-Cartan theory. The equation we must solve is (3.54) making $\beta = 0$. So we have

$$\begin{aligned} \frac{1}{2}\epsilon_{ab}{}^{cd} C_c{}^f \wedge e_f \wedge e_d &= -\kappa \star J_{ab} \\ &= -\kappa \star \tilde{J}_{ab} + \frac{\kappa}{2}\{\star(C_a{}^f \phi_f)\phi_b - \star(C_b{}^f \phi_f)\phi_a\}, \end{aligned} \tag{3.65}$$

where we define

$$\star\tilde{J}_{ab} = -\frac{1}{2}\{\star(\tilde{\nabla}\phi_a)\phi_b - \star(\tilde{\nabla}\phi_b)\phi_a\}. \tag{3.66}$$

Taking the Hodge dual, remembering that for a p-form ω_p in euclidean space, $\star\star\omega_p = (-1)^{p(n-p)}\omega_p$ so in particular $\star\star e^i = -e^i$, defining $(\star\star\tilde{J}_{ab})_i e^i \equiv J_{abi}e^i$ and using the fact that $\eta_{ab} = \delta_{ab}$, (3.65) can be written in the following way:

$$\frac{1}{2}\epsilon_{abcd}C_{clj}\varepsilon_{ijld} - \frac{\kappa}{2}\{C_{ali}\phi_b\phi_l - C_{bli}\phi_a\phi_l\} - \kappa J_{abi} = 0, \tag{3.67}$$

remembering the fact that $C_{abi} = -C_{bai}$ and $J_{abi} = -J_{bai}$.
Now we introduce the following notation: Let $A_{ijk\ldots rst}$ be a generic tensor field. From now on we will call it $A_{ijk\ldots rst} \equiv A(i,j,k,\ldots,r,s,t)$. In this manner we will shorten the notation for objects like $A_{ijk\ldots rst}\varphi_i \equiv A(\varphi,j,k,\ldots,r,s,t)$ where $\varphi_i \equiv \varphi(i)$ is a generic vector field.
Our equation becomes

$$\begin{aligned}&\frac{1}{2}C(\phi,a,i)\phi(b)\kappa - \frac{1}{2}C(\phi,b,i)\phi(a)\kappa + \frac{1}{2}C(a,i,b) - \frac{1}{2}C(a,l,l)d(b,i)\\&-\frac{1}{2}C(b,i,a) + \frac{1}{2}C(b,l,l)d(a,i) - J(a,b,i)\kappa = 0,\end{aligned} \tag{3.68}$$

where we also denote $d(a,b) \equiv \delta_{ab}$.
In 4 dimensions C_{abi} has 24 independent components. According to $SO(4)$ it can always be decomposed as $24 = 4+4+16$ in the following way:

$$C_{ijk} = \frac{1}{3}\{T_j\delta_{ik} - T_i\delta_{jk}\} + \frac{1}{6}\epsilon_{ijkl}S_l + q_{ijk}, \tag{3.69}$$

where $T_j \equiv C_{iji}$ is a trace vector, $S_l = \epsilon_{ijkl}C_{ijk}$ is a pseudovector dual to the completely antisymmetric part of C_{ijk} and $q_{ijk} = -q_{jik}$ is a tensor whose trace and completely antisymmetric part are zero, i.e., $q_{iji} = \epsilon_{ijkl}q_{ijk} = 0$.
Aplying this splitting we get

$$\begin{aligned}&\frac{1}{6}T(a)\phi(b)\phi(i)\kappa - \frac{1}{6}T(b)\phi(a)\phi(i)\kappa - \frac{1}{3}d(a,i)T(b) - \frac{1}{6}d(a,i)\phi(b)T\cdot\phi\kappa + \frac{1}{3}d(b,i)T(a)\\&+\frac{1}{6}d(b,i)\phi(a)T\cdot\phi\kappa - \frac{1}{12}\varepsilon(S,\phi,a,i)\phi(b)\kappa + \frac{1}{12}\varepsilon(S,\phi,b,i)\phi(a)\kappa + \frac{1}{6}\varepsilon(S,a,b,i)\\&-J(a,b,i)\kappa + \frac{1}{2}q(\phi,a,i)\phi(b)\kappa - \frac{1}{2}q(\phi,b,i)\phi(a)\kappa + \frac{1}{2}q(a,i,b) - \frac{1}{2}q(b,i,a) \equiv E(a,b,i) = 0,\end{aligned} \tag{3.70}$$

where $E(a,b,i) = -E(b,a,i)$ has been defined and $X\cdot Y \equiv X_iY_i$. Now let us take the product $d(a,i)E(a,b,i) = 0$. This gives us

$$-T(b) - \frac{1}{6}T(b)\phi^2\kappa - \frac{1}{3}\phi(b)T\cdot\phi\kappa + J(b,l,l)\kappa - \frac{1}{2}q(\phi,b,\phi)\kappa = 0, \tag{3.71}$$

where $\phi^2 \equiv \phi\cdot\phi$.
Let us now consider $\phi(b)d(a,i)E(a,b,i) = 0$. This is

$$-\frac{1}{2}T\cdot\phi\phi^2\kappa - T\cdot\phi + J(\phi,l,l)\kappa = 0, \tag{3.72}$$

so we find that

$$T \cdot \phi = \frac{2\kappa}{[2+\kappa\phi^2]} J(\phi, l, l). \tag{3.73}$$

Now let us consider $\phi(a)\phi(i)E(a,b,i) = 0$. This is

$$\begin{aligned}&-\frac{1}{3}T(b)\phi^2 - \frac{1}{6}T(b)\phi^4\kappa + \frac{1}{6}\phi(b)T\cdot\phi\phi^2\kappa + \frac{1}{3}\phi(b)T\cdot\phi - J(\phi,b,\phi)\kappa \\ &+ \frac{1}{2}q(\phi,b,\phi) - \frac{1}{2}q(\phi,b,\phi)\phi^2\kappa = 0.\end{aligned} \tag{3.74}$$

From this we have that

$$\begin{aligned} q(\phi,b,\phi) &= \frac{1}{3}\frac{[2+\kappa\phi^2]}{[1-\kappa\phi^2]}\{T(b)\phi^2 - \phi(b)T\cdot\phi\} + \frac{2\kappa}{[1-\kappa\phi^2]}J(\phi,b,\phi) \\ &= \frac{1}{3}\frac{\phi^2[2+\kappa\phi^2]}{[1-\kappa\phi^2]}T(b) - \frac{2\kappa}{3[1-\kappa\phi^2]}\phi(b)J(\phi,l,l) + \frac{2\kappa}{[1-\kappa\phi^2]}J(\phi,b,\phi). \end{aligned} \tag{3.75}$$

Using these relations we finally obtain that

$$T(b) = \frac{2\kappa[1-\kappa\phi^2]}{[2-\kappa\phi^2]}J(b,l,l) + \frac{2\kappa^3\phi^2}{[2-\kappa\phi^2][2+\kappa\phi^2]}\phi(b)J(\phi,l,l) - \frac{2\kappa^2}{[2-\kappa\phi^2]}J(\phi,b,\phi), \tag{3.76}$$

and for completeness we give $q(\phi,b,\phi)$ as a function of ϕ only:

$$q(\phi,b,\phi) = \frac{2\kappa\phi^2[2+\kappa\phi^2]}{3[2-\kappa\phi^2]}J(b,l,l) + \frac{2\kappa[6-5\kappa\phi^2-\kappa^2\phi^4]}{3[1-\kappa\phi^2][2-\kappa\phi^2]}J(\phi,b,\phi) - \frac{2\kappa[2+\kappa\phi^2]}{3[2-\kappa\phi^2]}\phi(b)J(\phi,l,l). \tag{3.77}$$

Now we consider the following product $\varepsilon(a,b,i,m)E(a,b,i) = 0$, which is

$$-S(m) - \frac{1}{3}S(m)\phi^2\kappa + \frac{1}{3}\phi(m)S\cdot\phi\kappa + J(f,g,h)\varepsilon(m,f,g,h)\kappa - q(\phi,f,g)\varepsilon(\phi,m,f,g)\kappa = 0. \tag{3.78}$$

It is clear that taking $\phi(m)\varepsilon(a,b,i,m)E(a,b,i) = 0$ should tell us what $S\cdot\phi$ is. In doing so we get

$$S\cdot\phi = \kappa\varepsilon(\phi,f,g,h)J(f,g,h). \tag{3.79}$$

Now we consider the combination $E(a,b,i)+E(a,i,b)-E(b,i,a) \equiv H(a,b,i) = 0$. If we take the product $\phi(a)\varepsilon(\phi,m,b,i)H(a,b,i) = 0$, we get

$$\frac{1}{3}S(m)\phi^2 - \frac{1}{3}\phi(m)S\cdot\phi + J(f,g,\phi)\varepsilon(\phi,m,f,g)\kappa + q(\phi,f,g)\varepsilon(\phi,m,f,g) = 0. \tag{3.80}$$

Using these equations we find that

$$S(m) = \kappa\varepsilon(m,f,g,h)J(f,g,h) + \kappa^2\varepsilon(\phi,m,f,g)J(f,g,\phi). \tag{3.81}$$

Let us now focus on $\phi(a)H(a,b,i)=0$ because this will allow us to see what $q(\phi,b,i)$ is. This is because the explicit expression is

$$\begin{aligned}
&-\frac{2}{3}T(b)\phi(i)-\frac{1}{3}T(b)\phi(i)\phi^2\kappa+\frac{1}{3}d(b,i)T\cdot\phi\phi^2\kappa+\frac{2}{3}d(b,i)T\cdot\phi-\frac{1}{6}\varepsilon(S,\phi,b,i)\\
&-J(\phi,b,i)\kappa-J(\phi,i,b)\kappa+J(b,i,\phi)\kappa-\frac{1}{2}q(\phi,b,\phi)\phi(i)\kappa+q(\phi,b,i)\\
&-\frac{1}{2}q(\phi,b,i)\phi^2\kappa+\frac{1}{2}q(\phi,i,\phi)\phi(b)\kappa-\frac{1}{2}q(\phi,i,b)\phi^2\kappa\equiv D(b,i)=0. \qquad (3.82)
\end{aligned}$$

If we now consider $D(i,b)=0$ we can isolate $q(\phi,i,b)$ as

$$\begin{aligned}
q(\phi,i,b)=&\frac{2}{3}\frac{[2+\kappa\phi^2]}{[2-\kappa\phi^2]}\{T(i)\phi(b)-d(b,i)T\cdot\phi\}-\frac{1}{3[2-\kappa\phi^2]}\varepsilon(S,\phi,b,i)+\frac{2\kappa}{[2-\kappa\phi^2]}J(\phi,b,i)\\
&+\frac{2\kappa}{[2-\kappa\phi^2]}J(\phi,i,b)+\frac{2\kappa}{[2-\kappa\phi^2]}J(b,i,\phi)-\frac{\kappa}{[2-\kappa\phi^2]}q(\phi,b,\phi)\phi(i)\\
&+\frac{\kappa\phi^2}{[2-\kappa\phi^2]}q(\phi,b,i)+\frac{\kappa}{[2-\kappa\phi^2]}q(\phi,i,\phi)\phi(b). \qquad (3.83)
\end{aligned}$$

If we substitute the value of $q(\phi,b,\phi)$ we get that

$$\begin{aligned}
q(\phi,b,i)=&\frac{[2+\kappa\phi^2]}{3[1-\kappa\phi^2]}\{T(b)\phi(i)-d(b,i)T\cdot\phi\}+\frac{1}{6}\varepsilon(S,\phi,b,i)\\
&+\frac{\kappa}{[1-\kappa\phi^2]}J(\phi,b,i)+\frac{\kappa}{[1-\kappa\phi^2]}J(\phi,i,b)-\kappa J(b,i,\phi)\\
&+\frac{\kappa^2}{[1-\kappa\phi^2]}\phi(i)J(\phi,b,\phi)-\frac{\kappa^2}{[1-\kappa\phi^2]}\phi(b)J(\phi,i,\phi). \qquad (3.84)
\end{aligned}$$

Explicitly in terms of ϕ this is

$$\begin{aligned}
q(\phi,b,i)=&\frac{2\kappa}{3}\frac{[2+\kappa\phi^2]}{[2-\kappa\phi^2]}\phi(i)J(b,l,l)-\frac{2\kappa}{3[1-\kappa\phi^2]}J(\phi,l,l)d(b,i)+\frac{\kappa}{3}\frac{[4-\kappa\phi^2]}{[1-\kappa\phi^2]}J(\phi,b,i)\\
&+\frac{\kappa}{3}\frac{[2+\kappa\phi^2]}{[1-\kappa\phi^2]}J(\phi,i,b)-\frac{\kappa[2+\kappa\phi^2]}{3}J(b,i,\phi)+\frac{2}{3}\frac{\kappa^3\phi^2}{[1-\kappa\phi^2][2-\kappa\phi^2]}\phi(b)\phi(i)J(\phi,l,l)\\
&-\frac{1}{3}\frac{\kappa^3\phi^2[2+\kappa\phi^2]}{[1-\kappa\phi^2][2-\kappa\phi^2]}\phi(i)J(\phi,b,\phi)-\frac{1}{3}\frac{\kappa^2[2+\kappa\phi^2]}{[1-\kappa\phi^2]}\phi(b)J(\phi,i,\phi). \qquad (3.85)
\end{aligned}$$

The reason we have considered $H(a,b,i)=0$ is that in doing so we can isolate the term $q(a,b,i)$. Explicitly $H(a,b,i)$ is

$$\begin{aligned}
&\frac{1}{3}T(a)\phi(b)\phi(i)\kappa-\frac{1}{3}T(b)\phi(a)\phi(i)\kappa-\frac{2}{3}d(a,i)T(b)-\frac{1}{3}d(a,i)\phi(b)T\cdot\phi\kappa+\frac{2}{3}d(b,i)T(a)\\
&+\frac{1}{3}d(b,i)\phi(a)T\cdot\phi\kappa-\frac{1}{6}\varepsilon(S,\phi,a,b)\phi(i)\kappa-\frac{1}{6}\varepsilon(S,a,b,i)-J(a,b,i)\kappa-J(a,i,b)\kappa+J(b,i,a)\kappa\\
&+\frac{1}{2}q(\phi,a,b)\phi(i)\kappa+\frac{1}{2}q(\phi,a,i)\phi(b)\kappa-\frac{1}{2}q(\phi,b,a)\phi(i)\kappa-\frac{1}{2}q(\phi,b,i)\phi(a)\kappa+\frac{1}{2}q(\phi,i,a)\phi(b)\kappa\\
&-\frac{1}{2}q(\phi,i,b)\phi(a)\kappa+q(a,b,i)=0. \qquad (3.86)
\end{aligned}$$

We see that we already have all the ingredients to express unequivocally $q(a,b,i)$ as a function of ϕ. So,

$$\begin{aligned}
q(a,b,i) &= \frac{1}{3}T(b)\{\kappa\phi(a)\phi(i)+2d(a,i)\} - \frac{1}{3}T(a)\{\kappa\phi(b)\phi(i)+2d(b,i)\} \\
&\quad + \frac{\kappa}{3}T\cdot\phi\{\phi(b)d(a,i)-\phi(a)d(b,i)\} + \frac{\kappa}{6}\varepsilon(S,\phi,a,b)\phi(i) + \frac{1}{6}\varepsilon(S,a,b,i) + \kappa J(a,b,i) \\
&\quad + \kappa J(a,i,b) - \kappa J(b,i,a) - \frac{\kappa}{2}\phi(i)\{q(\phi,a,b)-q(\phi,b,a)\} - \frac{\kappa}{2}\phi(b)\{q(\phi,a,i)+q(\phi,i,a)\} \\
&\quad + \frac{\kappa}{2}\phi(a)\{q(\phi,b,i)+q(\phi,i,b)\}.
\end{aligned} \tag{3.87}$$

Explicitly as a function of ϕ this is

$$\begin{aligned}
q(a,b,i) &= \frac{2\kappa}{3}\{2J(a,b,i)+J(a,i,b)-J(b,i,a)\} + \frac{\kappa^2}{3}\{2\phi(i)J(a,b,\phi)+\phi(b)J(a,i,\phi)-\phi(a)J(b,i,\phi)\} \\
&- \frac{\kappa^2}{[1-\kappa\phi^2]}\{\phi(b)J(\phi,a,i)+\phi(b)J(\phi,i,a)-\phi(a)J(\phi,b,i)-\phi(a)J(\phi,i,b)\} \\
&- \frac{2\kappa^2}{[2-\kappa\phi^2]}\{\phi(b)\phi(i)J(a,l,l)-\phi(a)\phi(i)J(b,l,l)\} + \frac{4\kappa^2}{3[2-\kappa\phi^2]}\{J(\phi,a,\phi)d(b,i)-J(\phi,b,\phi)d(a,i)\} \\
&- \frac{4\kappa[\frac{2}{3}-\kappa\phi^2+\frac{\kappa^3}{3}\phi^6]}{[1-\kappa^2][2+\kappa^2][2-\kappa^2]}\{J(a,l,l)d(b,i)-J(b,l,l)d(a,i)\} \\
&+ \frac{2\kappa^2[2-\frac{\kappa}{3}\phi^2-\frac{2\kappa^2}{3}\phi^4]}{[1-\kappa\phi^2][2-\kappa\phi^2][2+\kappa\phi^2]}J(\phi,l,l)\{d(a,i)\phi(b)-d(b,i)\phi(a)\} \\
&+ \frac{\kappa^4\phi^2}{[1-\kappa\phi^2][2-\kappa\phi^2]}\phi(i)\{\phi(b)J(\phi,a,\phi)-\phi(a)J(\phi,b,\phi)\}.
\end{aligned} \tag{3.88}$$

Finally we find our contorsion tensor, solution of the equation of motion:

$$\begin{aligned}
C(a,b,i) &= \kappa J(a,b,i) + \kappa J(a,i,b) - \kappa J(b,i,a) + \kappa^2\phi(i)J(a,b,\phi) \\
&\quad + \frac{\kappa^2}{[1-\kappa\phi^2]}\Big\{\phi(a)J(\phi,b,i)+\phi(a)J(\phi,i,b)-\phi(b)J(\phi,a,i)-\phi(b)J(\phi,i,a)\Big\} \\
&\quad + \frac{2\kappa}{[2-\kappa\phi^2]}d(b,i)\Big\{\kappa J(\phi,a,\phi) - \frac{\kappa[2-\kappa^2\phi^4]}{[1-\kappa\phi^2][2+\kappa\phi^2]}\phi(a)J(\phi,l,l) - [1-\kappa\phi^2]J(a,l,l)\Big\} \\
&\quad - \frac{2\kappa}{[2-\kappa\phi^2]}d(a,i)\Big\{\kappa J(\phi,b,\phi) - \frac{\kappa[2-\kappa^2\phi^4]}{[1-\kappa\phi^2][2+\kappa\phi^2]}\phi(b)J(\phi,l,l) - [1-\kappa\phi^2]J(b,l,l)\Big\} \\
&\quad + \frac{\kappa^2}{[2-\kappa\phi^2]}\phi(b)\phi(i)\Big\{\frac{\kappa^2\phi^2}{[1-\kappa\phi^2]}J(\phi,a,\phi) - 2J(a,l,l)\Big\} \\
&\quad - \frac{\kappa^2}{[2-\kappa\phi^2]}\phi(a)\phi(i)\Big\{\frac{\kappa^2\phi^2}{[1-\kappa\phi^2]}J(\phi,b,\phi) - 2J(b,l,l)\Big\}.
\end{aligned} \tag{3.89}$$

The total action is

$$\begin{aligned}
S_{\text{total}}[e,\omega,\phi] &= -\frac{1}{2\kappa}\int \star(e_a\wedge e_b)\wedge\tilde{R}^{ab} - \frac{1}{2}\int \star\tilde{F}_a\wedge\tilde{F}^a - \frac{1}{2\kappa}\int\star(e_a\wedge e_b)\wedge C^a{}_d\wedge C^{db} \\
&\quad + \frac{1}{2}\int \star C_a{}^f\wedge C^a{}_d\,\phi_f\phi^d.
\end{aligned} \tag{3.90}$$

The first two terms are the usual Riemannian ones. The other two are torsional contributions which depend upon the contorsion one-form. We see that we get a nontrivial interaction potential for the field

ϕ^a. The appearance of denominators which depend upon $\kappa\phi^2$ is a charasteristic feature of the problem. So for certain configurations of the field ϕ^a the interaction terms can grow enormously within the action or the dynamics, recalling that the equation of motion $\nabla(\star\nabla\phi_a) = 0$ contains torsional terms in the covariant derivative. However, it is almost an impossible task to solve at least analytically the equation of motion for our field and we do not have extra parameters to play with so we will not be able to control the dynamics for our own convenience if for example, we would like to use this toy model as a viable alternative for current Inflation Theory. The good thing about our toy model is that the "potential" is univoquely defined through the algebraic equation of motion for the spin connection and so would be a falsifiable self-contained proposal instead of an ad-hoc ansatz for the "inflaton" potential. It could be argued that a proper Wick rotation will give us back a faithful Lorentzian expression but in order to be sure we must repeat this calculation with a Lorentzian η from the beginning. The vacuum will be carefully analized elsewhere [73].

$\beta \neq 0$ case

Let us now consider the case when $\beta \neq 0$, but a finite constant. The equation of motion we have to solve is

$$\begin{aligned}&\frac{1}{2}C(\phi,a,i)\phi(b)\kappa - \frac{1}{2}C(\phi,b,i)\phi(a)\kappa + \frac{1}{2}C(a,i,b) - \frac{1}{2}C(a,l,l)d(b,i) + \frac{1}{2}C(a,f,g)\varepsilon(b,i,f,g)\beta \\ &- \frac{1}{2}C(b,i,a) + \frac{1}{2}C(b,l,l)d(a,i) - \frac{1}{2}C(b,f,g)\varepsilon(a,i,f,g)\beta - J(a,b,i)\kappa = 0. \qquad (3.91)\end{aligned}$$

We will not split the C tensor in terms of irreducible parts as before, mainly because doing so does not give us any new insights but makes the analysis more obscure. This is because β behaves as a pseudo-number (like a Vacuum Expectation Value (VEV) of a pseudo-scalar field) and mixes up the nature of them. We will just quote the result here because the difficulty grows exponentially when we consider a non-vanishing β term. After a pretty long and tedious calculation we get that

$$\begin{aligned}C(a,b,i) &= \frac{2\beta^4}{[1+\beta^2-2\beta^4]}\Big\{C(b,l,l)d(a,i) - C(a,l,l)d(b,i)\Big\} \\ &+ \frac{[1+\beta^2]}{[1+\beta^2-2\beta^4]}\Big\{A(a,b,i) + \beta A(i,f,g)\epsilon(a,b,f,g)\Big\} + \frac{\kappa\beta^2}{[1+\beta^2-2\beta^4]}\Big\{\phi(a)C(\phi,b,i) - \phi(b)C(\phi,a,i)\Big\} \\ &+ \frac{2\kappa\beta^2}{[1+\beta^2-2\beta^4]}J(a,b,i) + \frac{\beta^3}{[1+\beta^2-2\beta^4]}\Big\{A(b,f,g)\epsilon(a,i,f,g) - A(a,f,g)\epsilon(b,i,f,g)\Big\}. \qquad (3.92)\end{aligned}$$

In equation (3.92) all the terms in the right hand side are given by well established expressions which finally reduce to nonlinear functions of ϕ^a and β (See Appendix H). We see that β acquires classical "measurable" effects when matter that couples with the spin connection of gravity is taken into account. This we knew for Dirac fermions but if bosons could "feel" the gauge connection of gravity, they could also tell us something about the rather ambigous β parameter of LQG. We observe that β enters in the denominators of our expressions so certain configurations of the ϕ^a field for a given β could dominate the dynamics during the Inflation scenario.

The value for the Immirzi parameter is somehow fixed by the semiclassical value of black hole entropy $S = \frac{1}{4}A$ (in Planck units), where A stands for the area of the event horizon of the black hole. In LQG one gets that $S = \frac{\gamma_0}{4\gamma}A$, recalling that $\beta = -\frac{1}{\gamma}$. So we should set $\beta = -\frac{1}{\gamma_0}$ if we want to recover the famous Hawking's formula. Between several estimates of γ_0 one can find $\frac{\ln(2)}{\sqrt{3}\pi}$, $\frac{\ln(3)}{\sqrt{8}\pi}$, $\frac{\ln(3)}{\sqrt{2}\pi}$. Unfortunately since $|\gamma| < 1$, $|\beta| > 1$ so a "perturbative" approach does not apply and we must retain all powers of β in our expressions.

It has been argued that β should be thought of as a pseudoscalar field (like the axion) and what we call the Immirzi parameter is a vacuum expectation value $\langle\beta\rangle_0$. If this is the case the difficulty increases but also the richness of the solution.

Finally it must be stressed that in order to apply these ideas to a realistic scenario we should recover Lorentzian expressions. The Euclidean option is easier in dealing with **FORM** [63][64] but lacks of realism.

3.10.2. Equation of state for the Lorentz-valued scalar

Let us consider

$$S = -\frac{1}{2}\int d^4x\sqrt{-g}\nabla_\mu\phi_\nu\nabla^\mu\phi^\nu = -\frac{1}{2}\int d^4x\sqrt{-g}F_{\mu\nu}F^{\mu\nu}, \tag{3.93}$$

where $\nabla_\mu\phi_\nu = \partial_\mu\phi_\nu - \Gamma^\rho{}_{\nu\mu}\phi_\rho$. We define

$$\Pi^\nu = \frac{\delta S}{\delta\mathring{\phi}_\nu} = -N\sqrt{q}F^{0\nu}, \tag{3.94}$$

so

$$S = \int d^3x\int dt\left\{\Pi^\nu\mathring{\phi}_\nu - \frac{1}{2}\Pi^\nu F_{0\nu} - \frac{N\sqrt{q}}{2}F_{i\nu}F^{i\nu}\right\}, \tag{3.95}$$

where $\mathring{\phi}_\nu = \partial_0\phi_\nu - \Gamma^\rho{}_{\nu0}\phi_\rho$, or using the fact that

$$F_{0\lambda} = \frac{N}{\sqrt{q}}g_{\nu\lambda}\Pi^\nu + N^iF_{i\lambda}, \tag{3.96}$$

we get that

$$S = \int d^3x\int dt\left\{\Pi^\nu\mathring{\phi}_\nu - \frac{1}{2}N^i\Pi^\nu F_{i\nu} - N\left(\frac{1}{2\sqrt{q}}\Pi^\mu\Pi^\nu g_{\mu\nu} + \frac{\sqrt{q}}{2}F_{i\nu}F^{i\nu}\right)\right\}. \tag{3.97}$$

Now recalling that $\rho = \frac{1}{\sqrt{q}}\frac{\delta H}{\delta N}$ and $P = -\frac{2}{3N\sqrt{q}}q_{ab}\frac{\delta H}{\delta q_{ab}}$ (See Appendix E), we get that

$$\rho = \frac{1}{2q}\Pi^\mu\Pi^\nu g_{\mu\nu} + \frac{1}{2}F_{i\nu}F^{i\nu}. \tag{3.98}$$

A long but straightforward calculation gives

$$P = \frac{1}{3}\rho + B, \tag{3.99}$$

with

$$B = \frac{1}{3}\left\{\frac{1}{q}\Pi^\mu\Pi^\nu g_{\mu\nu} - \frac{1}{q}\Pi^a\Pi^b q_{ab} - 2F_{i\nu}F^{i\nu} + q_{ab}g^{ia}F_{i\nu}F^{b\nu} + q_{ab}g^{\nu a}F_{i\nu}F^{ib}\right\}, \tag{3.100}$$

so a gas of massless matter described by this dynamical theory would not respect the traditional radiation-like equation of state at the classical level as in the case of fermions [65].

3.10.3. Lorentz-valued scalar as the inflaton

Let us consider the action

$$S[g,\Gamma,\phi] = \frac{m_{Pl}^2}{16\pi}\int d^4x\sqrt{-g}R(\Gamma) + \frac{1}{2}\int d^4x\sqrt{-g}\nabla_\mu\phi_\nu\nabla^\mu\phi^\nu, \tag{3.101}$$

which after varying with respect to Γ, solving the algebraic equation of motion, and putting the solution back takes the form

$$S[g,\phi] = \frac{m_{Pl}^2}{16\pi}\int d^4x\sqrt{-g}\tilde{R}(\tilde{\Gamma}) + \int d^4x\sqrt{-g}\left\{\frac{1}{2}\tilde{\nabla}_\mu\phi_\nu\tilde{\nabla}^\mu\phi^\nu - V(\phi)\right\}. \tag{3.102}$$

The energy-momentum tensor $T^{\alpha\beta} \equiv \frac{-2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_M)}{\delta g_{\alpha\beta}}$ is

$$T^{\alpha\beta} = \nabla^\alpha\phi_\mu\nabla^\beta\phi^\mu + \nabla_\mu\phi^\alpha\nabla^\mu\phi^\beta - g^{\alpha\beta}\mathcal{L}_M. \tag{3.103}$$

We know that the connection Γ can always be decomposed as $\Gamma^\alpha{}_{\beta\gamma} = \tilde{\Gamma}^\alpha{}_{\beta\gamma} + K^\alpha{}_{\beta\gamma}$, where $\tilde{\Gamma}$ stands for the Levi-Civita connection (Christoffel symbol) and K is the contorsion tensor. Considering this, we can rewrite the energy-momentum tensor as

$$T^\alpha{}_\beta = \tilde{\nabla}^\alpha\phi^\mu\tilde{\nabla}_\beta\phi_\mu + \tilde{\nabla}^\mu\phi^\alpha\tilde{\nabla}_\mu\phi_\beta + W^\alpha{}_\beta - \delta^\alpha{}_\beta\mathcal{L}_M, \tag{3.104}$$

being W a symmetric tensor self-defined through the equation above and $\mathcal{L}_M$ is now taken to be

$$\mathcal{L}_M = \frac{1}{2}\tilde{\nabla}_\mu\phi_\nu\tilde{\nabla}^\mu\phi^\nu - V. \tag{3.105}$$

Recalling that $\rho \equiv T^0{}_0$ and defining $P^1 \equiv -T^1{}_1$ we find that

$$\rho = \frac{3}{2}\tilde{\nabla}_0\phi_0\tilde{\nabla}^0\phi^0 + \frac{1}{2}\tilde{\nabla}_0\phi_i\tilde{\nabla}^0\phi^i + \frac{1}{2}\tilde{\nabla}_i\phi_0\tilde{\nabla}^i\phi^0 - \frac{1}{2}\tilde{\nabla}_i\phi_j\tilde{\nabla}^i\phi^j + W^0{}_0 + V, \tag{3.106}$$

$$P^1 = -\tilde{\nabla}^1\phi_\mu\tilde{\nabla}_1\phi^\mu - \tilde{\nabla}_\mu\phi^1\tilde{\nabla}^\mu\phi_1 - W^1{}_1 + \mathcal{L}_M, \tag{3.107}$$

where "1" stands for any spatial index. We will be interested in the "mean pressure" $\bar{P} = \frac{\sum_i P^i}{3}$ which turns to be

$$\bar{P} = -\frac{1}{3}\tilde{\nabla}_i\phi_\mu\tilde{\nabla}^i\phi^\mu - \frac{1}{3}\tilde{\nabla}_\mu\phi_i\tilde{\nabla}^\mu\phi^i - \frac{1}{3}W^i{}_i + \mathcal{L}_M \tag{3.108}$$

$$= \frac{1}{2}\tilde{\nabla}_0\phi_0\tilde{\nabla}^0\phi^0 + \frac{1}{6}\tilde{\nabla}_0\phi_i\tilde{\nabla}^0\phi^i + \frac{1}{6}\tilde{\nabla}_i\phi_0\tilde{\nabla}^i\phi^0 - \frac{1}{6}\tilde{\nabla}_i\phi_j\tilde{\nabla}^i\phi^j - \frac{1}{3}W^i{}_i - V. \tag{3.109}$$

Now the only nonvanishing Christoffel symbols for the flat Friedmann-Robertson-Walker metric (I.3) are ${\Gamma^i}_{0i} = H$ and ${\Gamma^0}_{ii} = a^2 H$ (no summation in i) where H stands for the Hubble parameter and $a = a(t)$ is the scale factor (See Appendix H). Using these facts one can find the explicit expressions

$$\tilde{\nabla}_0 \phi_0 \tilde{\nabla}^0 \phi^0 = \partial_0 \phi_0 \partial^0 \phi^0 \tag{3.110}$$

$$\tilde{\nabla}_0 \phi_i \tilde{\nabla}^0 \phi^i = \partial_0 \phi_i \partial^0 \phi^i - H^2 \phi_i \phi^i \tag{3.111}$$

$$\tilde{\nabla}_i \phi_0 \tilde{\nabla}^i \phi^0 = \partial_i \phi_0 \partial^i \phi^0 - 2H \phi_i \partial^i \phi^0 + H^2 \phi_i \phi^i \tag{3.112}$$

$$\tilde{\nabla}_i \phi_j \tilde{\nabla}^i \phi^j = \partial_i \phi_j \partial^i \phi^j + 2H \partial_i \phi^i \phi^0 + H^2 \phi_0 \phi^0. \tag{3.113}$$

Since we require homogeneous fields we neglect spatial derivatives [66] and get

$$\rho = \frac{3}{2}\dot{\phi}_0^2 + \frac{1}{2}\dot{\phi}_i^2 - \frac{1}{2}H^2\phi_0^2 + {W'^0}_0 + V', \tag{3.114}$$

$$\bar{P} = \frac{1}{2}\dot{\phi}_0^2 + \frac{1}{6}\dot{\phi}_i^2 - \frac{1}{6}H^2\phi_0^2 - \frac{1}{3}{W'^i}_i - V', \tag{3.115}$$

where now W' and V' stand for the "potentials" restricted to the homogeneous condition as well. If we define the equation of state

$$\bar{P} = w\rho, \tag{3.116}$$

we see that w is given by

$$w = \frac{\frac{1}{2}\dot{\phi}_0^2 + \frac{1}{6}\dot{\phi}_i^2 - \frac{1}{6}H^2\phi_0^2 - \frac{1}{3}{W'^i}_i - V'}{\frac{3}{2}\dot{\phi}_0^2 + \frac{1}{2}\dot{\phi}_i^2 - \frac{1}{2}H^2\phi_0^2 + {W'^0}_0 + V'}. \tag{3.117}$$

We know that W' and V' are suppressed by powers of the Newton constant G. That is why considerable torsional effects are expected only at energies near the GUT scale. When the kinematical terms are negligible we get

$$w \simeq \frac{-\frac{1}{6}H^2\phi_0^2 - \frac{1}{3}{W'^i}_i - V'}{-\frac{1}{2}H^2\phi_0^2 + {W'^0}_0 + V'}. \tag{3.118}$$

We see that the first term in the last equation comes from the Christoffel symbol of our spacetime metric and has nothing to do with torsion. So even if the kinematical terms are negligible, we get the de Sitter limit $w \simeq -1$ only when $V' \gg \{W', -H^2\phi_0^2\}$ and the universe expands quasi-exponentially. Let us remember that for accelerated expansion of the universe all we need is $w < -\frac{1}{3}$ and with $W' \gg \{V', -H^2\phi_0^2\}$ we get the limit $w = -\frac{1}{3}$. Finally when $W', V' \ll 1$, we get $w = \frac{1}{3}$ and not $w = 1$ as in the usual scalar inflaton. Let us recall that there is no freedom in choosing W' and V' but they are determined by the dynamics of the theory. Even if we ignore them from the beginning, a "scalar" that couples to the Riemannian connection along with a suitable potential would deviate from the standard behavior of the scalar inflaton.

More generally, inflation will occur when $\epsilon \equiv \frac{3}{2}\left(\frac{P}{\rho} + 1\right) = \frac{3}{2}(w+1) < 1$. Apart of this, it is customary (but not imperative) that "friction" terms $\sim \dot{\phi}$ dominate over "acceleration" ones $\sim \ddot{\phi}$ in the equation of motion (See Appendix H). However, this equation is highly nontrivial due to torsional effects. To overcome this analysis we must first find the equation of motion for our ϕ field in the context of the new "effective" Lagrangian. This will be addressed elsewhere.

3.10.4. 3D gravity with torsion

Let us consider Lorentzian 3D gravity with a local Lorentz frame metric of the form $\eta_{ij} = (+,-,-)$. The normalization of the totally antisymmetric tensor is such that $\epsilon^{012} = 1$. Since in 3D an antisymmetric tensor is dual to a vector, we make the following definitions: $\omega^{ij} = -\epsilon^{ij}{}_k \omega^k$, $R^{ij} = -\epsilon^{ij}{}_k R^k$. Then, Cartan's structure equations become

$$T^i = de^i + \epsilon^i{}_{jk}\omega^j \wedge e^k, \tag{3.119}$$

$$R^i = d\omega^i + \frac{1}{2}\epsilon^i{}_{jk}\omega^j \wedge \omega^k. \tag{3.120}$$

As before, we can split the spin connection in such a way that $\omega^i = \tilde{\omega}^i + C^i$, where $\tilde{\omega}^i$ satisfies the homogeneous first structure equation and C^i is the contorsion one-form such that $T^i = \epsilon^i{}_{mn} C^m \wedge e^n$. Finally it is easy to show that

$$2R_i = 2\tilde{R}_i + 2\tilde{\nabla} C_i + \epsilon_{imn} C^m \wedge C^n, \tag{3.121}$$

where $\tilde{R}_i$ is the Riemannian curvature. We will consider a natural generalization of General Relativity with a cosmological constant, the so-called Mielke-Baekler model [67][68], namely,

$$S_G[e,\omega] = \int 2ae^i \wedge R_i - \frac{\Lambda}{3}\epsilon_{ijk} e^i \wedge e^j \wedge e^k + \alpha_3 L_{\mathrm{CS}}(\omega) + \alpha_4 e^i \wedge T_i, \tag{3.122}$$

where $a = \frac{1}{16\pi G}$ and $L_{\mathrm{CS}}(\omega) = \omega^i \wedge d\omega_i + \frac{1}{3}\epsilon_{ijk}\omega^i \wedge \omega^j \wedge \omega^k$ is the Chern-Simons Lagrangian for the Lorentz connection. The complete action will be $S_T[e,\omega,\Psi] = S_G[e,\omega] + S_M[e,\omega,\Psi]$ where S_M stands for the action of arbitrary matter fields $\Psi(x)$. We will consider the particular case

$$S_M[e,\omega,\Psi] \equiv S[e,\omega,\phi,2,1] = -\frac{\lambda}{2}\int \star F_a \wedge F^a, \tag{3.123}$$

where as before $F_a = \nabla\phi_a$ but now $\phi^a = \phi^a{}_\mu dx^\mu$ is a Lorentz-valued one-form. The equations of motion are

$$2aR_i + 2\alpha_4 T_i - \Lambda\epsilon_{ijk} e^j \wedge e^k = \Theta_i, \tag{3.124}$$

$$2\alpha_3 R_i + 2aT_i + \alpha_4\epsilon_{ijk} e^j \wedge e^k = \Sigma_i, \tag{3.125}$$

$$\nabla(\star F_a) = \nabla(\star\nabla\phi_a) = 0, \tag{3.126}$$

where $\Theta_i = -\frac{\delta L_M}{\delta e^i}$, $\Sigma_i = -\frac{\delta L_M}{\delta \omega^i}$ are the current 2-forms due to the presence of the matter field ϕ_a. Following [69] when $\Delta \equiv \alpha_3\alpha_4 - a^2 \neq 0$ the first two equations can be rewritten as

$$2T_i - p\epsilon_{ijk} e^j \wedge e^k = u\Theta_i - v\Sigma_i, \tag{3.127}$$

$$2R_i - q\epsilon_{ijk} e^j \wedge e^k = -v\Theta_i + w\Sigma_i, \tag{3.128}$$

where $p \equiv \frac{\alpha_3\Lambda+\alpha_4 a}{\Delta}$, $q \equiv -\frac{(\alpha_4)^2+a\Lambda}{\Delta}$, $u \equiv \frac{\alpha_3}{\Delta}$, $v \equiv \frac{a}{\Delta}$, $w \equiv \frac{\alpha_4}{\Delta}$. Remembering that the energy-momentum tensor is $\mathcal{T}^i_{\ k} \equiv \star(e^i \wedge \Theta_k)$ we can express the energy-momentum 2-form as

$$\Theta_i = \frac{1}{2}\mathcal{T}^k_{\ i}\,\epsilon_{kmn} e^m \wedge e^n = \epsilon_{imn} t^m \wedge e^n, \tag{3.129}$$

$$t^m = -\left(\mathcal{T}^m_{\ k} - \frac{1}{2}\delta^m_{\ k}\mathcal{T}\right) e^k, \tag{3.130}$$

where $\mathcal{T} = \mathcal{T}^k_{\ k}$.
Equivalently since $\mathcal{S}^k_{\ i} = \star(e^k \wedge \Sigma_i)$, we can write

$$\Sigma_i = \frac{1}{2}\mathcal{S}^k_{\ i}\epsilon_{kmn} e^m \wedge e^n = \epsilon_{imn} s^m \wedge e^n, \tag{3.131}$$

$$s^m = -\left(\mathcal{S}^m_{\ k} - \frac{1}{2}\delta^m_{\ k}\mathcal{S}\right) e^k, \tag{3.132}$$

where $\mathcal{S} = \mathcal{S}^k_{\ k}$. Using these results in the equation of motion for T_i we find that

$$C^j = \frac{1}{2}(pe^j + ut^j - vs^j). \tag{3.133}$$

Using this fact in the second equation of motion we get that

$$2R_i = q\epsilon_{ijk} e^j \wedge e^k - v\epsilon_{ijk} t^j \wedge e^k + w\epsilon_{ijk} s^j \wedge e^k. \tag{3.134}$$

Recalling the splitting between Riemannian and torsional contributions (3.121) we get that

$$\begin{aligned} 2R_i =& 2\tilde{R}_i + u\tilde{\nabla} t_i - v\tilde{\nabla} s_i \\ &+ \epsilon_{ijk}\left(\frac{p^2}{4} e^j \wedge e^k + \frac{u^2}{4} t^j \wedge t^m + \frac{v^2}{4} s^j \wedge s^m + \frac{up}{2} t^j \wedge e^k - \frac{vp}{2} s^j \wedge e^k - \frac{uv}{2} t^j \wedge s^k\right). \end{aligned} \tag{3.135}$$

In this form of the gravitational field equations, the role of ϕ_i as a source of gravity is clearly described by the one-forms t_i and s_i. Together with the equations of motion for the matter fields (3.126) and a suitable set of boundary conditions define the complete dynamics of the gravitational and matter fields.

3.10.5. Anti-restoration symmetry breaking

Let us consider the following action principle

$$S = -\frac{1}{16\pi G}\int d^4x\sqrt{-g}\{R(\omega) - 2\Lambda\} - \frac{1}{2}\int d^4x\sqrt{-g}\nabla_\mu\phi_a\nabla^\mu\phi^a \tag{3.136}$$

where $\nabla_\mu\phi^a = \partial_\mu\phi^a + \omega^a_{\ b\mu}\phi^b$.
We know that this can also be written as

$$S = -\frac{1}{16\pi G}\int d^4x\sqrt{-g}\{\tilde{R}(\tilde{\omega}) - 2\Lambda\} - \frac{1}{2}\int d^4x\sqrt{-g}\tilde{\nabla}_\mu\phi_a\tilde{\nabla}^\mu\phi^a - \int d^4x\sqrt{-g}V(\phi), \tag{3.137}$$

where $V(\phi)$ stands for a potential that we would like to be bounded from below. If V has a minimum at $\phi_a = \upsilon_a$, the expansion of V around the minimum yields the "mass matrix"

$$(\mu^2)_{ab} = \frac{1}{2}\left(\frac{\partial^2 V}{\partial\phi_a\partial\phi_b}\right)_{\phi_a=\upsilon_a}. \tag{3.138}$$

We can choose ϕ_a to be of the form

$$\phi_a = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \upsilon \end{pmatrix}. \tag{3.139}$$

All other solutions of ϕ_a are related to this one by a Lorentz transformation [70]. Then, the homogeneous Lorentz group $SO(3,1)$ is broken down to the spatial rotation group $O(3)$. The three rotation generators J_i $(i = 1, 2, 3)$ leave the vacuum invariant

$$J_i\upsilon_i = 0, \tag{3.140}$$

while the three Lorentz-boost generators K_i break the vacuum symmetry

$$K_i\upsilon_i \neq 0. \tag{3.141}$$

The J_i and K_i satisfy the commutation relations

$$[J_i, J_j] = i\epsilon_{ijk}J_k, \qquad [J_i, K_j] = i\epsilon_{ijk}K_k, \qquad [K_i, K_j] = -i\epsilon_{ijk}K_k. \tag{3.142}$$

There are three zero-mass Nambu-Goldstone bosons, the same as the number of massive bosons, and there are three massless degrees of freedom corresponding to the unbroken $O(3)$ symmetry. After the spontaneous breaking of the vacuum, one massive physical particle Φ remains [72]. No ghost particles will occur in the unitary gauge. The mass term in the Lagrangian density is given in the unitary gauge by

$$\mathcal{L}_M = \frac{1}{2}\sqrt{-g}\upsilon_b\upsilon_c(\omega_\mu)^{ab}(\omega^\mu)_a{}^c. \tag{3.143}$$

When Lorentz symmetry is restored for $E < E_c$, $\upsilon = 0$ and $\mathcal{L}_M = 0$ and we obtain the standard GR Lagrangian density with a massless spin-2 graviton, coupled minimally to a spin-0 Lorentz-valued particle.

A phase transition is assumed to occur at the critical temperature T_c, when $\upsilon_a \neq 0$ and the Lorentz symmetry is broken so the three gauge fields $(\omega_\mu)^{0i}$ become massive degrees of freedom (We know that the true degrees of freedom of gravity are the two states of polarization of the graviton. However there are alternative theories for the dynamics of spacetime that do consider the spin connection as a true physical field which is clearly not the case for the Riemannian Hilbert-Einstein action so, having said this, our arguments remain valid). Below T_c the Lorentz symmetry is restored, and we regain the usual classical gravitational field with massless gauge fields ω_μ. The symmetry breaking will extend to the singularity or the possible singularity-free initial state at $t = 0$, and since quantum effects associated with gravity do not become important before E_P, we expect that $E_c \sim 10^{19}$Gev.

After the symmetry is restored for $E < E_P$, the entropy will rapidly increase provided that no further phase transition occurs which breaks the Lorentz symmetry of the vacuum. Thus, the symmetry breaking mechanism could "explain" in a natural way the low entropy at the initial state at $t \sim 0$ and the large entropy in the present universe.

Since the ordered phase is at a much lower entropy than the disordered phase and due to the existence of a domain determined by the direction of the vev of the ϕ field, a "natural explanation" could be given for the cosmological arrow of time and the origin of the second law of thermodynamics. Thus, the spontaneous symmetry breaking of the gravitational vacuum corresponding to the breaking patters, $SO(3,1) \to O(3)$, leads to a manifold with the structure $R \times O(3)$, in which time appears as an absolute external parameter. The VEV, $\langle\phi\rangle_0$, points in a chosen direction of time to break the symmetry creating an arrow of time.

Capítulo 4

Conclusions

We calculated the chiral anomaly in the context of the Holst action plus a non-minimally coupled Dirac fermion to curved spacetime. Our aim was to relate the Immirzi parameter γ of the parity violating sector of the Holst action with the non-minimal coupling parameter α in the newly defined Dirac operator. The hope was to find that, upon the calculation of the anomaly, $\alpha = \gamma$ in a "natural" way, so that the Nieh-Yan topological invariant would arise giving no classical effects of the Immirzi parameter. The result was another, but it depends on some arbitrary identifications we must do.

During the final stage of this thesis, some authors have done a rather similar calculation although their starting point was already the action containing the Nieh-Yan term. However, adapting our calculation to this result implies that: I) There is another compelling reason for believing that the Immirzi parameter is "non-classical" and II) In order to fix γ so the calculation of black hole entropy in LQG coincides with Hawking's formula, we could in principle accept that chiral fermions (as neutrinos) do couple non-minimally.
Further discussion and possible phenomenology from this assertions must be extracted.

In the path to understanding why gauge bosons do not couple to spacetime torsion, we made a canonical analysis of the Maxwell action without neglecting the antisymmetric part of the Christoffel symbols right from the start. A well defined Hamiltonian $3+1$ decomposition lead us to redefine time derivatives and the Gauss law. Demanding invariance of the field strength we get a torsion tensor already found some 30 years ago by Hojman et al. This torsion is dynamically generated by a scalar field named "tlaplon". So we see that the tlaplon arises in a natural way from the Hamiltonian point of view. The same approach applied to Yang-Mills theory, however, fails. In the literature this problem has been somehow "solved" introducing a non-canonical field strength for the Yang-Mills field. We have just quoted what would be the implications of such a change. Finally we have noticed how the introduction of the tlaplon field in the LQG scenario would account for new **exclusive** interaction terms between the tlaplon and the fermions and thus, a new way to "measure" γ.

Finally we have systematically introduced a new kind of bosonic fields that serve as source of spacetime torsion. These bosonic field however would not correspond to gauge boson fields because this procedure of coupling them to the inner Lorentz space would break gauge invariance. Along with this we have proved that this kind of torsion tensor is the Noether charge associated with the invariance of the action under rotations in the inner space. Finally we have tried to get this mathematical framework near the physical phenomenology of LQG and Inflation Theory. Further issues will be addressed in the future.

The conclusion that we could extract from all this is that Einstein-Cartan theory should be revisited since it could be a more complete “limit” of an underlying quantum theory of gravity like LQG or even the String Theories. Even if spacetime torsion is unobservable in the present universe because it is suppressed by Newton’s constants, it may have played a major role in the distant past, near GUT scale and even near Planck scale.

Apéndice A

Hermiticity of $\nabla\!\!\!/$ and $\nabla\!\!\!/'$

After the Wick rotation we have that $\gamma^{\mu\dagger} = -\gamma^\mu$. Then, we use the notion of inner product in the Hilbert space spanned by the eigenfunctions of $\nabla\!\!\!/ = \gamma^\mu \nabla_\mu$. By definition,

$$\begin{aligned}
(\varphi_n, \nabla\!\!\!/ \varphi_m) &= \int d^4x |e| \varphi_n^\dagger \nabla\!\!\!/ \varphi_m \\
&= \int d^4x |e| \varphi_n^\dagger \gamma^\mu \nabla_\mu \varphi_m \\
&= -\int d^4x |e| (\nabla_\mu \varphi_n)^\dagger \gamma^\mu \varphi_m \\
&= -\int d^4x |e| \varphi_n^\dagger \nabla_\mu^\dagger \gamma^\mu \varphi_m \\
&= \int d^4x |e| \varphi_n^\dagger \nabla_\mu^\dagger \gamma^{\mu\dagger} \varphi_m \\
&= \int d^4x |e| \varphi_n^\dagger (\nabla\!\!\!/)^\dagger \varphi_m,
\end{aligned} \tag{A.1}$$

where an integration by parts has been done neglecting a total derivative. So we get that under these circumstances,

$$\nabla\!\!\!/^\dagger = \nabla\!\!\!/. \tag{A.2}$$

For the case of the operator $\nabla\!\!\!/' = \gamma^\mu \left(1 - \frac{i}{\alpha}\gamma_5\right) \nabla_\mu$ all we have to do is to prove that $\gamma^\mu \left(1 - \frac{i}{\alpha}\gamma_5\right)$ is anti-Hermitian. In effect,

$$\begin{aligned}
\left(\gamma^\mu (1 - \tfrac{i}{\alpha}\gamma_5)\right)^\dagger &= \left(1 - \tfrac{i}{\alpha}\gamma_5\right)^\dagger \gamma^{\mu\dagger} \\
&= -(1 + \tfrac{i}{\alpha}\gamma_5)\gamma^\mu \\
&= -\gamma^\mu (1 - \tfrac{i}{\alpha}\gamma_5),
\end{aligned} \tag{A.3}$$

where we have used the facts that $\{\gamma^5, \gamma^\mu\} = 0$ and that $\alpha \in \Re$. This ensures that $\nabla\!\!\!/'^\dagger = \nabla\!\!\!/'$.

Apéndice B

Nieh-Yan topological invariant

The Nieh-Yan term is the only Lorentz invariant exact 4-form including torsion. It is given by

$$\mathcal{N} = d(e_a \wedge T^a) = T^a \wedge T_a - e_a \wedge e_b \wedge R^{ab}, \tag{B.1}$$

identity that can be proven using the first Cartan structure equation along with its consistency condition, the first Bianchi identity.

One can show that in the presence of torsion the relevant tangent group of rotations is $SO(5)$ instead of $SO(4)$ so we have to consider the Pontryagin density associated with the curvature 2-form

$$\mathcal{R}^{AB} = d\mathcal{W}^{AB} + \mathcal{W}^A{}_C \wedge \mathcal{W}^{CB} \tag{B.2}$$

of the $SO(5)$ connection

$$\mathcal{W}^{AB} = \begin{pmatrix} \omega^{ab} & \frac{1}{\ell} e^a \\ -\frac{1}{\ell} e^b & 0 \end{pmatrix} \qquad a, b = 1, 2, 3, 4 \qquad A, B = 1, 2, 3, 4, 5, \tag{B.3}$$

where ℓ is a lenght scale known as "the radius of the Universe", which is neccesary in order to consider the vielbein as a part of a connection. In this way,

$$\mathcal{R}_{AB} \wedge \mathcal{R}^{AB} = R_{ab} \wedge R^{ab} + \frac{2}{\ell^2}(T_a \wedge T^a - e_a \wedge e_b \wedge R^{ab}). \tag{B.4}$$

Taking the integral of the last expression,

$$\frac{2}{\ell^2} \int_{M_4} \mathcal{N} = P_4[SO(5)] - P_4[SO(4)], \tag{B.5}$$

proves that the Nieh-Yan four-form $\mathcal{N} = d(e_a \wedge T^a)$ is indeed a topological invariant since it is the difference of two Pontryagin classes.

Apéndice C

Large gauge transformations

C.1. Yang-Mills gauge theories

Let us review the case of large gauge transformations in Yang-Mills gauge theories. Let the $SU(N)$ valued connection $A_\mu = A^I_\mu \lambda^I$ and its associated electric field $E^\gamma = E^\gamma_K \lambda^K$ (where $I, J, K, \cdots = 1, 2, 3, \ldots, N^2 - 1$ are internal indices) be a couple of conjugate variables in the framework of a canonical formulation of Yang-Mills gauge theories. The evolution of the system is limited to a restricted region of the phase space by the first class Gauss constraint, expressed by the following weak equation

$$G_I := D_\alpha E^\alpha_I = \partial_\alpha E^\alpha_I + f_{IJ}{}^K A^J_\alpha E^\alpha_K \approx 0. \tag{C.1}$$

According to the Dirac quantization procedure, the state functional describing the quantum physical system must satisfy the Gauss constraint, namely we have to require that

$$\widehat{G}_I \Phi(A) = -iD_\alpha \frac{\delta}{\delta A^I_\alpha} \Phi(A) = 0, \tag{C.2}$$

where the usual quantum representation of the operators has been assumed.

The Gauss constraint formalizes the request of gauge invariance of the quantum state describing the physical system, namely it is equivalent to requiring that the state functional be invariant under the small component of the gauge group $G = SU(N)$, as can be easily realized. Since the global structure of the gauge group in non-trivial, in view of quantization, it is particularly interesting to study the behavior of the state functional under the large gauge transformations. A non-trivial global structure of the gauge group, in fact, can produce striking effects in the non-perturbative theory, as, e.g, P and CP violations, physically motivating this extension of the theory.

In this respect, let $\widehat{\mathscr{G}}$ be the generator of the large gauge transformations, acting on the state functional $\Phi(A)$. Considering that the Hamiltonian operator, $\widehat{\mathscr{H}}$, is invariant under the full gauge group (or, more formally, it commutes with the operator $\widehat{\mathscr{G}}$), we can construct a set of eigenstates for the quantum theory by diagonalizing simultaneously $\widehat{\mathscr{H}}$ and $\widehat{\mathscr{G}}$. In other words, the following equation

$$\widehat{\mathscr{G}}\Phi_W(A) = \Phi_W(A^g) = e^{i\theta W}\Phi_W(A), \quad \text{where} \quad A^g = gAg^{-1} + gdg^{-1}, \tag{C.3}$$

is a super-selection rule for the states of the theory, which are now labeled by the *winding number* $W = W(g)$, according to their behavior under the action of the large gauge transformation operator. The constant θ is an angular parameter, which indicates how much the state functional "rotates" under the action of the large gauge transformations operator. Specifically, it represents a quantization ambiguity connected with the non-trivial global structure of the gauge group.

Equation (C.3) implies that the wave functionals either have to satisfy suitable θ-dependent boundary conditions passing from one "slab" to the next in the configuration space; or, a fully gauge invariant state functional can be constructed, trasnferring the θ-dependence in the momentum operator. In this respect, we recall that the so-called *Chern-Simons functional*,

$$\mathscr{Y}(A) = \frac{1}{8\pi^2}\int \operatorname{tr}\left(F\wedge A - \frac{1}{3}A\wedge A\wedge A\right), \tag{C.4}$$

is characterized by the following remarkable property:

$$\mathscr{Y}(A^g) = \mathscr{Y}(A) + W(g). \tag{C.5}$$

In other words, the Chern-Simons functional under a large gauge transformation turns out to be modified by a quantity exactly corresponding to the winding number, expressed by the Maurer-Cartan integral

$$W(g) = \frac{1}{24\pi^2}\int \operatorname{tr}(g^{-1}dg)\wedge(g^{-1}dg)\wedge(g^{-1}dg). \tag{C.6}$$

This directly implies that the new state functional,

$$\Phi'(A) = e^{-i\theta\mathscr{Y}(A)}\Phi_W(A), \tag{C.7}$$

will be invariant under the full gauge group, as can be easily demonstrated. In other words we have

$$\widehat{\mathscr{G}}\Phi'(A) = \Phi'(A). \tag{C.8}$$

So, by using the rescaling (C.7), we have obtained a new fully gauge invariant quantum state functional, at the price of modifying the momentum operator, namely, the θ-dependence has been transferred from the boundary conditions to the momentum operator, which becomes:

$$E'^{\alpha}\Phi'(A) = e^{-i\theta\mathscr{Y}(A)}E^{\alpha}e^{i\theta\mathscr{Y}(A)}\Phi'(A) = -i\left[\frac{\delta}{\delta A_\alpha} - \frac{i\theta}{8\pi^2}\epsilon^{\alpha\beta\gamma}F_{\beta\gamma}\right]\Phi'(A). \tag{C.9}$$

The above modification in the conjugate momentum reflects on the Hamiltonian operator, i.e

$$H' = \int d^3x \operatorname{tr}\left[\frac{1}{2}\left(E^{\alpha} - \frac{\theta}{8\pi^2}\epsilon^{\alpha\beta\gamma}F_{\beta\gamma}\right)\left(E_{\alpha} - \frac{\theta}{8\pi^2}\epsilon_{\alpha}{}^{\rho\sigma}F_{\rho\sigma}\right) + \frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}\right], \tag{C.10}$$

generating a pseudo-vectorial term which prevents the new Hamiltonian H' from being invariant under the CP discrete symmetry.

The new Hamiltonian corresponds to a topological modification of the classical action, consisting in the presence of an additional term belonging to the Pontryagin class, i.e.

$$S_{\text{new}}(A) = -\frac{1}{4}\int \operatorname{tr} \star F \wedge F + \frac{\theta}{8\pi^2}\int \operatorname{tr} F \wedge F. \tag{C.11}$$

The θ parameter appears as a multiplicative constant in front of the modification. It is worth mentioning that the new term does not affect the classical equations of motion but modifies the vacuum to vacuum amplitude in the path-integral formulation of the quantum theory. In other words, it allows to take into account possible tunneling phenomena between distinct vacua characterized by different winding numbers, violating the CP discrete symmetry.

C.2. Partially gauge fixed gravity

It has been argued that the Barbero-Immirzi parameter can have a topological origin analogous to that of the θ-angle of Yang-Mills theories and the Nieh-Yan functional,

$$\mathscr{Y}[e,\omega] = \int e_i \wedge T^i, \tag{C.12}$$

plays the role of the Chern-Simons functional $\mathscr{Y}(A)$. The situation though, is less clear here. The missing point in this construction is the relation existing between the Nieh-Yan and the large gauge sector of the theory, in analogy with the requirement of invariance under the large sector of the $SU(N)$ gauge group pertaining to the case of Yang-Mills gauge theories.

The Ashtekar-Barbero first class constraints are extracted from the fully covariant theory after having fixed the temporal gauge. This fixes the zeroth component of the local basis, e^0, in such a way that it remains parallel to the normal vector, n, along the evolution and, simultaneously, reduces the gauge group from $SO(3,1)$ to $SO(3)$. Therefore, once the gauge has been partially fixed, the local symmetry group reduces to the group of spatial rotations, $SO(3)$, so that one is immediately induced to think that the large gauge sector is merely related to the non-trivial global structure of $SO(3)$. But, physically, also the action of the T discrete operator, which acts on the zeroth component of the local basis by flipping its orientation with respect to the normal vector, represents a large gauge transformation. As a consequence the full gauge group is $\mathcal{G} = SO(3) \times \mathbb{Z}_2 \simeq S^3$. Namely, it consists of two copies of $SO(3)$, correlated with the two orientations of the zeroth component of the local basis. In particular, recalling that $\Pi_3(S^3) = \mathbb{Z}$, the disconnected components of the large gauge group are labeled by an integer, which is the winding number of the $SU(2) \simeq S^3$ group. Noting that $\mathcal{G} = SO(3) \times \mathbb{Z}_2 = SO(4)/SO(3)$, the connection for $SO(4)$, Ω^{AB}, can be written in the MacDowell-Mansouri form

$$\Omega^{AB} = \begin{pmatrix} \omega^{ij} & \frac{1}{\ell}e^i \\ -\frac{1}{\ell}e^j & 0 \end{pmatrix} \tag{C.13}$$

where $A, B, C, \dots$ are indexes valued on $SO(4)$, while $i, j, k, \dots$ are valued on $SO(3)$. The constant ℓ has the dimension of a length and can be associated with the radius of the spheres obtained compactifying the tangent planes. It is easy to show that

$$\begin{aligned} \mathscr{Y}[\Omega] &= F^{AB} \wedge \Omega_{AB} + \frac{1}{3}\Omega^{A}{}_{B} \wedge \Omega^{B}{}_{C} \wedge \Omega^{C}{}_{A} \\ &= R^{ij} \wedge \omega_{ij} + \frac{1}{3}\omega^{i}{}_{j} \wedge \omega^{j}{}_{k} \wedge \omega^{k}{}_{i} - \frac{2}{\ell^2} T^i \wedge e_i = \mathscr{Y}[\omega] - \frac{2}{\ell^2}\mathscr{Y}[e,\omega], \end{aligned} \tag{C.14}$$

where F^{AB} is the curvature 2-form associated with the connection Ω^{AB}, while R^{ij} is associated with the 3-dimensional connection ω^{ij}. Due to the fact that now $\mathcal{G} = SO(4)/SO(3)$, we can construct a Chern-Simons functional for the large gauge group of gauge fixed gravity as the difference between $\mathscr{Y}[\Omega]$ and $\mathscr{Y}[\omega]$, but this is exactly the Nieh-Yan functional since

$$\mathscr{Y}[e,\omega] = \frac{\ell^2}{2}(\mathscr{Y}[\omega] - \mathscr{Y}[\Omega]). \tag{C.15}$$

A new state functional, fully invariant under the large gauge group, can be obtained by rescaling the original state functional of the Einstein-Cartan theory by the Nieh-Yan functional. The new state functional satisfies the Ashtekar-Barbero constraints for General Relativity, revealing the topological origin of the Barbero-Immirzi parameter.

Apéndice D

Dirac Genus

Alvarez-Gaumé and Ginsparg [42] give us the following expression for the Dirac Genus up to fourth order in the Pontryagin classes for the curvature 2-form

$$\begin{aligned}
&\hat{A}(M) = 1 + \frac{1}{(4\pi)^2}\frac{1}{12}\operatorname{tr} R^2 + \frac{1}{(4\pi)^4}\left[\frac{1}{288}(\operatorname{tr} R^2)^2 + \frac{1}{360}\operatorname{tr} R^4\right] \\
&+ \frac{1}{(4\pi)^6}\left[\frac{1}{10368}(\operatorname{tr} R^2)^3 + \frac{1}{4320}\operatorname{tr} R^2 \operatorname{tr} R^4 + \frac{1}{5670}\operatorname{tr} R^6\right] \\
&+ \frac{1}{(4\pi)^8}\left[\frac{1}{497664}(\operatorname{tr} R^2)^4 + \frac{1}{103680}(\operatorname{tr} R^2)^2 \operatorname{tr} R^4 + \frac{1}{68040}\operatorname{tr} R^2 \operatorname{tr} R^6 + \frac{1}{259200}(\operatorname{tr} R^4)^2 + \frac{1}{75600}\operatorname{tr} R^8\right] \\
&+ \dots
\end{aligned} \tag{D.1}$$

Apéndice E

Canonical Formalism

In a canonical formulation, the Hamiltonian H rather than the action is used to determine equations of motion of any function f on the phase space by means of Poisson brackets, $\dot{f} = \{f, H\}$. The Poisson structure defines the kinematical arena which follows from the field variables and momenta.

A canonical formalism (Hamiltonian framework) is achieved by performing a Legendre transform of the action S, replacing time derivatives of configuration variables by momenta. This, as always, requires one to treat space and time differently and is the reason why the canonical formulation is no manifestly covariant. We introduce a foliation of the spacetime $(M, g_{\mu\nu})$ by a family of spacelike hypersurfaces $\Sigma_t : t = \text{const}$ in terms of a time function t on M. Canonical variables will depend on which time function one chooses, but the resulting dynamics of observable quantities will remain covariant. Furthermore, let t^μ be a timelike vector field whose integral curves intersect each leaf Σ_t of the foliation precisely once and which is normalized such that $t^\mu \nabla_\mu t = 1$. This t^μ is the "evolution vector field" along whose orbits different points on all $\Sigma_t \equiv \Sigma$ can be identified. This allows us to write all spacetime fields in terms of t-dependent components defined on a spatial manifold Σ. Lie derivatives of spacetime fields along t^μ are identified with "time derivatives" of the spatial fields (for instance, if $P = P_\mu dx^\mu$ is a one-form its time derivative reads $\dot{P}_\mu \equiv \pounds_t P_\mu = t^\nu \partial_\nu P_\mu + P_\nu \partial_\mu t^\nu$).

Let us decompose t^μ into normal and tangential parts with respect to Σ_t by defining the lapse function N and the shift vector N^μ as $t^\mu = N n^\mu + N^\mu$ with $N^\mu n_\mu = 0$, where n^μ is the unit normal vector field to the hypersurfaces Σ_t. The spacetime metric $g_{\mu\nu}$ induces a spatial metric $q_{\mu\nu}$ by the formula $g_{\mu\nu} = q_{\mu\nu} - n_\mu n_\nu$. Then one uses $n^\mu = N^{-1}(t^\mu - N^\mu)$ and $q^{\mu\nu} = g^{\mu\nu} + n^\mu n^\nu$ to project fields normal and tangential to Σ_t.

Having said this we now make a comment about a fact that is usually not known. The Lie derivative depends only on the structure of the manifold and not on the connection. In abstract matrix notation it acts as an operator

$$\pounds_\xi = \xi \cdot \partial - [\partial\xi,\ ]. \tag{E.1}$$

For a Riemannian connection, which is torsion-free, we can replace the ordinary by the covariant derivative

$$\pounds_\xi = \xi \cdot \nabla - [\nabla\xi,\ ], \tag{E.2}$$

since

$$\begin{aligned} \xi \cdot \nabla - [\nabla\xi,\ ] &= \xi^\nu \partial_\nu + \xi^\lambda [\Gamma^\alpha{}_{\lambda\beta},\ ] - [\partial_\beta \xi^\alpha + \Gamma^\alpha{}_{\beta\lambda}\xi^\lambda,\ ] \\ &= \xi \cdot \partial - [\partial_\beta \xi^\alpha,\ ] = \pounds_\xi, \end{aligned}$$

recalling that torsion-freedom implies symmetric Christoffel symbols. However, in the case of a nonvanishing torsion tensor

$$T^\alpha{}_{\lambda\beta} = \Gamma^\alpha{}_{\lambda\beta} - \Gamma^\alpha{}_{\beta\lambda} \neq 0 \tag{E.3}$$

we find instead

$$\pounds_\xi = \xi \cdot \nabla - [\nabla_\beta \xi^a + T^\alpha{}_{\lambda\beta}\,\xi^\lambda,\ ], \tag{E.4}$$

which can be quickly verified

$$\begin{aligned} \xi \cdot \nabla - [\nabla_\beta \xi^a + T^\alpha{}_{\lambda\beta}\,\xi^\lambda,\ ] &= \xi \cdot \partial + \xi^\lambda [\Gamma^\alpha{}_{\lambda\beta},\ ] - [\partial_\beta \xi^\alpha,\ ] - [\Gamma^\alpha{}_{\beta\lambda}\xi^\lambda,\ ] - [(\Gamma^\alpha{}_{\lambda\beta} - \Gamma^\alpha{}_{\beta\lambda})\xi^\lambda,\ ] \\ &= \xi \cdot \partial - [\partial_\beta \xi^\alpha,\ ] = \pounds_\xi. \end{aligned}$$

This justifies our rather strange definition of "time derivative" in the Hamiltonian decomposition of gauge theories in the presence of spacetime torsion.

Apéndice F

Energy density and pressure

The matter Hamiltonian is directly related to energy density by

$$\rho = \frac{1}{\sqrt{q}} \frac{\delta H_M}{\delta N}. \tag{F.1}$$

This is the usual term for energy per volume, and does not mean that ρ is a geometrical density.

The general, thermodynamical definition of pressure is the negative change of energy by volume, which we can write as

$$P = -\frac{1}{N} \frac{\delta H}{\delta \sqrt{q}} \tag{F.2}$$

whenever the Hamiltonian H depends isotropically on the metric. Otherwise, one has to use all components of the stress tensor $\frac{\delta H}{\delta q^{ab}}$ which is not proportional to the identity. The derivative by the determinant of the metric can be expressed in terms of metric components by using a suitable change of variables which includes q as an independent one. We thus introduce $q_{ab} \equiv q^{1/3}\bar{q}_{ab}$ with $\det \bar{q}_{ab} = 1$ such that $\frac{\partial q_{ab}}{\partial q} = \frac{1}{3} q^{-1} q_{ab}$ where all components of $\bar{q}_{ab}$ are kept fixed in the partial derivative. This is exactly what we need to compute pressure since only the volume but not the shape of the fluid is varied. This change of variables implies

$$\frac{\delta}{\delta \sqrt{q}} = 2\sqrt{q} \frac{\delta}{\delta q} = 2\sqrt{q} \sum_{a,b} \frac{\partial q_{ab}}{\partial q} \frac{\delta}{\delta q_{ab}} = \frac{2}{3\sqrt{q}} \sum_{a,b} q_{ab} \frac{\delta}{\delta q_{ab}},$$

and thus

$$P = -\frac{2}{3N\sqrt{q}} q_{ab} \frac{\delta H}{\delta q_{ab}}. \tag{F.3}$$

Apéndice G

Noether's theorem

Consider a d-form Lagrangian $L(\varphi, d\varphi)$, where φ denotes collectively a set of p-form fields. An arbitrary variation of the action under a local change $\delta\varphi$ is given by the integral of

$$\delta L = (\mathscr{E} - \mathscr{L})\delta\varphi + d\Theta(\varphi, \delta\varphi), \tag{G.1}$$

where $\mathscr{E} - \mathscr{L}$ stands for equations of motion and Θ is a corresponding boundary term. The total change in φ ($\bar{\delta}\varphi = \varphi'(x') - \varphi(x)$) can be decomposed as a sum of a local variation and the change induced by a diffeomorphism, that is, $\bar{\delta}\varphi = \delta\varphi + \pounds_\xi\varphi$, where $\pounds_\xi$ is the Lie derivative operator. In particular, a symmetry transformation acts on the coordinates of the manifolds as $\delta x^\mu = \xi^\mu(x)$, and on the field as $\delta\varphi$, leading a change in the Lagrangian given by $\delta L = d\Omega$.
Noether's theorem states that there exists a conserved current given by

$$\star J = \Omega - \Theta(\varphi, \delta\varphi) - I_\xi L, \tag{G.2}$$

which satisfies $d \star J = 0$. This, in turn, implies the existence of the conserved charge

$$Q = \int_\Sigma \star J, \tag{G.3}$$

where Σ is the spatial section of the manifold, when a manifold is assumed to be of topology $R \times \Sigma$.
The proof goes as follows:
Under the variation $\varphi^A \to \varphi^A + \delta\varphi^A$, the Lagrangian will vary in the form

$$\delta L^{(d)} = E_A\delta\varphi^A + d(B_A\delta\varphi^A), \tag{G.4}$$

where now we make explicit the fact that the Lagrangian is a d-form and A is the collective index that labels the set of fields involved. Here,

$$E_A(\varphi) = 0 \tag{G.5}$$

correspond to the Euler-Lagrange equations of motion and

$$B_A(\varphi)\delta\varphi^A|_{\partial M} = 0 \tag{G.6}$$

are the boundary conditions. In this case we must stress that δ implies a functional variation of the form

$$\delta\varphi^A = \varphi^{A\prime}(x) - \varphi^A(x). \tag{G.7}$$

Let us consider that the Lagrangian $L^{(d)}$ possesses two symmetries: one is the symmetry under diffeomorphisms, the other one will be a gauge symmetry.

G.1. On-shell and off-shell diffeomorphism current

Under an infinitesimal diffeomorphism, $x^\mu \to x^\mu + \xi^\mu$, the functional variation of an arbitrary differential p-form $\alpha = \frac{1}{p!}\alpha_{\mu_1\mu_2\ldots\mu_p}dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_p}$ is given by

$$\delta_{\text{diff}}\,\alpha = -\pounds_\xi\alpha, \tag{G.8}$$

where $\pounds_\xi$ stands for the Lie derivative operator defined by

$$\pounds_\xi = dI_\xi + I_\xi d \tag{G.9}$$

and I_ξ is the contraction operator which acting on a p-form α gives

$$I_\xi\alpha = \frac{1}{(p-1)!}\xi^{\mu_1}\alpha_{\mu_1\mu_2\ldots\mu_p}dx^{\mu_2}\wedge\cdots\wedge dx^{\mu_p}. \tag{G.10}$$

We now replace in (G.4) the functional variation asociated with a diffeomorphism, so we get

$$-\pounds_\xi L^{(d)} = -E_A\pounds_\xi\varphi^A - d(B_A\pounds_\xi\varphi^A). \tag{G.11}$$

Since $L^{(d)}$ is a d-form, $\pounds_\xi L^{(d)} = dI_\xi L^{(d)}$, so we get the identity

$$d(B_A\pounds_\xi\varphi^A - I_\xi L^{(d)}) + E_A\pounds_\xi\varphi^A = 0. \tag{G.12}$$

Defining

$$\star J^{(\text{diff-on})} = B_A\pounds_\xi\varphi^A - I_\xi L^{(d)}, \tag{G.13}$$

we have that

$$d\star J^{(\text{diff-on})} + E_A\pounds_\xi\varphi^A = 0. \tag{G.14}$$

When φ^A corresponds to a on-shell configuration, i.e., satisfies $E_A(\varphi)=0$, $\star J^{(\text{diff-on})}$ is conserved,

$$d\star J^{(\text{diff-on})} = 0. \tag{G.15}$$

Of course, if the term $E_A \pounds_\xi \varphi^A$ is an exact form,

$$E_A \pounds_\xi \varphi^A = dX, \tag{G.16}$$

it is possible to define

$$\star J^{(\text{diff-off})} = \star J^{(\text{diff-on})} + X, \tag{G.17}$$

which will be conserved without demanding an on-shell configuration,

$$d \star J^{(\text{diff-off})} = 0. \tag{G.18}$$

G.2. On-shell and off-shell gauge current

When the Lagrangian is invariant under an infinitesimal symmetry transformation $\varphi^A \to \varphi^A + \epsilon^A$, we get from (G.4) that

$$E_A \epsilon^A + d(B_A \epsilon^A) = 0. \tag{G.19}$$

So when we define $\star J^{(\text{gauge-on})} = B_A \epsilon^A$, we have

$$d \star J^{(\text{gauge-on})} + E_A \epsilon^A = 0, \tag{G.20}$$

so, when φ^A corresponds to an on-shell configuration, $J^{(\text{gauge-on})}$ is conserved,

$$d \star J^{(\text{gauge-on})} = 0. \tag{G.21}$$

We notice than when $E_A \epsilon^A$ is an exact form,

$$E_A \epsilon^A = dY, \tag{G.22}$$

it is possible to define the conserved current

$$\star J^{(\text{gauge-off})} = \star J^{(\text{gauge-on})} + Y, \tag{G.23}$$

$$d \star J^{(\text{gauge-off})} = 0, \tag{G.24}$$

for any configuration of φ^A.

Apéndice H

$\beta \neq 0$ case formulas

For completeness we show the formulas in the implicit solution of the contorsion tensor $C(a,b,i)$ (equation (3.92)) for the "scalar" case ($p=1, q=1$) when $\beta \neq 0$.

$$\begin{aligned}A(a,b,i) = &-\frac{1}{2}C(\phi,a,b)\phi(i)\kappa - \frac{1}{2}C(\phi,a,i)\phi(b)\kappa + \frac{1}{2}C(\phi,b,a)\phi(i)\kappa + \frac{1}{2}C(\phi,b,i)\phi(a)\kappa - \frac{1}{2}C(\phi,i,a)\phi(b)\kappa \\ &+ \frac{1}{2}C(\phi,i,b)\phi(a)\kappa + C(a,l,l)d(b,i) - C(b,l,l)d(a,i) + \kappa J(a,b,i) - \kappa J(b,i,a).\end{aligned} \tag{H.1}$$

$$\begin{aligned}&C(\phi,b,i) = \frac{1}{2}C(\phi,b,\phi)\phi(i)\kappa[1+\beta^2]R^{-1} - \frac{1}{2}C(\phi,i,\phi)\phi(b)\kappa[1+\beta^2]R^{-1} + C(\phi,l,\phi)\epsilon(\phi,b,i,l)\kappa\beta^3R^{-1} \\ &+ C(\phi,l,l)d(b,i)[1-\beta^2][1-\kappa\phi^2-\beta^2]^{-1} - \frac{1}{2}C(b,l,l)\phi(i)[1+\beta^2-2\beta^4]R^{-1} - \frac{1}{2}C(b,l,l)\phi(i)[1-\beta^2][1-\kappa\phi^2-\beta^2]^{-1} \\ &+ \frac{1}{2}C(i,l,l)\phi(b)[1+\beta^2-2\beta^4]R^{-1} - \frac{1}{2}C(i,l,l)\phi(b)[1-\beta^2][1-\kappa\phi^2-\beta^2]^{-1} - C(l,f,l)\epsilon(\phi,b,i,f)\beta[1-\beta^2]R^{-1} \\ &+ \kappa J(\phi,b,i)[1-\kappa\phi^2-\beta^2]^{-1} + \kappa\beta^2 J(\phi,b,i)R^{-1} + J(\phi,i,b)\kappa[1-\kappa\phi^2-\beta^2]^{-1} - J(\phi,i,b)\kappa\beta^2R^{-1} \\ &- J(b,i,\phi)\kappa[1+\beta^2]R^{-1} + J(f,g,\phi)\varepsilon(b,i,f,g)\kappa\beta^3R^{-1} - \frac{1}{2}J(f,g,b)\epsilon(\phi,i,f,g)\kappa\beta[1-\kappa\phi^2-\beta^2]^{-1} \\ &+ \frac{1}{2}J(f,g,b)\epsilon(\phi,i,f,g)\kappa\beta R^{-1} - \frac{1}{2}J(f,g,i)\epsilon(\phi,b,f,g)\kappa\beta[1-\kappa\phi^2-\beta^2]^{-1} - \frac{1}{2}J(f,g,i)\epsilon(\phi,b,f,g)\kappa\beta R^{-1}.\end{aligned} \tag{H.2}$$

$$R \equiv 1 - \kappa\phi^2\beta^2 + \beta^2 - 2\beta^4. \tag{H.3}$$

$$C(\phi,l,l) = \kappa\beta\epsilon(\phi,f,g,h)J(f,g,h)[2+\kappa\phi^2-2\beta^2]^{-1} - 2\kappa J(\phi,f,f)[2+\kappa\phi^2-2\beta^2]^{-1}. \tag{H.4}$$

$$\begin{aligned}&C(b,l,l) = \kappa^3\phi^2\beta[1-\beta^2][2+\kappa\phi^2-2\beta^2]^{-1}X^{-1}\phi(b)\epsilon(\phi,f,g,h)J(f,g,h) \\ &- 2\kappa^3\phi^2[1-\beta^2]X^{-1}[2+\kappa\phi^2-2\beta^2]^{-1}\phi(b)J(\phi,f,f) - \kappa^3\phi^2\beta X^{-1}\epsilon(\phi,b,f,g)J(f,g,\phi) \\ &+ 2\kappa^2[1-\beta^2]X^{-1}J(\phi,b,\phi) + \kappa\beta X^{-1}[1-\kappa\phi^2-\beta^2]\epsilon(b,f,g,h)J(f,g,h) - 2\kappa X^{-1}[1-\kappa\phi^2-\beta^2]J(b,f,f).\end{aligned} \tag{H.5}$$

$$X \equiv 2 - \kappa\phi^2 - 4\beta^2 + 2\beta^4 + \kappa\phi^2\beta^2 - \kappa^2\phi^4\beta^2. \tag{H.6}$$

$$C(\phi, b, \phi) = C(\phi, l, l)\phi(b)[1 - \kappa\phi^2 - \beta^2]^{-1} - C(\phi, l, l)\phi(b)\beta^2[1 - \kappa\phi^2 - \beta^2]^{-1} - C(b, l, l)\phi^2[1 - \kappa\phi^2 - \beta^2]^{-1} + C(b, l, l)\phi^2\beta^2[1 - \kappa\phi^2 - \beta^2]^{-1} + 2J(\phi, b, \phi)\kappa[1 - \kappa\phi^2 - \beta^2]^{-1} - J(f, g, \phi)\epsilon(\phi, b, f, g)\kappa\beta[1 - \kappa\phi^2 - \beta^2]^{-1}. \tag{H.7}$$

Apéndice I

Inflation from scalar fields

Let us consider the canonical scalar action in curved spacetime

$$S = \int d^4x \sqrt{-g}\mathcal{L}_\phi, \tag{I.1}$$

with

$$\mathcal{L}_\phi = \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi). \tag{I.2}$$

For simplicity, we assume a flat spacetime,

$$g_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -a^2(t) & & \\ & & -a^2(t) & \\ & & & -a^2(t) \end{pmatrix}, \tag{I.3}$$

and the equation of motion for the field ϕ is

$$\ddot{\phi} + 3H\dot{\phi} - \nabla^2\phi + \frac{\delta V}{\delta\phi} = 0, \tag{I.4}$$

where an overdot indicates a derivative with respect to the coordinate time t, and $H = \frac{\dot{a}}{a}$ is the Hubble parameter. We will be particularly interested in the homogeneous mode of the field, for which the gradient term vanishes, $\nabla\phi = 0$, so that the functional derivative $\frac{\delta V}{\delta\phi}$ simplifies to an ordinary derivative, and the equation of motion simplifies to

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0. \tag{I.5}$$

The stress-energy for a scalar field is given by

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\mathcal{L}_\phi, \tag{I.6}$$

and, for a homogeneous field, it takes the form of a perfect fluid with energy density ρ and pressure p, with

$$\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi), \tag{I.7}$$

$$p = \frac{1}{2}\dot{\phi}^2 - V(\phi). \tag{I.8}$$

We see that the de Sitter limit, $p \simeq -\rho$, is just the limit in which the potential energy of the field dominates the kinetic energy, $\dot{\phi} \ll V(\phi)$. This limit is referred to as *slow roll*, and under such conditions the universe expands quasi-exponentially,

$$a(t) \propto \exp\left(\int H dt\right) \equiv e^{-N}, \tag{I.9}$$

where it is conventional to define the number of e-folds N with the sign convention

$$dN \equiv -H dt, \tag{I.10}$$

so that N is large in the far past and decreases as we go forward in time and as the scale factor a increases. Recalling that Friedmann and Raychaudhuri equations (Einstein's equations) are respectively

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{\kappa}{a^2} = \frac{8\pi}{3m_{Pl}^2}\rho, \tag{I.11}$$

$$\left(\frac{\ddot{a}}{a}\right) = -\frac{4\pi}{3m_{Pl}^2}(\rho + 3p), \tag{I.12}$$

we can rewrite them in a convenient form ($\kappa = 0$)

$$H^2 = \frac{8\pi}{3m_{Pl}}\left[\frac{1}{2}\dot{\phi}^2 + V(\phi)\right], \tag{I.13}$$

$$\left(\frac{\ddot{a}}{a}\right) = H^2(1 - \epsilon), \tag{I.14}$$

where ϵ

$$\epsilon \equiv \frac{3}{2}\left(\frac{p}{\rho} + 1\right) = \frac{4\pi}{m_{Pl}}\left(\frac{\dot{\phi}}{H}\right)^2, \tag{I.15}$$

specifies the equation of state. It can be shown that

$$\epsilon = -\frac{d\ln H}{d\ln a} = \frac{1}{H}\frac{dH}{dN}. \tag{I.16}$$

This is a useful parametrization because the condition for accelerated expansion $\ddot{a} > 0$ is simply equivalent to $\epsilon < 1$. The de Sitter limit $p \to -\rho$ is equivalent to $\epsilon \to 0$, so that the potential $V(\phi)$ dominates the energy density, and

$$H^2 \simeq \frac{8\pi}{3m_{Pl}}V(\phi). \tag{I.17}$$

We make the additional approximation that the friction term in the equation of motion (I.5) dominates,

$$\ddot{\phi} \ll 3H\dot{\phi}, \tag{I.18}$$

so that the equation of motion for the scalar field is approximately

$$3H\dot{\phi} + V'(\phi) \simeq 0. \tag{I.19}$$

This last equation together with Friedmann equation are referred to as *slow roll approximation*. Condition (I.18) can be expressed in terms of a second dimensionless parameter, conventionally defined as

$$\eta \equiv -\frac{\ddot{\phi}}{H\dot{\phi}} = \epsilon + \frac{1}{2\epsilon}\frac{d\epsilon}{dN}. \tag{I.20}$$

The parameters ϵ and η are referred to as *slow roll parameters*, and the slow roll approximation is valid as long as both are small, $\epsilon, |\eta| \ll 1$. It is not obvious that this will be a valid approximation for situations of physical interest. η need **not** be small for inflation to take place. Inflation takes place when $\epsilon < 1$, regardless of the value of η. In the limit of slow roll, we can use (I.17),(I.19) to write the parameter ϵ approximately as

$$\epsilon = \frac{4\pi}{m_{Pl}^2}\left(\frac{\dot{\phi}}{H}\right)^2 \simeq \frac{m_{Pl}^2}{16\pi}\left(\frac{V'(\phi)}{V(\phi)}\right)^2. \tag{I.21}$$

The inflationary limit, $\epsilon \ll 1$ is then just equivalent to a field evolving on a flat potential, $V'(\phi) \ll V(\phi)$. The second slow roll parameter η can likewise be written approximately as:

$$\begin{aligned}\eta &= -\frac{\ddot{\phi}}{H\dot{\phi}} \\ &\simeq \frac{m_{Pl}^2}{8\pi}\left[\frac{V''(\phi)}{V(\phi)} - \frac{1}{2}\left(\frac{V'(\phi)}{V(\phi)}\right)^2\right],\end{aligned} \tag{I.22}$$

so that the curvature V'' of the potential must also be small for slow roll to be a valid approximation. Similarly, we can write the number of e-folds as a function $N(\phi)$ of the field as:

$$\begin{aligned}N \simeq -\int H dt = -\int \frac{H}{\dot{\phi}} d\phi = \frac{2\sqrt{\pi}}{m_{Pl}} \int \frac{d\phi}{\sqrt{\epsilon}} \\ \simeq \frac{8\pi}{m_{Pl}^2}\int_{\phi_e}^{\phi} \frac{V(\phi)}{V'(\phi)} d\phi.\end{aligned} \tag{I.23}$$

The limits on the last integral are defined such that ϕ_e is taken to be the end on inflation, and N increases as we go *backward* in time, representing the number of e-folds of expansion which take place between field value ϕ and ϕ_e.

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