OSCILLATORY INSTABILITIES IN BENARD-MARANGONI CONVECTION IN A FLUID BOUNDED ABOVE BY A FREE SURFACE

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ABSTRACT. We study numerically the linear stability theory of a fluid bounded above by a free deformable surface and below by a rigid or free plane surface. Oscillatory instabilities are found which do not exist when surface deformation is neglected. Analytical results are given for a long wavelength oscillatory instability which we identify as gravity waves.

Introduction

In many analytical studies of convection it is assumed that the boundaries of the fluid are free but plane. This boundary condition is chosen since it is simpler to treat analytically than the rigid boundary condition.[1] A real free surface, however, is deformed due to the fluid motion, effect which is frequently neglected for the sake of simplicity. If the full boundary conditions on the free surface are used, the monotonicity principle is no longer valid in Rayleigh-Bénard convection and we may expect oscillatory instabilities. We have studied this problem numerically and analytically[2–3]; the results are described below.

Mathematical Formulation

Let us consider a two dimensional fluid bounded above by a thermally insulating passive gas and below by a plane stress-free perfect thermally conducting medium which, at rest, lies between $z = 0$ and $z = d$. Upon it acts gravity $\vec{g} = -g\hat{z}$. In the Boussinesq approximation the equations that describe the motion of the fluid are

$$\nabla \cdot \vec{v} = 0$$

$$\rho_o \frac{d\vec{v}}{dt} = -\vec{\nabla} p + \mu \nabla^2 \vec{v} + \vec{g}\rho$$

$$\frac{dT}{dt} = \kappa \nabla^2 T$$

$$\rho = \rho_o[1 - \alpha(T - T_o)]$$

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where $d/dt = \partial/\partial t + \vec{v}\cdot\vec{\nabla}$ is the convective derivative; $\vec{v} = (u, 0, w)$ is the fluid velocity, p is the pressure, and T is the temperature. T_o and ρ_o are a reference temperature and density respectively. The viscosity, μ, thermal diffusivity, κ, and coefficient of thermal expansion, α are constant.

On the upper free surface $z = d + \eta(x,t)$ the boundary conditions are

$$\eta_t + u\eta_x = w$$
$$p - p_a - \frac{2\mu}{N^2}[w_z + u_x\eta_x^2 - \eta_x(u_z + w_x)] = 0$$
$$\mu(1-\eta_x^2)(u_z + w_x) + 2\mu\eta_x(w_z - u_x) = 0$$

and

$$T_z - \eta_x T_x = -FN/k$$

Subscripts x and z denote derivatives with respect to the horizontal and vertical coordinates respectively. Here $N = (1+\eta_x^2)^{1/2}$, F is the normal heat flux, k is the thermal conductivity, and p_a is a constant pressure exerted on the upper free surface.

We shall assume that the lower surface is either stress-free and plane or rigid, and it may be at constant temperature T_b or at constant heat flux. The boundary conditions on the lower surface $z = 0$ are then $w = u_z = 0$, if it is stress-free, or $w = u = 0$ if it is rigid; and $T = T_b$ if it is held at constant temperature or $dT/dz = -F/k$ if the heat flux F is held fixed.

The static solution to these equations is given by $T_s = -F(z-d)/k + T_o$, $\rho_s = \rho_o[1 + (\alpha F/k)(z-d)]$, and $p_s = p_a - g\rho_o[(z-d) + (\alpha F/2k)(z-d)^2]$. It is convenient to adopt d as unit of length, d^2/κ as unit of time, $\rho_o d^3$ as unit of mass, and $T_s(0) - T_s(d)$ as unit of temperature. Then only three dimensionless parameters are involved in the problem, the Prandtl number $\sigma = \mu/\rho_o\kappa$, the Rayleigh number $R = \rho_o g\alpha(T_s(0) - T_s(d))d^3/\kappa\mu$ and the Galileo number $G = gd^3\rho_o^2/\mu^2$. Introducing a stream function $\psi(x,z,t)$ in terms of which the velocity is given by $\vec{v} = (\psi_z, 0, -\psi_x)$, the linear equations for the perturbations to the static state may be reduced to

$$(D^2 - a^2)(D^2 - a^2 - \lambda/\sigma)\psi = iaR\theta$$
$$(D^2 - a^2 - \lambda)\theta = ia\psi$$

where $D = d/dz$, θ is the perturbation to the static temperature profile, and where we have assumed that all perturbations evolve in time as $\exp(\lambda t)$ and in the horizontal variable as $\exp(iax)$. The linearized boundary conditions become

$$\lambda(D^2 - 3a^2 - \lambda/\sigma)D\psi - a^2(\sigma G + a^2/C)\psi = 0$$
$$(D^2 + a^2)\psi - \frac{R\Gamma}{\sigma G + a^2/C}(D^2 - 3a^2 - \lambda/\sigma)D\psi + iaR\Gamma\theta = 0$$

The dimensionless numbers that have appeared are the Capillary number $C = \mu\kappa/\tau_o d$ and $\Gamma = \gamma/\rho_o g\alpha d^2$. The Marangoni number is given by $M = \Gamma R$, we have chosen to use Γ as an independent parameter instead of M. The thermal boundary conditions are either $\theta = 0$ or $D\theta = 0$. We have solved the linear equations for different boundary conditions on the lower surface in search for oscillatory instabilities.

Results

The effect of surface deformation on the eigenvalues λ and R is measured by the size of $\sigma G + a^2/C$. When this coefficient tends to infinity we recover the simpler stress-free boundary conditions. In the case when the lower surface is rigid, an oscillatory instability is found at finite wavenumber. The critical R for overstability decreases with increasing Γ and increases with G (figs.1-2). This shows that surface tension is a driving mechanism for this instability. The critical Rayleigh number remains higher than the corresponding value for the onset of steady convection within the range of validity of the Boussinesq approximation. When the lower surface is stress-free an oscillatory instability with similar features to the case just described is present. In addition we have found a long wavelength instability at critical R considerably lower than the corresponding value for the onset of steady convection (figs.3-4). The driving mechanism for this instability is buoyancy alone. An asymptotic analysis of this long wave instability shows that the leading order stream function and temperature perturbations are given by $\psi = z$ and $\theta = a(z^3 - 3z)/6$. The critical Rayleigh number is given by $R_c = 30/(1 - 5\Gamma/2)$ and the frequency at criticality is given by $\omega = a\sqrt{\sigma^2 G - \sigma \Gamma R_c}$. This instability corresponds to gravity waves in a shallow viscous fluid.

Acknowledgments

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References

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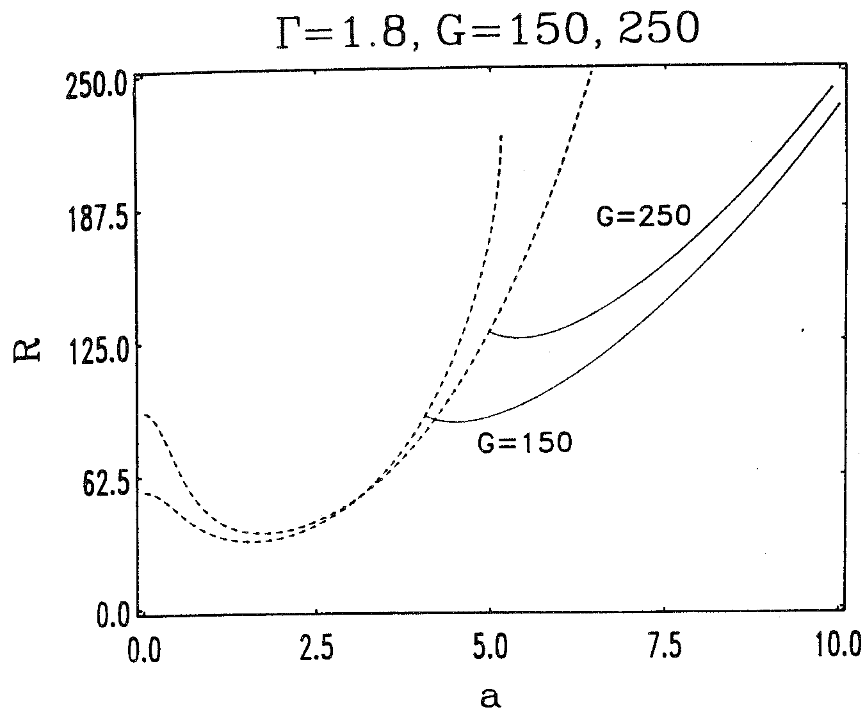

Figure 1. Rayleigh number vs. wavenumber for different values of the Galileo number, with $\Gamma = 1.8$. The dashed lines correspond to the marginal curves.

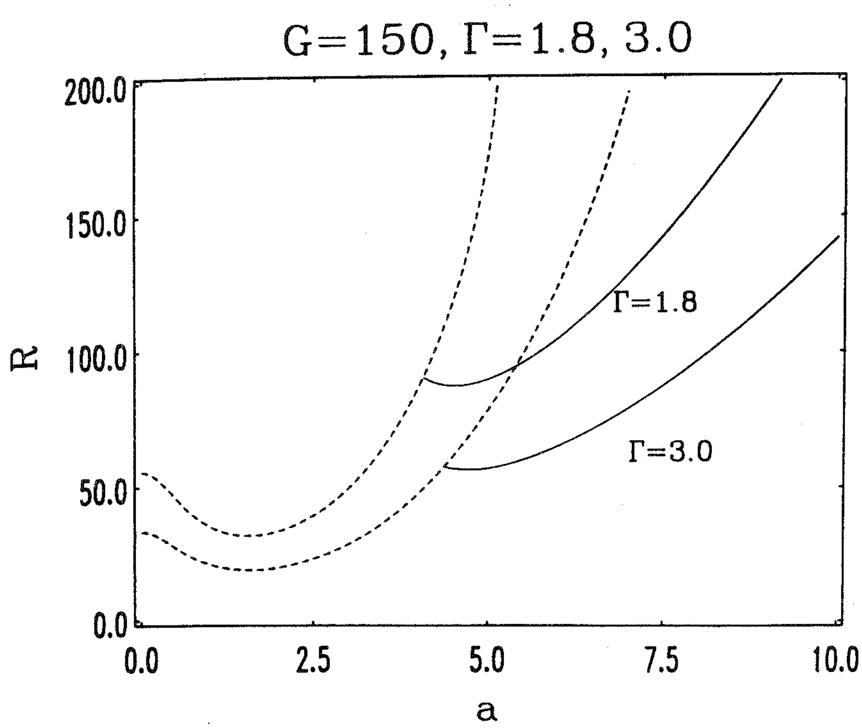

Figure 2. Rayleigh number vs. wavenumber for $G = 150$ and different values of Γ.

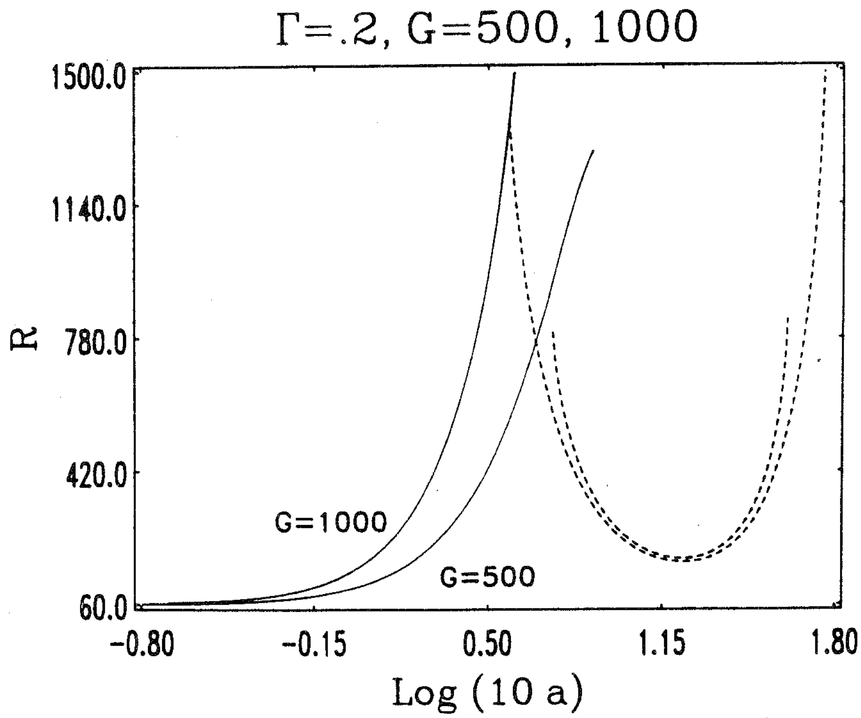

Figure 3. Rayleigh number vs. wavenumber for a small value of Γ (= 0.2) and different values of G.

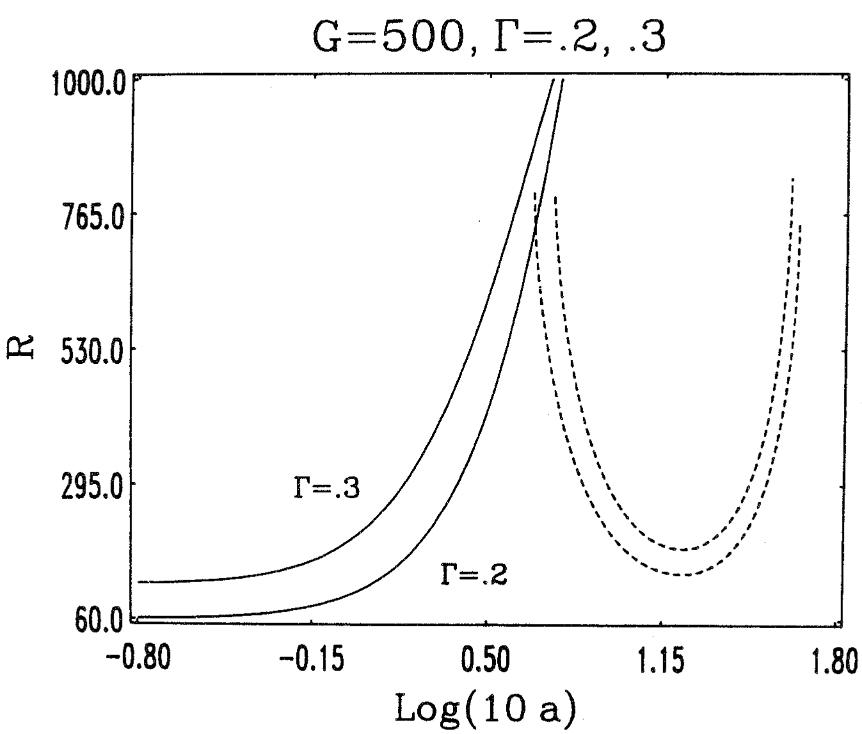

Figure 4. The same as in Fig. 3 for two different (small) values of Γ and fixed G.