Upper and lower bounds for eigenvalues of nonlinear elliptic equations: I. The lowest eigenvalue

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We give a method for finding bounds for the lowest eigenvalue of nonlinear elliptic equations with monotone, local, nonlinearities. This is an extension to nonlinear problems of Barta's method for linear elliptic operators.

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1. INTRODUCTION

There are two main variational characterizations for the fundamental eigenvalue of linear elliptic problems, namely the (integral) Rayleigh–Ritz principle and the (local) Barta method.[1] The Rayleigh principle provides upper bounds for the fundamental eigenvalue whereas Barta's principle gives both upper and lower bounds.[2] The Rayleigh principle is a special case, for the fundamental eigenvalue, of the Courant–Fisher variational characterization[3] for all the eigenvalues. As for Barta's method, it also extends to the rest of the eigenvalues but only for one dimensional problems.[4] When going into nonlinear problems the Courant–Fisher principle goes into the Ljusternik–Schnirelman category theory,[5] where the solutions to nonlinear (eigenvalue) elliptic equations are critical points of a given functional subject to certain constraints. However, the numerical values of this functional at the critical points do not coincide with the eigenvalues corresponding to the solutions aforementioned. Thus the Ljusternik–Schnirelman theory does not directly give bounds for the eigenvalues of nonlinear elliptic problems. In this article we show that Barta's principle for the eigenvalue of linear problems remains practically unchanged when going into nonlinear elliptic eigenvalue equations with local, monotone, nonlinearities. Therefore, this method directly provides with bounds for the eigenvalues of these nonlinear problems. More precisely, it gives bounds for the graphs of the eigenvalues as functions of the norm of the corresponding solutions. We believe the requirement on the nonlinearities to be local can somehow be relaxed and thus, we conjecture that Barta's method should extend to equations such as the Hartree equation,[6] the Thomas–Fermi–von Weizsäcker equation,[7] some equations connected with non-Boussinesq convection,[8] etc. The monotonicity requirement is more stringent as we show in the Ex. 2 below. There are other ways of getting bounds for the eigenvalue of the equations considered here; in particular Amann's method[9] of proving existence of solutions to equations of this type by constructing upper and lower solutions yields as a byproduct bounds on the eigenvalues. The advantage of the extended Barta's method is that the conditions on the trial functions used in the variational inequalities are easier to satisfy. In

Sec. 2 we prove our main theorem, i.e., the bounds for the lowest eigenvalue, and we give some examples and applications.

2. BOUNDS ON THE LOWEST EIGENVALUE

Let us consider the problem[10]

$$\mathcal{L}u + f(x, u) = \lambda au \quad \text{in } \Omega,$$
$$u = 0 \qquad\qquad \text{on } \partial\Omega, \tag{1}$$

where $\Omega \subset \mathcal{R}^N$ is a bounded domain with smooth boundary. Here $\mathcal{L}$ denotes a self-adjoint elliptic operator defined by

$$\mathcal{L}u = - \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial u}{\partial x_j} \right) + c(x)u, \tag{2}$$

with $a_{ij} = a_{ji} \in C^{1,\alpha}(\overline{\Omega})$, $c \in C^{0,\alpha}(\overline{\Omega})$, $c \geqslant 0$ and

$$\sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \geqslant \eta |\xi|^2,$$

all $x \in \overline{\Omega}$, all $\xi \in \mathcal{R}^N$ with $\eta > 0$, the ellipticity constant. Moreover, $a \in C^0(\overline{\Omega})$ and $a > 0$ in $\overline{\Omega}$. We assume $f: \overline{\Omega} \times \mathcal{R}^+ \to \mathcal{R}$ is continuous, $f(x, 0) = 0$, $f(x, s) = o(s)$ in the neighborhood of $s = 0$, uniformly with respect to $x \in \overline{\Omega}$, and $s \to (f(x, s)/s)$ (defined to be 0 in $s = 0$) is a strictly increasing function on $\mathcal{R}^+$, all $x \in \Omega$. Moreover, $\lim_{s \to +\infty} (1/s)f(x, s) = +\infty$, uniformly with respect to $x \in \overline{\Omega}$. Let λ_1 denote the lowest eigenvalue of the linear problem $\mathcal{L}\emptyset = \lambda a\emptyset$. It is known[10] that under the above hypothesis on f, there is a unique positive solution u_λ to Eq. (1) for every $\lambda > \lambda_1$. Furthermore, the mapping $\lambda \to u_\lambda$ is continuous from $(\lambda_1, +\infty)$ into $C^{1,\alpha}(\overline{\Omega})$. We are interested in getting bounds on the graph $\lambda = \lambda(\|u_\lambda\|)$, where $\|u_\lambda\|$ denotes the norm of u_λ [without loss of generality we will work with the L^2-norm, i.e., $\|u\| \equiv (\int |u|^2 dx)^{1/2}$]. Our main result is the following:

Theorem 1: Fix $r > 0$ and let $u \in C^2(\overline{\Omega})$ be any function on Ω such that $u > 0$ almost everywhere in Ω, $u \geqslant 0$ on $\partial\Omega$ and $\|u\| = r$. Then,

$$\lambda(r) \geqslant \inf_{\overline{\Omega}} \{ (\mathcal{L}u + f(x, u))/au \}, \tag{3}$$

where $\lambda(r)$ is the eigenvalue corresponding to the positive solution u_λ of Eq. (1) with norm $\|u_\lambda\| = r$.

Proof: We need only consider the case $u \neq u_\lambda$, for if $u = u_\lambda$, all $x \in \Omega$, then Eq. (1) implies

$$[\mathcal{L}u + f(x, u)]/au = \lambda(r) \tag{4}$$

for all $x \in \Omega$ and (3) follows from here. So let $u \in C^2(\bar{\Omega})$, $u \not\equiv u_\lambda$ and define

$$A = \min_{\bar{\Omega}} (\mathscr{L}u + f(x, u) - \lambda(r)au). \tag{5}$$

Since $u \in C^2(\bar{\Omega})$ and $\bar{\Omega}$ is compact A is finite and there exists $y \in \bar{\Omega}$ such that

$$A = \mathscr{L}u(y) + f(y, u(y)) - \lambda(r)a(y)u(y).$$

We consider separately the two possibilities i) $A < 0$ and ii) $A > 0$ (as in Ref. 4). If i) holds, the continuity of $\mathscr{L}u + f(x, u) - \lambda(r)au$ implies that there is an open neighborhood U of y such that

$$\mathscr{L}u + f(x, u) - \lambda(r)au < 0, \tag{6}$$

all $x \in U \cap \bar{\Omega}$, and in particular u cannot vanish identically on any open neighborhood contained in $U \cap \bar{\Omega}$ so it follows that

$$u(z) > 0 \tag{7}$$

for some $z \in U \cap \bar{\Omega}$. Since $a > 0$ in $\bar{\Omega}$, Eqs. (6) and (7) imply

$$\lambda(r) > \{\mathscr{L}u(z) + f(z, u(z))\}/a(z)u(z),$$

which proves the proposition in case i). Now we conclude the theorem by showing that ii) cannot hold unless $u \equiv u_\lambda$. If ii) holds we have

$$\mathscr{L}u + f(x, u) - \lambda(r)au \geqslant 0, \tag{8}$$

all $x \in \bar{\Omega}$. Eqs. (1) and (8) together with the maximum principle imply

$$u \geqslant u_\lambda, \quad \text{all } x \in \bar{\Omega}. \tag{9}$$

In fact let $D = \{x \in \Omega \mid u(x) < u_\lambda(x)\}$; since u and u_λ are continuous in $\bar{\Omega}$, D is an open subset of Ω. Let $W \equiv (f(x, u_\lambda)/u_\lambda) - \lambda(r)a$ and $V \equiv (f(x, u)/u) - \lambda(r)a$. By hypothesis $s \to (1/s)f(x, s)$ is strictly increasing so $V < W$ in D. Moreover, $u > 0$ a.e. in D (in fact in Ω), $u = u_\lambda$ in ∂D, $D \cap \partial\Omega = \Phi$, $\mathscr{L}u + Vu \geqslant 0$ and $\mathscr{L}u_\lambda + Wu_\lambda = 0$ in D. Therefore, by the comparison theorem 2.1 in Ref. 11 (see also the remark below), $u \geqslant u_\lambda$ in D, hence D is empty and (9) follows. Since $\|u\| = \|u_\lambda\| = r$, (9) is impossible unless $u \equiv u_\lambda$ and the theorem follows from here.

Remark: In order to better understand how (9) follows from (1) and (8) we give a heuristic argument. For simplicity we take the following particular case: $\mathscr{L} = -\Delta$, $a = 1$, $f(x, u) = u^2$. Then (1) and (8) read,

$$-\Delta u_\lambda + u_\lambda^2 - \lambda u_\lambda = 0 \tag{10}$$

and

$$-\Delta u + u^2 - \lambda u \geqslant 0. \tag{11}$$

Multiplying (10) by u, (11) by u_λ and subtracting we get

$$-\nabla(u_\lambda^2 \nabla g) + u_\lambda^2 ug \geqslant 0, \tag{12}$$

where

$$g = (u/u_\lambda) - 1. \tag{13}$$

Since u_λ^2 and uu_λ^2 are positive, the operator $-\nabla(u_\lambda^2 \nabla \cdot) + u_\lambda^2 u \cdot$ is elliptic. Thus (12) implies $g \geqslant 0$ and (13) implies $u \geqslant u_\lambda$.

A proof completely analogous to that of Theorem 1 gives the following upper bound

Theorem 2: Fix $r > 0$ and let $u \in C^2(\bar{\Omega})$ be any function on Ω such that $u > 0$ almost everywhere in Ω, $u = 0$ on $\partial\Omega$ and

$\|u\| = r$. Then,

$$\lambda(r) \leqslant \sup_{\bar{\Omega}} \{(\mathscr{L}u + f(x, u))/au\}, \tag{14}$$

where $\lambda(r)$ is the eigenvalue corresponding to the positive solution u_λ of Eq. (1) with norm $\|u_\lambda\| = r$.

We now give some applications of Theorems 1 and 2.

Example 1: Consider the equation

$$-\frac{d^2u}{dx^2} + u^3 = \lambda u, \tag{15}$$

defined on $\Omega = (o, b) \subset \mathscr{R}$, with $u(o) = u(b) = 0$. The positive solution to Eq. (15), with the above boundary conditions, is given parametrically by

$$u_\lambda(x) = \frac{2\sqrt{2}}{b} k K(k) \operatorname{sn}\left(\frac{2x}{b} K(k), k\right), \tag{16}$$

with

$$\lambda = (4/b^2)(1 + k^2)K(k)^2, \quad 0 \leqslant k < 1. \tag{17}$$

Here and below K and E denote the complete elliptic integrals. From (16) we get the L^2-norm of u_λ,

$$\|u_\lambda\| = \{(8/b)K(k)[K(k) - E(k)]\}^{1/2}. \tag{18}$$

Thus, the graph $\lambda(\|u_\lambda\|)$ for the positive solution of (15) is given parametrically by equations (17) and (18) with $0 \leqslant k < 1$. λ is an increasing and convex function of $\|u_\lambda\|$, $\lambda(0) = (\pi/b)^2$. Moreover, as $\|u_\lambda\| \to 0(k \to 0)$, $\lambda \approx (\pi/b)^2 + (3\|u_\lambda\|^2/2b)$, whereas if $\|u_\lambda\| \to \infty$ $(k \to 1)$, $\lambda \approx \|u_\lambda\|^2/b$. In order to get a lower bound for $\lambda(\|u_\lambda\|)$ we use in Theorem 1 the trial function $u(x) = rb^{-1/2}$, all $x \in \Omega$. Note that $\|u\| = r$, so by Eq. (3) we get $\lambda(r) \geqslant r^2/b$. Also, trying $u(x) = (2/b)^{1/2}r \sin(\pi x/b)$ in Eq. (3) gives $\lambda(r) \geqslant (\pi/b)^2$. Therefore, $\lambda(r) \geqslant \max\{(\pi/b)^2, r^2/b\}$. As for upper bounds, trying $u(x) = (2/b)^{1/2}r \sin(\pi x/b)$ in Eq. (14) yields $\lambda(r) \leqslant (\pi/b)^2 + 2(r^2/b)$. So the curve $\lambda(\|u_\lambda\|)$, given parametrically by (17) and (18), lies between the curves $\lambda_-(r) = \max\{(\pi/b)^2, r^2/b\}$ and $\lambda_+(r) = (\pi/b)^2 + 2(r^2/b)$.

As the next example shows, if the nonlinearity in problem (1) is not monotone, Theorems 1 and 2 are not valid anymore.

Example 2: Consider the equation

$$-\frac{d^2u}{dx^2} - u^3 = \lambda u \tag{19}$$

defined on $\Omega = (o, b) \subset \mathscr{R}$, with $u(o) = u(b) = 0$. The problem (19) violates one of the hypothesis of Theorem 1 and 2, namely, $f(x, u)/u = -u^2$ is not strictly increasing (in fact, $-u^2$ is strictly decreasing). The positive solution to Eq. (19) is given parametrically by

$$u_\lambda(x)$$
$$= (kK(k)/b) \cdot (8(1 - k^2))^{1/2} \operatorname{sn}(2xK(k)/b)/dn(2xK(k)/b), \tag{20}$$

with

$$\lambda = 4K(k)^2(1 - 2k^2)/b^2, \quad 0 \leqslant k < 1. \tag{21}$$

Therefore, the graph $\lambda(\|u_\lambda\|)$ is given parametrically by (21) and

$$\|u_\lambda\| = \{(8/b)K(k)[E(k) - (1 - k^2)K(k)]\}^{1/2}. \tag{22}$$

λ is a decreasing and concave function of $\|u_\lambda\|$, $\lambda(o) = (\pi/$

b)^2. Moreover, as $\|u_\lambda\| \to 0$ $(k \to 0)$, $\lambda \approx (\pi/b)^2 - (2\|u_\lambda\|^2/b)$, whereas if $\|u_\lambda\| \to \infty$ $(k \to 1)$, $\lambda \approx -\frac{1}{16}\|u_\lambda\|^4$. Our purpose here is to show that Theorem 1 is not valid for this problem. In fact, let us assume (3) holds and take the trial function (with $\|u\| = r$) $u(x) = (2/b)^{1/2} r \sin(\pi x/b)$; then Eq. (3) gives $\lambda(r) \geqslant (\pi/b)^2 - (2r^2/b)$ which contradicts the asymptotic behavior $\lambda(r) \approx -\frac{1}{16}r^4$, for large r, found above.

Example 3: (Monotonicity of λ in $\|u_\lambda\|$). It is well known[10] that u_λ is strictly increasing with λ, i.e., if $\lambda < \nu$, then $u_\lambda < u_\nu$ in Ω. This, of course, implies that $\|u_\lambda\|$ is strictly increasing with λ. This last fact can also be obtained directly from Theorem 1: we now show that $s > r$ implies $\lambda(s) \geqslant \lambda(r)$. Denote by u_λ the positive solution of (1) with norm $\|u_\lambda\| = r$ [and thus, eigenvalue $\lambda(r)$], and choose $u = (s/r)u_\lambda$ as a trial function on Eq. (3). Note that $\|u\| = s$, and $u > u_\lambda$ in Ω, which implies $f(x, u)/u \geqslant f(x, u_\lambda)/u_\lambda$ in Ω. So,

$$[\mathscr{L}u + f(x, u)]\frac{1}{au} = \frac{\mathscr{L}u_\lambda}{au_\lambda} + \frac{f(x, u)}{au}$$

$$= \lambda(r) - \frac{f(x, u_\lambda)}{au_\lambda} + \frac{f(x, u)}{au}$$

$$\geqslant \lambda(r), \tag{23}$$

where the second equality follows from Eq. (1). Introducing (23) in Eq. (3) we find $\lambda(s) \geqslant \lambda(r)$, hence λ is monotone nondecreasing in $\|u_\lambda\|$.

In the Ex. 1 above, λ is not only an increasing function of $\|u_\lambda\|$ but it is also convex. We conjecture that λ is a convex function of $\|u_\lambda\|$ for the general problem (1), at least for convex domains Ω.

Note added in proof: Equation (8) above says that u is an upper solution for Eq. (1) with $\lambda = \lambda(r)$. Form here (9) follows.

ACKNOWLEDGMENT

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